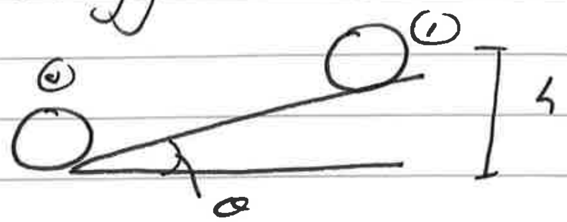
Dr G Wells / Dr G Csanyi

Part IB Paper 1 2011 

1 a) Conservation of energy: $KE_0 = PE_1$




$$\tfrac{1}{2} m v_0^2 + \tfrac{1}{2} I_g \Omega_0^2 = mgh$$

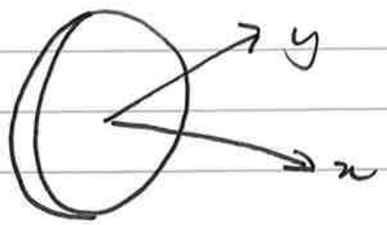
$$I_g = \frac{ma^2}{2}, \quad \Omega_0 = v_0/a$$

$$\rightarrow \frac{1}{2}\cancel{m}v^2 + \frac{1}{2}\frac{\cancel{ma^2}}{2}\frac{v^2}{\cancel{a^2}} = \cancel{m}gh$$

$$\frac{3v^2}{4} = gh \rightarrow h = \frac{3v^2}{4g}$$

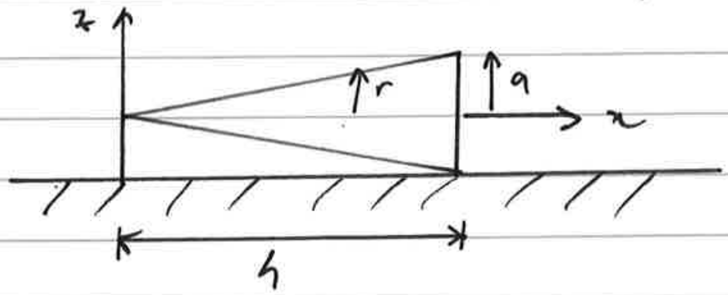
$$\text{distance } d = \frac{3v^2}{4g\sin\theta}$$

b) i)




For a disk: $I_{xx} = \dfrac{ma^2}{2}$ (data book)

$I_{zz} = \dfrac{ma^2}{4}$



$r = ax/h$ , $V = \pi a^2 h/3$

$\rightarrow \rho = m/V = \dfrac{3m}{\pi a^2 h}$

$$dI_{xx} = \frac{r^2}{2} dm$$

$$= \frac{r^2}{2} \underbrace{\cancel{\pi} r^2 \frac{3m}{\cancel{\pi} a^2 h} dx}_{dm}$$

$$= \frac{3mr^4}{2a^2 h} dx$$

$$I_{xx} = \int_0^h \frac{3mr^4}{2a^2h} dx = \underline{\frac{3ma^2}{10}}$$

$$dI_{zz} = dI_{z'z'} + x^2 dm$$

$$= \frac{r^2}{4} dm + x^2 dm$$

$$= \frac{3r^4 m}{4a^2 h} dx + \frac{3mr^2}{a^2 h} x^2 dx$$

$$I_{zz} = \int_0^h \frac{3r^4 m}{4a^2h} + \frac{3mr^2}{a^2h} x^2 \; dx$$

$$= \frac{3ma^2}{20} + \frac{3mh^2}{5}$$

Both $I_{xx}$ and $I_{zz}$ can be verified against data book.

5) ii)


$$v_E = \omega a = \Omega h \rightarrow \omega = \frac{\Omega h}{a}$$

angular velocity about $x$-axis

$$KE = \frac{1}{2} I_{xx} \frac{\Omega^2 h^2}{a^2} + \frac{1}{2} I_{zz} \Omega^2$$

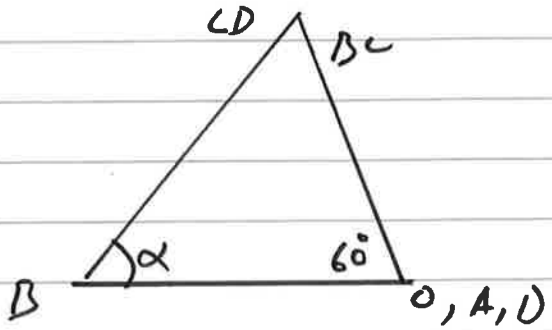
$$= \frac{1}{2} \frac{3m\cancel{a^2}}{10} \frac{\Omega^2 h^2}{\cancel{a^2}} + \frac{1}{2}\left(\frac{3ma^2}{20} + \frac{3mh^2}{5}\right)\Omega^2$$

