

## IB Paper 6, 2011: Solutions

### SECTION A

1. *Physical system transfer functions & responses*

(a)(i) The dynamics equations can be written

$$\begin{aligned}\tau_b \dot{\theta}_b &= \frac{\bar{q}_i - \bar{q}_o}{h} + (\theta_w - \theta_b) \\ \tau_w \dot{\theta}_w &= \theta_b - \theta_w\end{aligned}$$

where

$$\tau_b = \frac{m_b c_a}{h} = 150\text{s} \quad \tau_w = \frac{m_w c_w}{h} = 700\text{s}$$

Taking Laplace transforms

$$\begin{aligned}\bar{\theta}_b &= \frac{1}{s\tau_b + 1} \left( \bar{\theta}_w + \frac{\bar{q}_i - \bar{q}_o}{h} \right) \\ \bar{\theta}_w &= \frac{1}{s\tau_w + 1} \bar{\theta}_b\end{aligned}$$

Hence

$$[(s\tau_b + 1)(s\tau_w + 1) - 1] \bar{\theta}_b = \frac{s\tau_w + 1}{h} (\bar{q}_i - \bar{q}_o)$$

Rearranging:

$$\begin{aligned}\bar{\theta}_b &= \frac{(s\tau_w + 1)}{s(\tau_b\tau_w s + \tau_b + \tau_w)} \frac{(\bar{q}_i - \bar{q}_o)}{h} \\ \bar{\theta}_w &= \frac{1}{s(\tau_b\tau_w s + \tau_b + \tau_w)} \frac{(\bar{q}_i - \bar{q}_o)}{h}\end{aligned}$$

For the case that

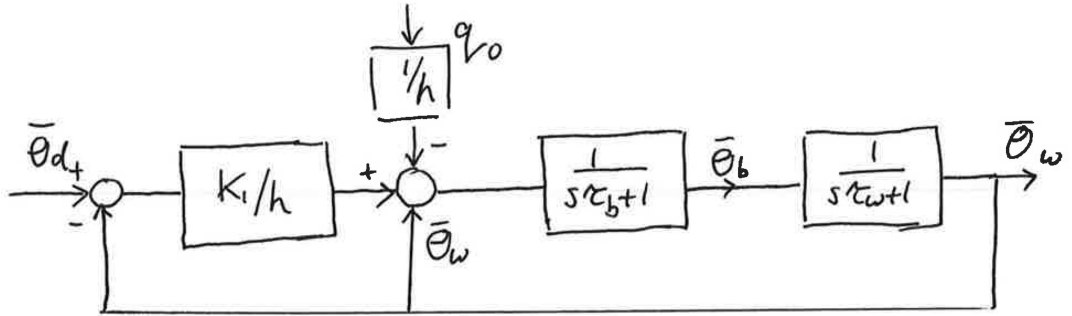
$$\bar{q}_i = k_1 (\bar{\theta}_d - \bar{\theta}_w)$$

Then block-diagram can be drawn from the initial Laplace transforms of the dynamics showing the inputs of blocks as  $\bar{\theta}_d$  and outputs as  $\bar{\theta}_b, \bar{\theta}_w$ , with  $\bar{q}_i$  defined as above.

The corresponding CLTF is

[7]

$$\bar{\theta}_w = \frac{k_1/h}{s(\tau_b\tau_w s + \tau_b + \tau_w) + k_1/h} (\bar{\theta}_d - \bar{q}_o/k_1)$$



(a)(ii) From the denominator of the CLTF

$$\omega_n^2 = \frac{k_1/h}{\tau_b\tau_w} \quad 2\zeta\omega_n = \frac{\tau_b + \tau_w}{\tau_b\tau_w}$$

Since  $\zeta = 1/\sqrt{2}$  then

$$2\frac{k_1/h}{\tau_b\tau_w} = \left(\frac{\tau_b + \tau_w}{\tau_b\tau_w}\right)^2 \quad \frac{k_1}{h} = \frac{1}{2} \cdot \frac{17^2}{14 \times 3}$$

(dividing top and bottom by  $50^2$ ), and hence  $k_1/h = 289/84$  and  $k_1 = 10.3$ .

