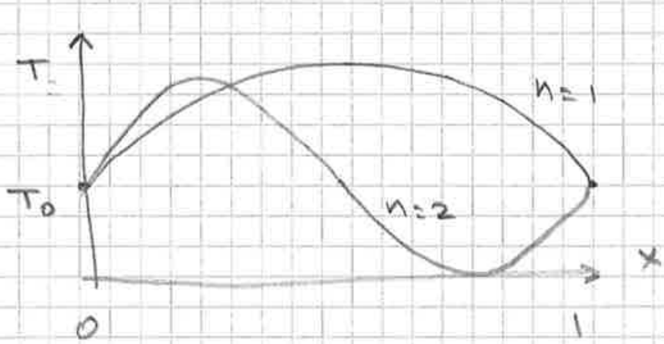
P7 Mathematical Methods Part Ib

Q1. (a) Initially



If $T(x,t) = T_0 + X(x)F(t)$

$$\Rightarrow \alpha \frac{X''}{X} = \frac{F'}{F} = \text{constant}$$

To get correct behaviour as $t \to \infty$, choose $-$ve constant, say $-\lambda^2$.

$$\therefore \left\{ \begin{array}{l} F = e^{-\lambda^2 \alpha t} \\ X = A \cos kx + B \sin kx \end{array} \right\}$$

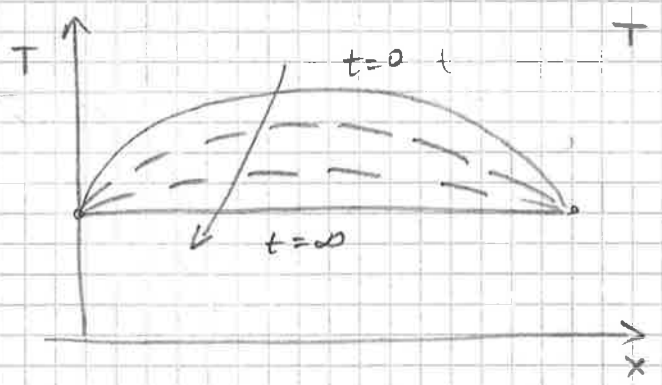
To satisfy b.c.: $X=0$ at $x=0=1$, $A=0$

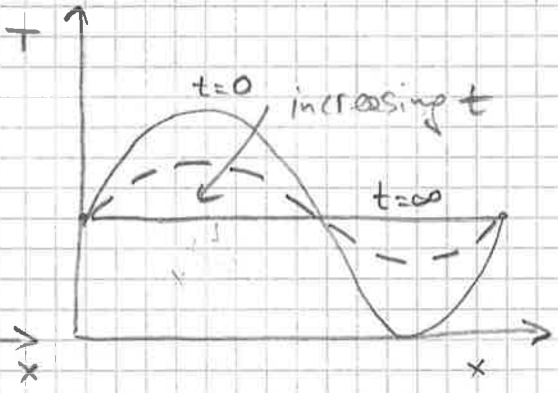
To satisfy initial condition at $t=0$, $B=2T_0$

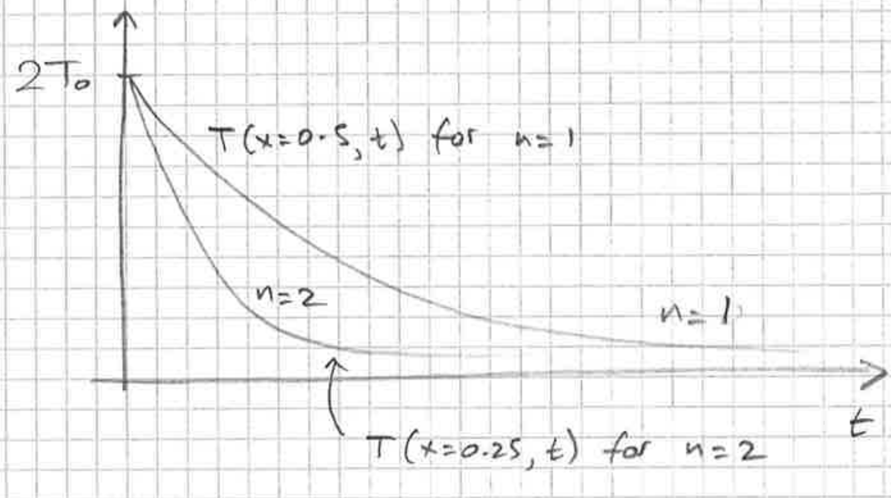
$\lambda = n\pi$

$$\Rightarrow T = T_0 \left[1 + 2 e^{-\alpha n^2 \pi^2 t} \sin(n\pi x) \right]$$

(b)