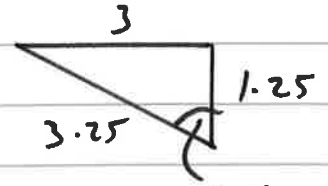
$$= \frac{\Omega^2 3m}{20}\left(3h^2 + \frac{a^2}{2}\right)$$

2. a) $\dot{r} = Q/A \rightarrow r = \int \frac{Q}{A}\, dt = \left. \frac{Qt}{A} \right|_{t_1}^{t_2}$

$\rightarrow \Delta r = Q(t_2 - t_1)/A$

b)




$\alpha \approx 67.38$

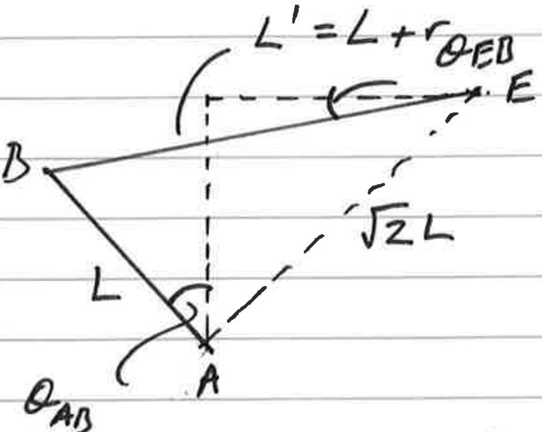
$$\frac{v_{DC}}{\sin\alpha} = \frac{\dot{r}}{\sin(120-\alpha)} \rightarrow v_{DC} = \frac{\dot{r}\sin\alpha}{\sin(120-\alpha)}$$

$$v_{DC} = \omega_{DC}\, 4.5L$$

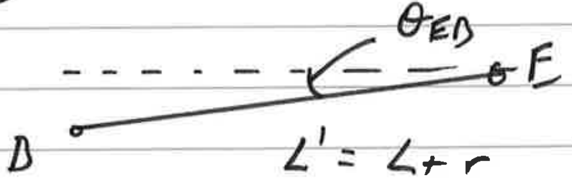
$$\rightarrow \omega_{DC} = \frac{\dot{r}\sin\alpha}{4.5L\sin(120-\alpha)}$$

$$\approx 0.258\,\dot{r}/L$$

c)




i)




$$x_B = -(L+r)\cos(\theta_{EB})$$

$$\dot{x}_B = -\dot{r}\cos(\theta_{EB}) + (L+r)\dot{\theta}_{EB}\sin(\theta_{EB})$$

$$\ddot{x}_B = -\cancel{\ddot{r}}^{\,0} \cos\theta_{EB} + \dot{r}\,\dot{\theta}_{EB} \sin\theta_{EB}$$

$$+ \dot{r}\,\dot{\theta}_{EB} \sin(\theta_{EB}) + (L+r)\,\dot{\theta}_{EB}^2 \cos\theta_{EB}$$
$$+ (L+r) \sin\theta_{EB}\, \ddot{\theta}_{EB}$$

$$= 2\dot{r}\,\dot{\theta}_{EB} \sin\theta_{EB} + (L+r)\,\dot{\theta}_{EB}^2 \cos\theta_{EB}$$
$$+ (L+r)\,\ddot{\theta}_{EB} \sin\theta_{EB}$$

ii) $$x_B = -L\sin\theta_{AB}$$

$$\dot{x}_B = -L\,\dot{\theta}_{AB} \cos\theta_{AB}$$

Need to eliminate $\dot{\theta}_{AB}$. Use cosine rule:

$$L'^2 = (L+r)^2 = L^2 + 2L^2 - \underbrace{2\sqrt{2}L^2 \cos(\theta_{AB} + \pi/4)}_{= 2L^2(\cos\theta_{AB} - \sin\theta_{AB})} \quad (1)$$

Take time derivative:
$$2L'\dot{L}' = 2\sqrt{2}L^2\,\dot{\theta}_{AB} \sin(\theta_{AB} + \pi/4)$$