Steady-state error =  $q_o/k_1$  (either from final value theorem or noting that at steady-state input and output power is equal). Hence error is  $4.8^\circ\text{C}$ . [5]

(b) In this case the feedback is from the temperature of the boiler. So that

$$\bar{q}_i = k_2(\bar{\theta}_d - \bar{\theta}_b)$$

Substituting this in the expression for  $\bar{\theta}_b$  and then keeping the final output as  $\bar{\theta}_w$  gives as CLTF

$$\bar{\theta}_w = \frac{k_2/h}{s(\tau_b\tau_w s + \tau_b + \tau_w) + k_2/h(1 + s\tau_w)} (\bar{\theta}_d - \bar{q}_o/k_2)$$

Thus to halve the steady-state error need to set  $k_2 = 2k_1 = 20.6$ .

To find closed loop poles solve

$$150 \times 700s^2 + (150 + 700 \times (1 + k_2/h))s + k_2/h = 0$$

Poles are both real, and one is very slow. [5]

(c) Increasing the gain in part (b) makes the steady state error small but at the cost of a very sluggish response to a step change in demanded temperature. Whereas in part (a) the response is much faster but the steady state error is larger and there is some oscillatory response and overshoot in the step response. [3]

## 2. Bode plots

(a) The Bode plot shows the magnitude (in dB) and phase (in degrees) of the steady state frequency response for a range of frequencies for the open loop system. To construct, need a sinusoidal input to the open loop system, when at steady state, measure the magnitude and phase. This should be done for a range of frequencies (often with logarithmic spacing) over the range of interest. [3]

(b)(i) First find the value of  $n$ . From the gradient at low frequency, and the low frequency phase of  $\approx -90^\circ$ , it is clear that  $n = 1$ .

For  $\alpha$  examine the frequency that the phase is  $-45^\circ$  ( $+45^\circ$  due to the numerator term). This happens at  $\omega = 0.2$  and hence  $\alpha = 0.2$ .

For  $\gamma$ , look at the resonant frequency. It is a fairly sharp peak at  $\omega = 2$ , hence  $\gamma = 2^2 = 4$ .

For  $\beta$  examine the height of the resonant peak. Although difficult to measure accurately from the plot it is  $\approx 14\text{dB}$ , which is a gain of 5. This is equal to  $\frac{1}{2\zeta}$  and hence  $\zeta = 0.1$  and  $\beta = 2\zeta\omega_n = 0.4$ .

Finally for  $A$ , need to match the overall gain. At  $\omega = 0.01$  the gain is  $\approx 56\text{dB} \approx 600$ . Hence

$$\begin{aligned}\frac{A\alpha}{0.01 \times 4} &= 600 \\ A &= 120\end{aligned}$$

Note that a reasonable range of values for  $A$  is acceptable. [6]

(b)(ii) It can be seen that the phase margin is very small, but the phase is slightly less than  $-180^\circ$  when the magnitude is equal to 1. Hence the closed loop system is stable, but the phase margin is about  $1^\circ$ . [2]

