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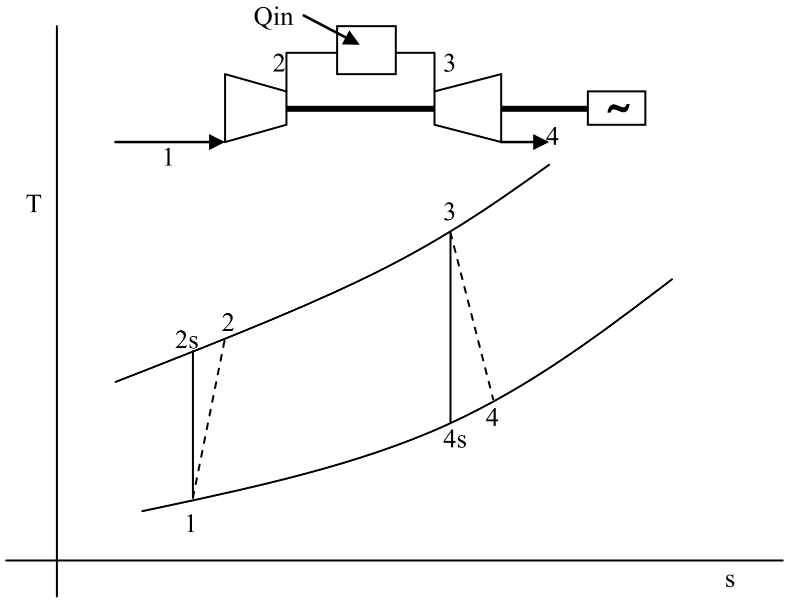
2011 IB Paper 4 Solutions

Qu 1

a)



b) $T_1 = 293.15K$, $T_3 = 1773.15K$

Compressor – 1 to 2s $\frac{T_{2s}}{T_1} = \left(\frac{p_2}{p_1}\right)^{\gamma-1/\gamma}$ $\therefore T_{2s} = 689.9K$

Real process $\eta_C = \frac{T_{2s} - T_2}{T_2 - T_1}$ $\therefore T_2 = T_1 + (T_{2s} - T_1)/\eta_C = 760.0K$

Turbine – 3 to 4s $\frac{T_{4s}}{T_3} = \left(\frac{p_1}{p_2}\right)^{\gamma-1/\gamma}$ $\therefore T_{4s} = 753.4K$

Real process $\eta_T = \frac{T_3 - T_4}{T_3 - T_{4s}}$ $\therefore T_4 = T_3 - \eta_T(T_3 - T_{4s}) = 906.4K$

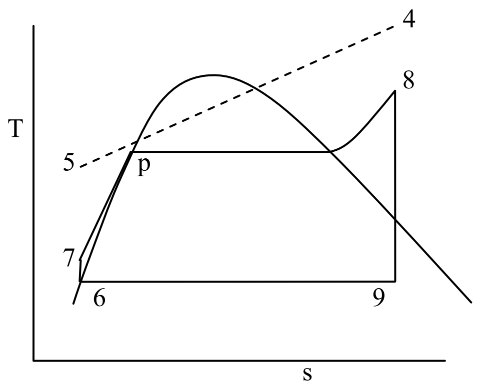
Power balance:- $\dot{m}c_p(T_3 - T_4) - \dot{m}c_p(T_2 - T_1) = 350E6$ W

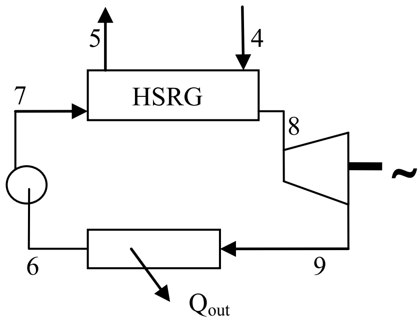
$\dot{m} * 1.01E3((1773.15 - 906.4) - (760.0 - 293.15)) = 350E6$ W

$\therefore \dot{m}_{gas} = 866.4$ kg/s

Heat flow rate $\dot{Q}_{23} = \dot{m}_{gas}c_p(T_3 - T_4) = 866.4 * 1010 * (1773.15 - 760.0) = 886.6$ MW