Since $\dot{L}' = \dot{r}$,

$$\dot{\theta}_{AB} = \frac{L'\dot{r}}{L^2(\sin\theta_{AB} + \cos\theta_{AB})}$$

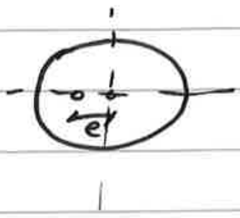
using $\sin(\theta_{AB} + \pi/4) = (\sin\theta_{AB} + \cos\theta_{AB})/\sqrt{2}$

$$\rightarrow \dot{x}_B = \frac{-\cancel{L}\, L'\dot{r} \cos\theta_{AB}}{L^{\cancel{2}}(\sin\theta_{AB} + \cos\theta_{AB})}$$

Use (1) to eliminate $r$ in $L'$

$$\dot{x}_B = \frac{-\dot{r}\cos\theta_{AB}\,\cancel{L}\sqrt{3 - 2(\cos\theta_{AB} - \sin\theta_{AB})}}{\cancel{L}\,\sin\theta_{AB} + \cos\theta_{AB}}$$

3 a)

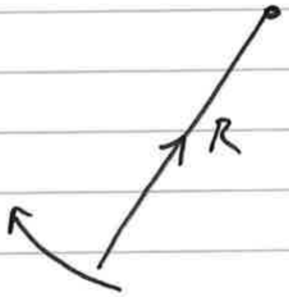



$$|F_A + F_B| = m\bar{\omega}^2 e$$

$$\therefore |F_A| = \frac{m\bar{\omega}^2 e}{2}$$

$\bar{\omega}$ : rad/s

b)




$$a = v^2/R$$

$$F_A = \frac{mv^2}{2R}$$

towards plane

c) Gyroscopic couple $\underline{Q}$

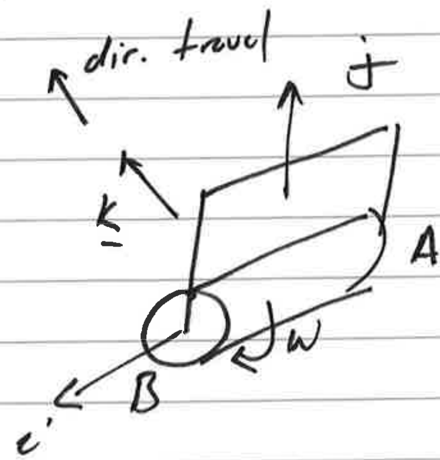
$$\underline{Q} = I_1 \omega_1 \omega_2 \, \underline{k} \times \underline{i}$$

spin ↗ ($\omega_1$)    precession ↑ ($\omega_2$)

$$= I_1 \omega_1 \omega_2 \underline{j}$$

$$I_1 = \frac{mr^2}{2}$$

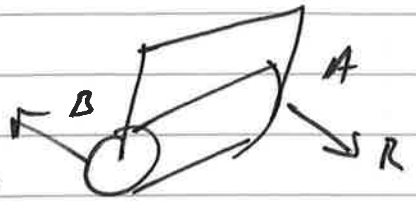
<u>Roll</u>

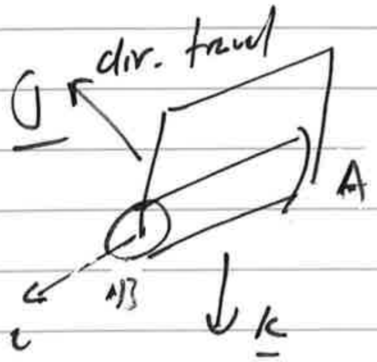


$$\underline{Q}_{roll} = -\frac{mr^2\bar{\omega}\,\theta_1}{2}\,\underline{j}$$

↑ +ve, rolls

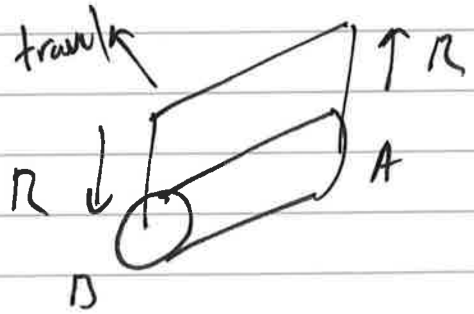
$$\left|\frac{\underline{Q}_{roll}}{L}\right| = \frac{mr^2\bar{\omega}\theta_1}{2L} = R$$




$\underline{\text{Yaw}}$ $$\underline{Q}_{yaw} = -\frac{mr^2\bar{\omega}\theta_2}{2}\,\underline{j}$$

$$\left|\frac{\underline{Q}_{yaw}}{L}\right| = \frac{mr^2\bar{\omega}\theta_2}{2L} = R$$