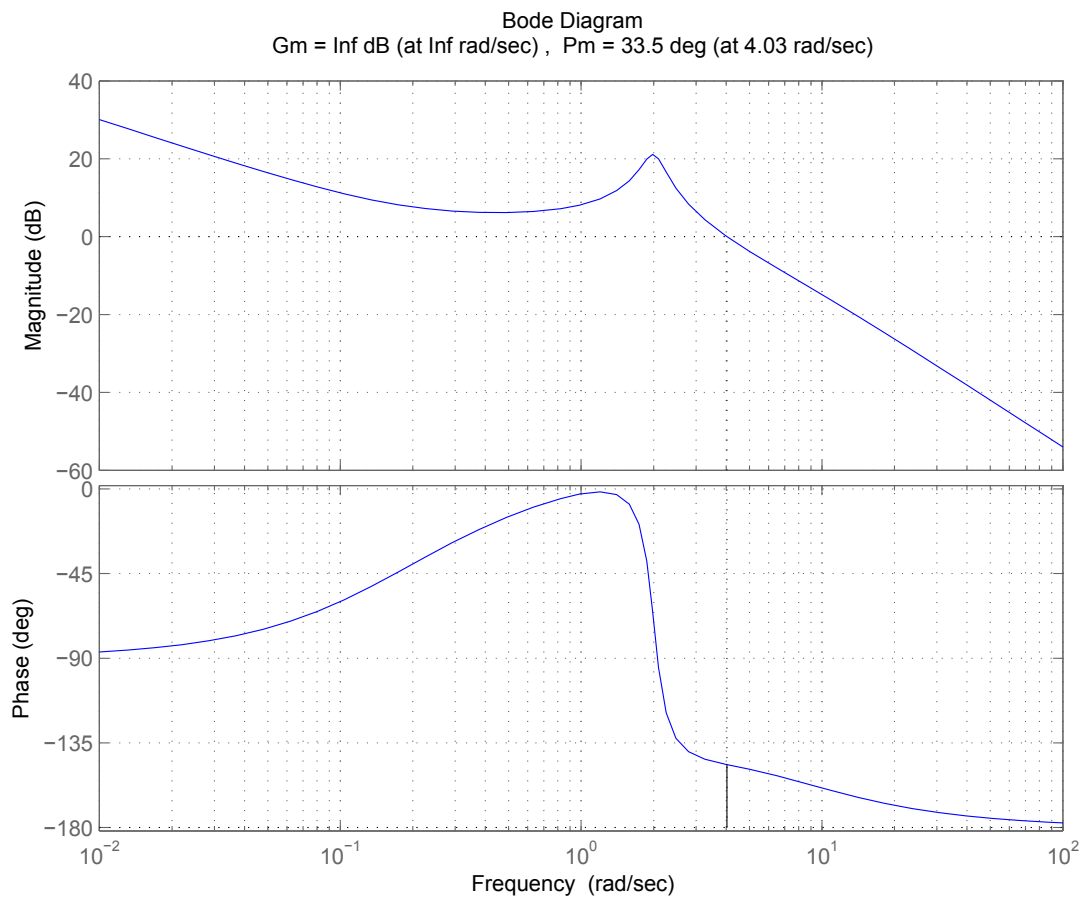
(c) At  $\omega = 4$  from the plot we have for  $|G(j4)| \approx 20\text{dB} = 10$ . To make  $|K(j\omega)G(j\omega)| \approx 1$  when  $\omega = 4$ , then the magnitude of the compensator response = 0.1. Taking the squared magnitude value with  $\omega = 4$ : [3]

$$\begin{aligned}\frac{\omega^2 + 4}{36(\omega^2 + B^2)} &= 0.01 \\ 16 + B^2 &= \frac{2000}{36} \\ B &= 6.29\end{aligned}$$

(d) We already know that the phase margin of the compensated system will be at  $\omega = 4$ . The phase of the compensator at this frequency is

$$\tan^{-1} \frac{\omega}{2} - \tan^{-1} \omega B = 31^\circ$$

Hence the overall phase margin is original phase difference to  $180^\circ$  at  $\omega = 4$  added to the compensator phase. The total new phase margin is  $33.5^\circ$ . The combined Bode plot of is should be drawn on the additional sheet: modified loop shown below. The modified loop has lower gain but will be much less oscillatory step response. [6]