GT cycle efficiency $\eta_{GT} = 350/886.6 = 0.395$





Enthalpy balance across HSRG:-

$$\dot{m}_{GT} c_p (T_4 - T_5) = \dot{m}_{H_2O} (h_8 - h_7)$$

$$\dot{m}_{H_2O} = \frac{\dot{m}_{GT} c_p (T_4 - T_5)}{(h_8 - h_7)} = \frac{866.4 * 1.01(906.4 - 403)}{(3675 - 121.4)} = 123.9 \text{ kg/s}$$

[Check on the pinch point:- the $h_{f,30bar} = h_p = 1008.4$ kJ/kg. Heat transferred between 7 and p is $\dot{m}_{H_2O}(h_p - h_7) = 123.9(1008.4 - 121.4) = 109.9$ MW. The steam temperature at the pinch point (tables) is 233.85ºC. The gas temperature there will be given by $109.9E6 = \dot{m}_{gas} c_p (t_p - 110) = 866.4 * 1010 (t_p - 130)$. Hence $t_p = 255.9$ ºC – OK]

Power output $\dot{W}_x = \dot{m}_{H_2O}(h_8 - h_9)$

$$h_8 = 3675\, kJ/kg,\ h_9 = 2260\, kJ/kg$$

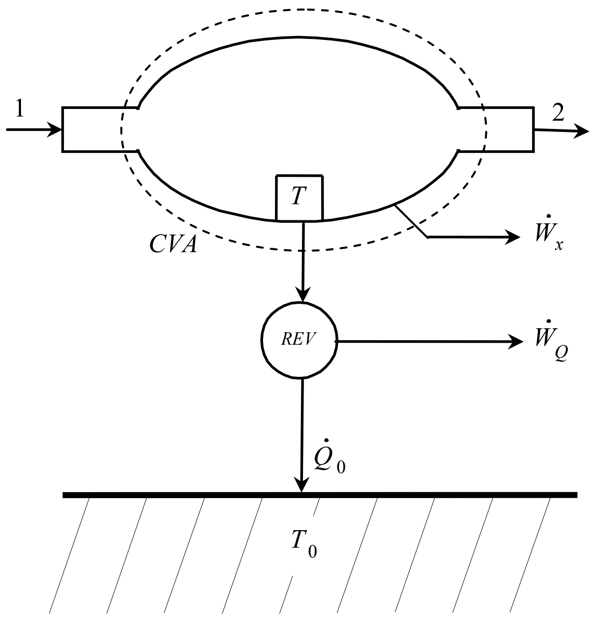
$$\therefore \dot{W}_x = 123.9(3675 - 2260) = 175.3\, MW$$

c) Total power = 350 + 175.3 = 525.3 MW, and the overall cycle efficiency is

$$\therefore \eta_{CC} = 525.3 / 886.6 = 59.3\%$$

d) Increase GT inlet temperature
Reduce condenser pressure
Reheat in HRSG

Q2 Soln



1[st] Law: $-\dot{Q}_0 - \left(\dot{W}_X + \dot{W}_Q\right) = \dot{m}(h_2 - h_1)$

2[nd] Law: $\dot{m}(s_2 - s_1) = \frac{-Q_0}{T_0}$

Combine the two equations:

$$\dot{m}T_0(s_2 - s_1) - \left(\dot{W}_X + \dot{W}_Q\right) = \dot{m}(h_2 - h_1)$$
$$-\left(\dot{W}_X + \dot{W}_Q\right) = (h_2 - h_1) - T_0(s_2 - s_1)$$

$$b_1 = h_1 - T_0 s_1$$

$b_2 = h_2 - T_0 s_2$ so

$$b_2 - b_1 = (h_2 - h_1) - T_0(s_2 - s_1)$$

Tidy up. so $b_2 - b_1 = -\left(\dot{W}_x + \dot{W}_Q\right)$

$$b_1 - b_2 = \dot{W}_x + \dot{W}_Q$$
$$\Delta b = \dot{W}_{MAX}$$

b) Surroundings 1 bar, 300 K

i) Entropy change $s_2 - s_1 = c_p \ln\left\{\frac{T_2}{T_1}\right\} - R\ln\left\{\frac{P_2}{P_1}\right\}$