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July 2011

2. The forces acting on the pencil are gravity and the reaction force at the point of contact (which is composed of a normal force $N$ and a horizontal frictional force $F$).

a) The moment of inertia of the pencil about its end point is $ml^2/3$, either from the Data Book or by integration,

$$I = \frac{m}{l}\int_0^l x^2 dx = ml^2/3.$$

b) After rotating through an angle $\theta$, the centre of mass of the pencil has moved down a distance $l(1-\cos\theta)/2$. Conservation of energy during the rotation implies that

$$\begin{aligned} \frac{1}{2}\frac{ml^2}{3}\dot{\theta}^2 &= mg\frac{l}{2}(1-\cos\theta) \\ \dot{\theta} &= \sqrt{3g(1-\cos\theta)/l} \end{aligned}$$

The angular acceleration $\ddot{\theta}$ is obtained by considering the torque around the point of contact, to which only the moment of the gravitational force contributes, since the reaction force goes through the point of contact. Thus

$$\begin{aligned} \frac{l}{2}g\sin\theta &= \frac{l^2}{3}\ddot{\theta} \\ \ddot{\theta} &= \frac{3g}{2l}\sin\theta \end{aligned}$$

c) Resolving the equation of motion $F = ma$ first in the vertical direction gives

$$\begin{aligned} mg - N &= m\frac{l}{2}\dot{\theta}^2\cos\theta + m\frac{l}{2}\ddot{\theta}\sin\theta \\ N &= mg - \frac{3}{2}mg(1-\cos\theta)\cos\theta - mg\frac{3}{4}\sin^2\theta \\ &= \frac{1}{4}mg\left(1 - 6\cos\theta + 6\cos^2\theta - 3\sin^2\theta\right) \\ &= \frac{mg}{4}\left(4 - 6\cos\theta + 9\cos^2\theta - 3\right) \\ &= \frac{mg}{4}(1-3\cos\theta)^2 \end{aligned}$$

And resolving in the horizontal direction gives

$$\begin{aligned} F &= -m\frac{l}{2}\dot{\theta}^2\sin\theta + m\frac{l}{2}\ddot{\theta}\cos\theta \\ &= -\frac{3}{2}mg\,(1-\cos\theta)\sin\theta + mg\frac{3}{4}\sin\theta\cos\theta \\ &= \frac{3}{4}mg\sin\theta\,(3\cos\theta - 2) \end{aligned}$$

d) Looking at the expression for the normal force, if the pencil has not slipped before, then at $\cos\theta = 1/3$, we have $N = 0$, so the pencil must slip.

e) The frictional force goes through zero and changes sign when $\cos\theta = 2/3$. The ratio $F/N$ is

$$\frac{3\sin\theta(3\cos\theta - 2)}{(1-3\cos\theta)^2}$$

which starts at zero, has a maximum, comes back to zero at $\cos\theta = 2/3$ and diverges at $\cos\theta = 1/3$

3. Static balance equation: $\sum m_i \boldsymbol{r}_i = 0$, dynamic balance equation: $\sum m_i \boldsymbol{r}_i z_i = 0$.

a) The total out of mass balance is $7R/6 \times 30\text{g} = 35R$ g, so static balance would be achieved by placing a 35 g mass directly opposite the out-of-balance mass, so at an angle of 225°. To achieve dynamic balance, we need to split this mass on the two rims, suppose we split it in the ratio $p$ and $1-p$. Then the dynamic balance equation is

$$30\frac{7}{6}R\frac{d}{6} + p35R\frac{d}{2} - (1-p)35R\frac{d}{2} = 0$$

which yields $p = 1/3$, so the two masses are $35/3 = 11.666$ g and $70/3 = 23.333$ g, again directly opposite the original out of balance mass.

b) Now we need to achieve the balances in part (a) using 6 g and 12 g masses. The 6 g masses are used to achieve the 11.666 g balance and the 12 g masses are used for the 23.333 g balance. The angles of the 6 g masses are symmetrical on either side of 225° deviating by an angle given by $6\cos\phi = 5.8333$, so $\phi \approx 13.5°$, and the 12 g masses are again symmetrical around 225° deviating by an angle $\psi$ given by $12\cos\psi = 11.666$, so $\psi \approx 13.5°$ also.

c) With just the 6 g weights, the out of balance is $(7/6 \times 30\text{g} - 11.666)R = 23.3333\,R\text{g}$ in the plane of the tyre along 45°.