### 3. Nyquist Diagrams

(a) The closed-loop system is stable if and only if the complete Nyquist diagram of the open loop system  $K(s)G(s)$  has no encirclements of the '-1' point . The Nyquist diagram is a plot of the open loop steady state frequency response of the system in the complex plane with the frequency varying along the plot. [4]

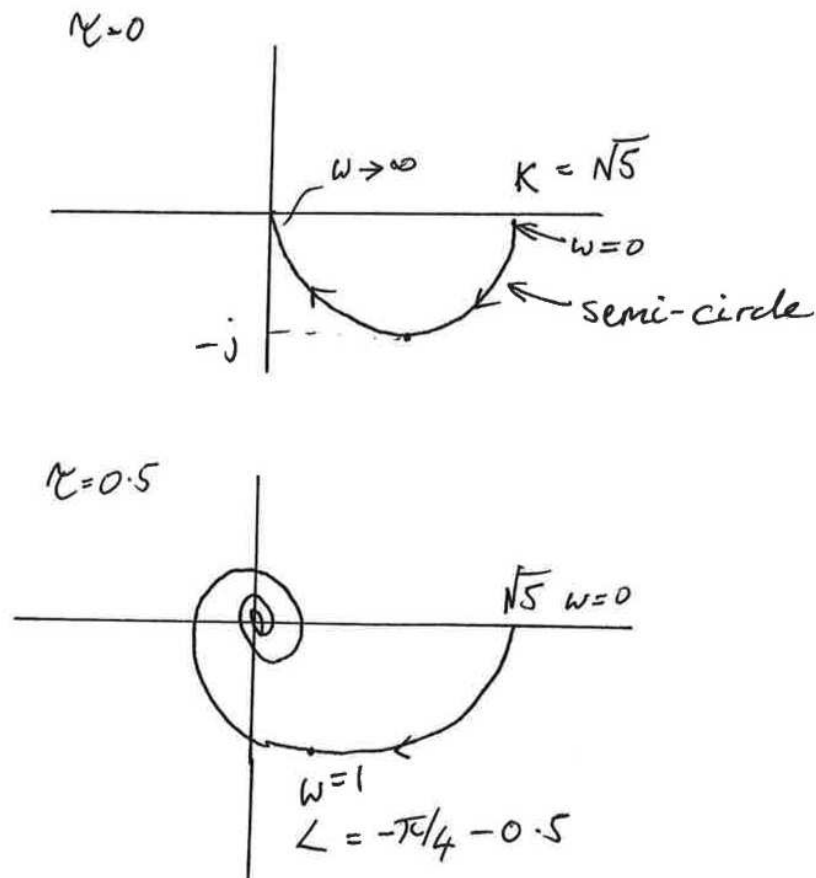
(b)(i) if  $|KG(j\omega)| > 1$  for all  $\omega < 2$ .

At  $\omega = 2$

$$\begin{aligned} |KG(j2)| &= \frac{1}{\sqrt{1+2^2}} \\ &= \frac{1}{\sqrt{5}} \end{aligned}$$

Hence  $K = \sqrt{5}$ . [4]

(b)(ii)



For stability considerations, examine the case when the magnitude is equal to unity and the phase angle is  $-\pi$ . With  $K = \sqrt{5}$ , then unit magnitude is at  $\omega = 2$ . [6]

$$\begin{aligned} -2\tau_{\text{crit}} - \tan^{-1} 2 &= -\pi \\ \tau_{\text{crit}} &= 1.017 \end{aligned}$$

(b)(iii) Now the gain margin is equal to 2. Hence the new value of  $K = K'$

$$K' = \frac{\sqrt{5}}{2}$$

For phase margin, find the frequency at which magnitude is unity,  $\omega_{\text{pm}}$

$$\begin{aligned} \frac{\sqrt{5}}{2\sqrt{\omega_{\text{pm}}^2 + 1}} &= 1 \\ \omega_{\text{pm}}^2 + 1 &= \frac{5}{4} \\ \omega_{\text{pm}} &= \frac{1}{2} \end{aligned}$$

At this frequency  $\omega_{\text{pm}}$ , the phase angle is

$$\frac{-1.017}{2} - \tan^{-1} \frac{1}{2} = -55.7^\circ$$

Hence the phase margin is  $180^\circ - 55.7^\circ = 124.3^\circ$ .