$$-\frac{\dot{W}_{\mathrm{MAX}}}{\dot{m}} = c_p(T_2 - T_1) - T_0(s_2 - s_1)$$

$$\frac{\dot{W}_{\mathrm{MAX}}}{\dot{m}} = 325.6 \text{ kW kg}^{-1}$$

ii) Isentropic turbine

$$-\frac{\dot{W}_{\mathrm{TURB}}}{\dot{m}} = c_p(T_2 - T_1)$$

$$T_2 = T_1\left\{\frac{p_2}{p_1}\right\}^{\frac{\gamma-1}{\gamma}} = 700\left\{\frac{1}{8}\right\}^{\frac{1.4-1}{1.4}} = 386.4\text{K}$$

$$\rightarrow \frac{\dot{W}_{\mathrm{TURB}}}{\dot{m}} = 315.14 \text{ kW kg} - 1$$

So fraction of max $\frac{315.14}{325.6} = 96.8\%$

iii) Theory

Turbine exit is above ambient so a heat engine (reversible) could be run reducing the stream to the dead state.

c) Thermal equilibrium $T_1 = 300\text{K} = T_{\text{amb}}$

i) Isentropic turbine

(Cold reservoir) $T_{2s} = T_{\text{amb}}\{p_2/p_1\}^{\frac{\gamma-1}{\gamma}}$

$$= 300\left\{\frac{1}{8}\right\}^{\frac{1.4-1}{1.4}}$$

$$= 165.6 \text{ K}$$

$$-\frac{\dot{W}_{\text{TURB}}}{\dot{m}} = c_p(T_{2S} - T_1)$$

$$\frac{\dot{W}_{\text{TURB}}}{\dot{m}} = 135.1 \text{ kW kg}^{-1}$$

Fraction of lost max available power $\dfrac{135.1}{325.6} = 41.5\%$

ii) Real turbine

(Cold reservoir) $\eta_T = 80\%$ $\eta_T = \frac{T_1 - T_2}{T_1 - T_{2s}}$

$$\eta_T = \frac{T_{\text{amb}} - T_2}{T_{\text{amb}} - T_{2s}}$$

So $T_2 = T_{\text{amb}} - \eta_T(T_{\text{amb}} - T_{2s})$ = 192.5K

$$-\frac{\dot{W}_{\text{TURB}}}{\dot{m}} = c_p(T_1 - T_2)$$

$$\frac{\dot{W}_{\text{TURB}}}{\dot{m}} = 108.1 \text{ kW kg}^{-1}$$

Qu 3 Soln

a) The fraction of the radiation leaving body 1 and reaching body 2 is the shape factor F_{12}. For a black body of area A_1, the radiation leaving 1 and reaching 2 is $A_1F_{12}\sigma T_1^4$. Similarly the radiation leaving 2 and reaching 1 is $A_2F_{21}\sigma T_2^4$. Now, as the bodies are black, all the energy reaching a body is absorbed, so the net exchange between body 1 and 2 is

$$\dot{Q}_{12} = A_{12}F_{12}\sigma T_1^4 - A_{21}F_{21}\sigma T_2^4$$

Now by the reciprocity relationship, we know that $A_1F_{12} = A_2F_{21}$(which must be true, as the net heat exchange must be zero when $T_1 = T_2$). So finally we have

$$\dot{Q}_{12} = A_{12}F_{12}\sigma\left(T_1^4 - T_2^4\right)$$

b) i) from the result above (the filament is in radiation exchange with a single body), we have

$$\dot{Q} = A_{wire}\sigma\left(T_{wire}^{\ 4} - T_{glassinner}^{\ 4}\right),$$

so per unit length

$$\frac{\dot{Q}}{l} = \pi * 0.002 * 5.67E-8\left(3000^4 - 700^4\right) = 2.88kW/m$$

b) ii) Thick walled cylinder. Thermal resistance per unit length is $R_{th} = \dfrac{ln\left(r_{outer}/r_{inner}\right)}{2\pi\lambda}$, so