To find out where the closed loop response has unit magnitude need

$$\Re \{K'G(j\omega)\} = -0.5$$

For the case considered here, since the gain margin is equal to 2, then this occurs at  $\omega = 2$ . The real part of the frequency response is

$$\frac{\sqrt{5}}{2(1 + \omega^2)} (\cos \omega\tau - \omega \sin \omega\tau)$$

To find the other value when this is equal to -0.5 we need to use an iterative/trial and error solution and get  $\omega \approx 1.5$ . Hence the closed loop frequency response is greater than 1 between  $\approx 1.5$  and 2. [6]

## SECTION B

### 4. *Fourier Transform*

(a) This the familiar result about the Fourier transform of the rectangle pulse, but here stated in its dual form.

It is easiest to work back from the Fourier transform:

$$\begin{aligned}x(t) &= \frac{1}{2\pi} \int X(\omega) \exp(+j\omega t) d\omega \\&= \frac{1}{2\pi} \int_{-\Omega}^{+\Omega} 1 \cdot \exp(+j\omega t) d\omega \\&= \frac{1}{2\pi} \frac{1}{jt} [\exp(+j\omega t)]_{-\Omega}^{\Omega} \\&= \frac{1}{2\pi} \frac{1}{jt} 2j \sin(\Omega t) \\&= \frac{\sin(\Omega t)}{\pi t}\end{aligned}$$

as required.

[5]

Examiner's comment: Some candidates used the data book and duality result to get the answer. Generally very well answered.

(b) Now, use the result that convolution in the time domain is multiplication in the frequency domain. Then we have that

$$FT\{f_1(t) * f_2(t)\} = F_1(\omega)F_2(\omega) = \begin{cases} 1, & |\omega| < \min(\Omega_1, \Omega_2) \\ 0, & \text{Otherwise} \end{cases}$$

and hence we have the FT of the original sinc-type function with  $\Omega_3 = \min(\Omega_1, \Omega_2)$ .

[5]

Examiner's comment: Well answered by most. A few were confused about  $\Omega_3$  and gave some silly answers:  $\sqrt{\Omega_1\Omega_2}$  was a common guess.

(c) Now use the standard frequency response result that to find the FT of the filter output multiply the FT of the input by the filter frequency response.

This output FT can be split up into two parts: a rectangle pulse of width  $\Omega$  and amplitude  $-j$ , shifted to the right by  $\Omega/2$ , and a similar rectangle pulse having amplitude  $j$ , shifted to the left by  $\Omega/2$ .

We can find the inverse FT of this summation using the result from part (a) and the

frequency shifting theorem:

$$\begin{aligned}y(t) &= -j \exp(j\Omega/2t) \frac{\sin(\Omega t/2)}{\pi t} + j \exp(-j\Omega/2) \frac{\sin(\Omega t/2)}{\pi t} \\ &= 2 \sin(\Omega/2t) \frac{\sin(\Omega t/2)}{\pi t} \\ &= \frac{(1 - \cos(\Omega t))}{\pi t}\end{aligned}$$

as required. [6]

Examiner's comment: Again, well answered by many. Most people worked out the inverse FT of  $Y(\omega)$  directly, which is probably simpler than the above solution.

(d) Denote the final output as  $z(t)$ , then:

$$\begin{aligned}Z(\omega) &= H(\omega)Y(\omega) \\ &= \begin{cases} -1, & |\omega| < \Omega \\ 0, & \text{Otherwise} \end{cases}\end{aligned}$$

and therefore by inspection from the result of part a),

$$z(t) = -x(t)$$

i.e. negative the original input function. This result is general for all signals with zero mean, i.e. no  $\delta$ -functions at  $\omega = 0$  and  $H(\omega)^2 = -1$  for all frequencies except  $\omega = 0$ . [4]

Examiner's comment: well answered by a small minority of candidates. The subtlety about zero-mean signals was not required in the exam. Note though that under exam pressure *most* candidates were unable to correctly obtain  $H(\omega)^2$ , thinking for example that  $(-j)^2 = 1$ , which is a basic error that cost many number of candidates marks.



## 5. DFT

(a) The steps are as in the lecture notes.

Start off with the sampled signal

$$x_s(t) = \sum_n x(nT)\delta(t - nT)$$

Take its FT:

$$X_s(\omega) = \int \sum_n x(nT)\delta(t - nT) \exp(-j\omega t) dt = \sum_{p=-\infty}^{\infty} x(pT) e^{-j\omega pT}$$

which is the Discrete Time Fourier Transform (DTFT). This expresses the spectrum of a sampled signal in terms of the signal samples but is not computable on a digital computer for two reasons:

- The frequency variable  $\omega$  is continuous.
- The summation involves an infinite number of samples

These problems can be overcome by the following steps:

1. evaluating the DTFT at a finite collection of discrete frequencies
2. Performing the summation over a finite number of data points.

The discrete set of frequencies chosen is arbitrary. However, since the DTFT is *periodic* we generally choose a uniformly spaced grid of  $N$  frequencies covering the range  $\omega T = 0 \rightarrow 2\pi$ . If the summation is then truncated to just  $N$  data points, we get the DFT:

$$G_p = G(e^{j\frac{2\pi}{N}p}) = \sum_{n=0}^{N-1} g_n e^{-j\frac{2\pi}{N}np}$$

The frequency of component  $p$  is obtained by uniformly splitting the Nyquist sampling frequency into  $N$  equal parts,  $1/(NT)$ . Then the  $p$ th component is obtained as  $p/(NT)$ Hz. [5]

Examiner's comment: This was very poorly understood and answered. Most candidates either did not know the material, or did not understand what was being asked for.

(b)

$$\begin{aligned}
X_{N-p} &= \sum_{n=0}^{N-1} x_n \exp(-jn(N-p)2\pi/N) \\
&= \sum_{n=0}^{N-1} x_n \exp(jnp2\pi/N) \exp(-jnN2\pi/N) \\
&= \sum_{n=0}^{N-1} x_n \exp(jnp2\pi/N) \quad [\text{since } \exp(-jnN2\pi/N) = 1] \\
&= X_p^* \quad [(\text{since})(\sum \text{complex})^* = \sum \text{complex}^*]
\end{aligned}$$

as required. [2]

(c)(i) The Nyquist sampling theorem states that sampling must be at a frequency at least double that of the maximum frequency component of the signal, i.e. 20Hz. Allowing say 10% for filter transition bands (other reasonable percentages acceptable), we should go for 22Hz sampling. [2]

(c)(ii) Taking the maximum amplitude as 1, the average energy of the sinusoid is  $1/2$ . The quantisation noise is  $\delta^2/12$ , where  $\delta$  is the ADC step size. 30dB corresponds to an energy ratio of 1000, since  $10\log_{10}(1000) = 30$ . Hence,

$$SNR = \frac{0.5}{\delta^2/12} = 1000, \quad \delta = \sqrt{0.006} = 0.0775$$

Then, only  $2/0.0775=26$  (approximately) levels are required. But,  $2^5 = 32$ , so hence a 5-bit ADC may be used to achieve the desired SNR. [4]

(c)(iii) Case 1:

$$\sin(0.5\pi n) = [0, 1, 0, -1]$$

Plugging into the DFT formula we get:

$$[0, -2i, 0, 2i]$$

This is as expected, since component  $p = 1$  in the DFT corresponds exactly to frequency  $\omega = 2\pi/(NT) = 2\pi/0.04 = \pi/0.02$ , i.e. exactly the frequency of the sine-wave. Also, we expect the  $p = 3$  component to be the conjugate of the  $p = 1$  component since  $X_p = X_{N-p}^*$ , which it is.