$R_{th} = \dfrac{ln\left(12/10\right)}{2\pi 1.38} = 0.021K/(W/m)$. Thus $\left(T_{inner} - T_{outer}\right) = 0.021 * 2.877E3 = 60.5K$, and the outer temperature is 639.5 K.

b) iii) The energy leaving by radiation will be $\dfrac{\dot{Q}}{l} = \pi 0.024\sigma\left(639.6^4 - 293^4\right)$ = 683.9W/m, which is 683.9/2877 = 23.8% of the total.

b) iv) The Nusselt number is defined by $Nu_D = \dfrac{hD}{\lambda}$; the rate of heat transfer by convection per unit length is $\dfrac{\dot{Q}}{l} = h\dfrac{A_{outer}}{l}\left(T_{outer} - T_\infty\right)$.

The Reynolds number is $Re_D = \dfrac{\rho U_\infty D}{\mu}$. Thus

$$\frac{\dot{Q}}{l} = Nu_D\frac{\lambda}{D}\frac{A_{outer}}{l}\left(T_{outer} - T_\infty\right) = 0.2\left(\frac{\rho U_\infty D}{\mu}\right)^{0.6} Pr^{.33}\frac{\lambda}{D}\pi D\left(T_{outer} - T_\infty\right)$$
$$= \left(2877 - 683.9\right)$$

Thus the Reynolds' number is found from

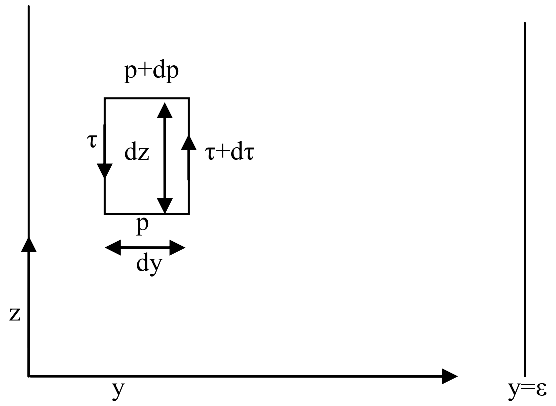
$2193.1 = 0.2 Re_d^{0.6}\, 0.71^{0.33} 0.026\pi(639.6 - 293)$, from which we find that $Re_d =$ 2.48E4.

Finally, we have $\frac{\rho U_\infty D}{\mu} = \frac{1.25 * 0.024}{2E-5} U_\infty = 2.48E4$, and thus $U_\infty = 16.56$ m/s.

Note that the values of C and n in the Nusselt number correlation $Nu_D = C\, Re^n\, Pr^{1/3}$ (for cylinders in cross flow) vary considerably with the Reynolds' number. In this question, the choice of the appropriate correlation – which would be required in practice, and might involve iteration – was not considered for simplicity.

Q4 Soln

(a)



A force balance on the element gives (assuming unit distance into the page):-

$$-\tau dz + (\tau + d\tau)dz - dp dy = 0$$

But by definition, $\tau = \mu \dfrac{du}{dy}$

$$\therefore \frac{dp}{dz} = \mu \frac{d^2u}{dy^2}$$

b (i) The forces on the piston must sum to zero. The three forces are those due to pressure, skin friction, and the piston's mass. To determine the skin friction, we need to integrate the last expression to get the velocity profile in the gap. Taking (arbitrarily) the y origin as at (and of course outward normal to) the piston surface, we have (noting that the pressure gradient in the z direction must be constant, as the gap and flow rate is uniform, and we are neglecting hydrostatic pressure effects)

$$y\frac{dp}{dz} = \mu \frac{du}{dy} + C_1$$

$$\frac{y^2}{2}\frac{dp}{dz} = \mu u + C_1 y + C_2$$

$$y = 0, u = 0, \therefore C_2 = 0$$

$$y = \varepsilon, u = 0, \therefore C_1 = \frac{\varepsilon}{2}\frac{dp}{dz}$$

$$\therefore \mu u = \frac{y}{2}(y - \varepsilon)\frac{dp}{dz}$$

The shear stress at the piston ($y = 0$) is given by

$\tau_w = \mu \left.\frac{du}{dy}\right)_w = \left(0 - \frac{\varepsilon}{2}\right)\frac{dp}{dz}$. Noting that $\frac{dp}{dz} = \frac{(p_2 - p_1)}{h}$, which is a negative quantity, we see that as expected the shear stress acting <u>on</u> the walls is in the +ve z direction for the piston and the cylinder. The shear stress acting on the piston is thus

$$\tau_w = \frac{\varepsilon}{2}\frac{(p_1 - p_2)}{h}.$$

Summing the forces on the piston, we have

$$mg = \frac{\pi}{4}(D - 2\varepsilon)^2(p_1 - p_2) + \tau_w \pi (D - 2\varepsilon)h$$

Or, substituting for τ_w and re-arranging we obtain

$$(p_1 - p_2) = \frac{4mg}{\pi D(D - 2\varepsilon)}$$

[$(p_1 - p_2) = \frac{4mg}{\pi D^2}$ is also acceptable as $D >> \varepsilon$]

b (ii) The flow rate, Q, is given by

$$Q = \pi D \int_0^\varepsilon u dy = \frac{\pi D}{2\mu}\frac{dp}{dz}\int_0^\varepsilon (y\varepsilon - y^2) dy$$

$$\therefore Q = \frac{\pi D \varepsilon^3}{12\mu}\frac{(p_1 - p_2)}{h}$$

b (iii) Rate of energy dissipation is $Q(p_1 - p_2)$.

This is the work that would have to be supplied by an ideal pump.

For discussion:-

- How would the results be affected if the variation in hydrostatic pressure was significant?

- will the piston automatically be centered in the cylinder?

Q 5 Soln

Applying continuity between 1 and 2, we have

$$\rho u\frac{\pi}{4}\left(D^2-d^2\right)+\rho v\frac{\pi}{4}d^2=\rho w\frac{\pi}{4}D^2$$

Dividing through by D^2, and making the substitution $k=d^2/D^2$ we obtain

$$w=kv+(1-k)u \qquad \text{(eq 1)}$$

As required.

Applying Bernoulli for a streamline which passes through the annulus, and originates far from section 1, we have

$$p_a=p_1+\frac{1}{2}\rho u^2 \qquad \text{(eq 2)}$$

Applying the force momentum equation between 1 and 2, we have, noting the streamlines exiting at 2 are parallel, so $p_2=p_a$, (and omitting the common term $\pi/4$, and dividing through by D^2)

$$p_1-p_a=\rho w^2-\rho v^2k-\rho u^2(1-k)$$

Eliminating p_1-p_a between these two expressions gives

$$-\frac{1}{2}\rho u^2=\rho w^2-\rho v^2k-\rho u^2(1-k)$$

Which can be written as

$$2w^2-2kv^2=(1-2k)u^2 \qquad \text{(eq 3)}$$

As required.

With $k=1/2$, and $v=10$ m/s, substitution into eq 3 gives $w=10/\sqrt{2}$ (7.07 m/s) and eq 1 then gives $u=10\left(\sqrt{2}-1\right)$ (4.14 m/s).

$(p_a-p_1)=1/2\rho u^2$ (from eq 2). $\frac{1}{2}\rho u^2=0.5*\ 1.2*\ 100\left(\sqrt{2}-1\right)^2$=10.3 N/m^2

[Clearly we could eliminate u or w using eqs 1 and 3, but the algebra is tedious – for the record, the results are

$$\frac{u}{v}=\frac{\sqrt{2k(1-k)}-2k(1-k)}{2k^2-2k+1} \quad \text{and} \quad \frac{w}{v}=\frac{\sqrt{2k(1-k)}+\left(2k^3-k\right)}{2k^2-2k+1}$$

]

For the last part, we need to apply the force-momentum equation – and we must use the control volume shown in the figure for the whole ejector pump (why?). Far upstream of the inlet, the velocities are so low that they can be neglected. Ambient pressure acts on all the control volume, except inside the tube. Since we are neglecting fluid friction, the pressure everywhere in the tube is p_1. Thus the only term in the force-momentum equation due to pressure is $(p_1 - p_a)\frac{\pi}{4}d^2$.