Case 2:

$$\sin(0.25\pi n) = [0, \sqrt{2}, 1.0000, \sqrt{2}]$$

Plugging into the DFT formula we get:

$$[2.4142, -1.0000, -0.4142, -1.0000 - 0.0000i]$$

This has the expected conjugacy properties, as before, but we have a ‘smearing’ of frequency components across all  $p$ . This is because  $\omega$  lies midway between two frequency ‘bins’, and hence we do not get an accurate frequency estimation. [There is a lot more that could be said here about resolution and smearing, etc, but this would be beyond 1B level].

[7]

## 6. Modulation of Wireless Signals

(a) Frequency division multiple access - FDMA - multiplexes users in frequency. Each user is assigned a bandwidth and it is modulated to a carrier such that there are no overlaps between spectra of adjacent users.

Time division multiple access - TDMA - multiplexes the users in the time domain. Each user is assigned a given time slot.

Code division multiple access - CDMA - multiplexes the users assigning different signatures to each user. Each user transmits the whole bandwidth over the whole duration of the frame. Signatures should be orthogonal to enable recovery of transmitted data from each user. [5]

Examiner's comment: very well known piece of bookwork.

(b) The spectrum of the rectangle pulse of height  $+5V$  is (from data book):

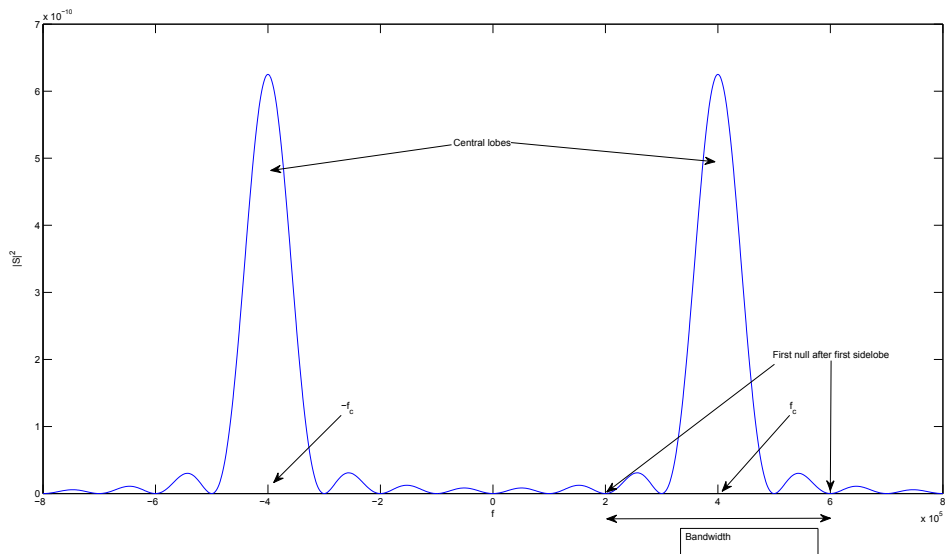
$$X(f) = 5T \operatorname{sinc}(\pi f T)$$

Therefore, from result in lecture notes, spectrum of ASK signal is

$$|S|^2 = 0.25(|X(f-f_c)|^2 + |X(f+f_c)|^2) = 6.25T^2(\operatorname{sinc}^2(\pi(f-f_c)T) + \operatorname{sinc}^2(\pi(f+f_c)T))$$

where  $T = 1/100000$  is the bit period.