If the axial compression in the tube where it crosses the control volume is F, (i.e. F is an external force acting left to right on the control volume) then

$$F + (p_1 - p_a)\frac{\pi}{4}d^2 = \rho w^2 \frac{\pi}{4}D^2 - \rho v^2 \frac{\pi}{4}d^2 \qquad \text{eq 4}$$

Because far upstream of the ejector, the flow being induced has negligible velocity where it crosses the control volume. Thus

$$F - 10.3\frac{\pi}{4}0.5 * 0.04 = 1.2 * 50\frac{\pi}{4}0.04 - 1.2 * 100\frac{\pi}{4}0.5 * 0.04$$

$$\therefore F = 0.162\, N$$

And the tube is in compression.

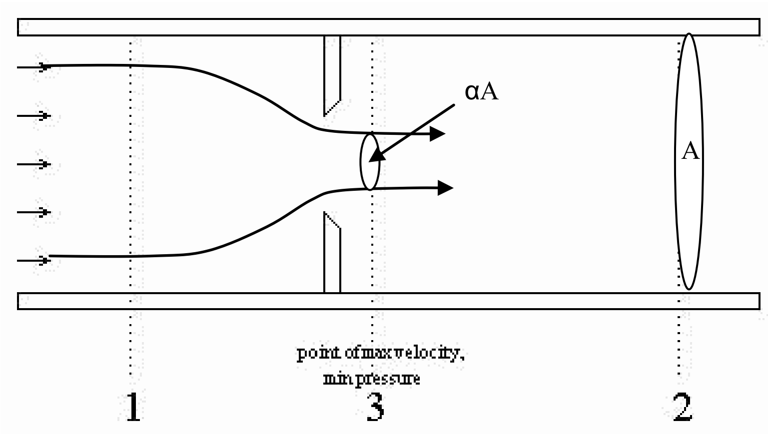
Note that the RHS of the equation equals zero. Some attempts at this question assumed that the momentum change was zero, and thus obtained the correct numerical answer – but if in eq 4 we substitute from eq 3 on the RHS, we obtain

$$F + (p_1 - p_a)\frac{\pi}{4}d^2 = \rho\frac{\pi}{4}D^2\frac{(1-2k)}{2}u^2$$

We see that only for $k = 1/2$ is RHS of eq 4 equal to zero.

Q6 Soln

(a)



Four physical principles are required:-

i) Continuity - $\rho A v$ is constant. [$Q_{orifice} = Av_1 = \alpha A v_3 = Av_2$]

ii) Bernoulli's equation applies between 1 and 3 [$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_3^2$]

iii) Streamlines straight at 3 – or outside the jet of low velocity – therefore pressure constant across plane 3

iv) The force momentum equation applied to the control containing the whole flow between 3 and 2 will allow determination of the "pressure recovery" as the flow mixes between 3 and 2. [$p_3 A - p_2 A = \dot{m}v_2 - \dot{m}v_3$] (No body forces as axial distance too small for significant wall friction effects.)

(b) (i) The minimum pump work for a given flow rate must be associated with the minimum throttling losses. The pressure at stations 1 and 2 must be greater than that at station 3, due to friction, so in order to get equal flows in each delivery pipe, the valve V_3 can, and should, be fully open $(\alpha_3 = 1)$.

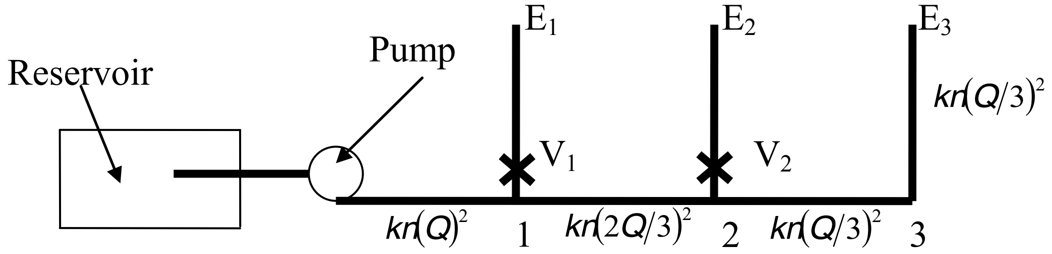
For pipe carrying a flow of Q, the pipe dynamic head is given by

$$\frac{1}{2}\rho v^2 = \frac{1}{2}\rho\left(\frac{4Q}{\pi d^2}\right)^2 = nQ^2 \quad \text{where} \qquad n = \frac{8\rho}{\pi^2 d^4} \ .$$

And the stagnation pressure drop due to friction is $\Delta p_{0,\,friction} = 4c_f \frac{L}{d}\frac{1}{2}\rho v^2 = knQ^2$,

where $k = 4c_f \frac{L}{d}$.

As there is no pressure loss across valve 3, we can determine the pressure drop from inlet to the open end, E_3, since we know all the flows. The stagnation pressure losses due to friction in each section are as shown on the figure.



The stagnation pressure entry to the pump is $p_{ambient}$ because there are no losses in the entry pipe (the static pressure will be $p_{ambient} - nQ^2$). Noting that the stagnation pressure is constant across all the junctions and bends, we see that the difference between the stagnation pressure after the pump $p_{ambient} + \Delta_{0,\,pump}$, and the stagnation pressure at E_3, $p_{ambient} + n(Q/3)^2$, is the sum of the stagnation pressure losses due to friction in the pipes between these locations, which are shown on the figure. Thus we can write

$$\left(p_{ambient} + \Delta p_{0,\,pump}\right) - \left(p_{ambient} + n(Q/3)^2\right) = 2kn(Q/3)^2 + kn(2Q/3)^2 + knQ^2 \quad \text{or}$$

$$\Delta p_{0,\,pump} = nQ^2\left(\frac{1+15k}{9}\right)$$

(b), (ii) For this part, we need to include the stagnation pressure losses across the valves. The valves each pass a flow of $Q/3$, so the valve pressure drop will be

$$\Delta p_{0,orifice} = \frac{1}{2}\rho\left(\frac{Q}{3}\frac{4}{\pi d^2}\right)^2\left(\frac{1-\alpha}{\alpha}\right)^2 = n\left(\frac{Q}{3}\frac{(1-\alpha)}{\alpha}\right)^2$$

Following the same method as in part (b) (i), we can thus write for the flow from the pump to E_1

$$\left(p_{ambient} + \Delta p_{0,\,pump}\right) - \left(p_{ambient} + n(Q/3)^2\right) = kn\left(\frac{Q}{3}\right)^2 + n\left(\frac{Q}{3}\frac{(1-\alpha_1)}{\alpha_1}\right)^2 + knQ^2$$

or $$\Delta p_{0,\,pump} = \frac{nQ^2}{9}\left(1 + 10k + \left(\frac{1-\alpha_1}{\alpha_1}\right)^2\right),$$

which when equated to the expression for $\Delta p_{0,\,pump}$ in part (b),(i), and we find

$$\alpha_1 = \left(1+\sqrt{5k}\right)^{-1}$$

For the flow to E_2 we have

$$\left(p_{ambient}+\Delta p_{0,\,pump}\right)-\left(p_{ambient}+n(Q/3)^2\right)=kn\left(\frac{Q}{3}\right)^2+n\left(\frac{Q}{3}\frac{(1-\alpha_2)}{\alpha_2}\right)^2+kn\left(\frac{2Q}{3}\right)^2$$

And once again substituting for $\Delta p_{0,\,pump}$ from part (b) (i), we obtain

$$\alpha_2 = \left(1+\sqrt{k}\right)^{-1}$$

Note that as k must be positive, the α_1, α_2 must be less than unity. Also we see that $\alpha_2 > \alpha_1$ as expected.

(b) (iii) The resistance to flow is proportional to Q^2, so if the pump pressure rise increases by 50%, there will be a 22.5% increase in flow $\left(\sqrt{1.5}=1.225\right)$. The relative flow rates remain the same since all losses scale as Q^2.