Nulls occur in  $X(f)$  when  $\pi f T = n\pi$ ,  $n \neq 0$ , i.e. at frequencies  $f = n/T$ . Hence nulls in  $S$  occur at frequencies  $\pm f_c + 100000n$ , where  $n = \pm 1, \pm 2, \dots$ . Central lobes occur at  $\pm f_c$  and sideobes at  $\pm f_c \pm (75000 + 100000n)$ , where  $n = 0, 1, \dots$ , (see diagram). [6]



Examiners comment: Many candidates plotted the power spectrum before modulation. Most were able to determine the position of the nulls.

(c) Databook gives channel capacity as

$$C_u = B_u \log_2(1 + P_u/N)$$

Now, for each user,  $B_u = B/K$  for FDMA. The noise power is  $N = N_0 B_u = N_0 B/K$ , since  $N_0$  is the noise power spectral density. The power for each user is split equally,  $P_u = P/K$ . Then:

$$C_u = B/K \log_2(1 + P/(N_0 B))$$

How to sketch this? A few illustrative points will do, but noting for the case of very small  $B$  that since  $\lim_{x \rightarrow 0} x \ln(1 + 1/x) = 0$  the sketch starts at  $(0,0)$ . Considering  $B$  increasing (and since there are no turning points), as  $B$  gets large we have

$$\log_2(1 + P/(N_0 B)) \approx \log_2(e) \frac{P}{N_0 B}$$

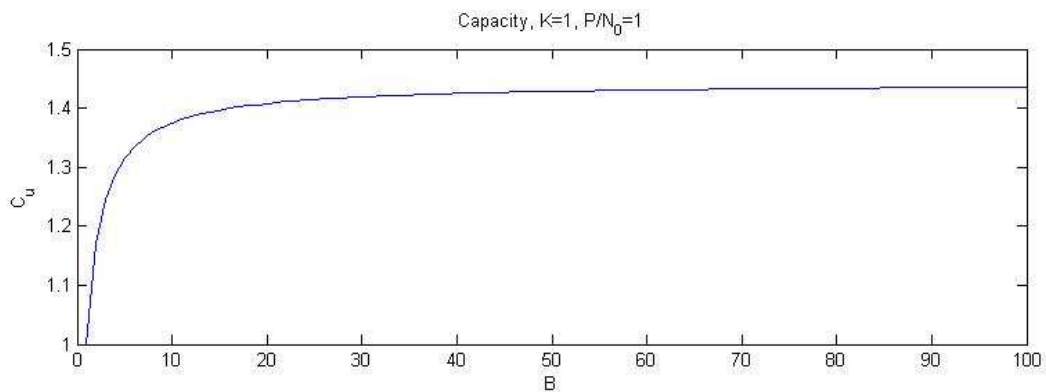
(using first order Taylor expansion of  $\ln(1 + x)$  and  $\log_2(x) = \log_2(e) \ln x$ )

Hence overall, as  $B \rightarrow \infty$

$$C_u = \log_2(e) \frac{P}{K N_0}$$

i.e it tends to a constant, see figure below:

[4]



Examiner's comment:

Most people knew or could find the basic capacity formula. However, there was a lack of understanding about the noise power spectral density. Most people used  $N = N_0$  instead of  $N = N_0 B_u$ , completely changing the form of the result. Intuitively, the noise must increase as the bandwidth of the channel increases. Only a few people

could then sketch the general form of the curve: a ‘first principles’ analysis was not really necessary (and in fact no single candidate achieved this), but plotting a few points to get the idea should have been feasible.

(d)(i) First null beyond first sidelobe is 200kHz from central lobe, see figure from part (b). Hence bandwidth of each user is  $2 \times 200 = 400\text{kHz}$ . Therefore max number of users is  $(200 - 100)/0.4 = 250$  [3]

(d)(ii) Spacing between spectrum nulls is 100kHz. Hence there is an additional 100kHz per channel and the number of channels reduces to  $200 - 100/0.5 = 200$ . [2]

Examiner’s comment: Generally very successfully answered.