(c) $q = -\lambda \dfrac{\partial T}{\partial x}$

$$\frac{\partial T}{\partial x} = 2T_0\, e^{-\alpha n^2 \pi^2 t}\; n\pi \cos(n\pi x)$$

At $x=0$, $n=0$: $\quad q_0 = -\lambda \cdot 2T_0\, e^{-\alpha\pi^2 t}\,\pi \quad$ (-ve x i.e. out of domain)

$x=1$, $n=1$: $\quad q_1 = \lambda\; 2T_0\, e^{-\alpha\pi^2 t}\,\pi \quad$ (i.e. out of domain) again

$$\Rightarrow \text{total} = \int_0^\infty (|q_0| + |q_1|)\,dt \qquad \text{(total out of domain)}$$

$$= \lambda\, 4T_0\,\pi \int_0^\infty e^{-\alpha\pi^2 t}\,dt$$

$$= 4T_0\pi\lambda \left(-\frac{1}{\alpha\pi^2}\right)\left[e^{-\alpha\pi^2 t}\right]_0^\infty$$

$$= 4T_0\pi\lambda \frac{1}{\frac{\lambda}{\rho C_p}\pi^2}$$

$$= \frac{4}{\pi}\rho C_p T_0$$

For $n=2$, due to symmetry, heat flux out of domain at $x=0$ is equal to heat flux in the domain at $x=1$ $\Rightarrow$ net $=0$

Q2 (a) From divergence theorem, flux of $\underline{B}$ crossing S is $\iint_S \underline{B}\cdot\underline{n}\,dA = \iiint_V \nabla\cdot\underline{B}\,dV$

$$= \iiint_V \nabla\cdot(\nabla\times\underline{A})\,dV$$

But $\nabla\cdot(\nabla\times\underline{A})=0 \Rightarrow \iint_S \underline{B}\cdot\underline{n}\,dA = 0$ QED

(b) $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

$$\iint_S \underline{r}\cdot\underline{n}\,dA = \iiint_V \nabla\cdot\underline{r}\,dV$$

$$= \iiint_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) dV$$

$$= 3\cdot\iiint_V dV$$

$$= 3V \qquad \text{QED}$$

(c) In polar coordinates:

(i) $\underline{u} = u_r\,\hat{e}_r + u_\theta\,\hat{e}_\theta$

$$\Rightarrow u_r = \frac{\partial\Phi}{\partial r} = \frac{\partial}{\partial r}\left[Ur\cos\theta + \frac{UR^2}{r}\cos\theta + \frac{\Gamma}{2\pi}\theta\right]$$

$$= U\left(1-\frac{R^2}{r^2}\right)\cos\theta$$

$$u_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \frac{1}{r}\left[Ur\left(1+\frac{R^2}{r^2}\right)(-\sin\theta) + \frac{\Gamma}{2\pi}\right] = -U\left(1+\frac{R^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}$$

At $r = R$, $\quad u_\theta = -2U\sin\theta + \dfrac{\Gamma}{2\pi R}$

$$\underline{u} \cdot d\underline{s} = u_\theta R \, d\theta$$

$$\Rightarrow \oint_c \underline{u} \cdot d\underline{s} = \int_0^{2\pi} u_\theta R \, d\theta$$

$$= \int_0^{2\pi} (-2UR) \sin\theta \, d\theta + \int_0^{2\pi} \frac{\Gamma}{2\pi} \, d\theta$$

$$= \Gamma \qquad \text{QED}$$

(ii) $\nabla \times \underline{u} = \left[\dfrac{1}{r} \dfrac{\partial}{\partial r}(r u_\theta) - \dfrac{1}{r} \dfrac{\partial u_r}{\partial \theta} \right] \underline{k}$

$$= \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[-Ur\left(1 + \frac{R^2}{r^2}\right) \sin\theta + \frac{\Gamma}{2\pi} \right] + \frac{1}{r} U \left(1 - \frac{R^2}{r^2}\right) \sin\theta \right\} \underline{k}$$

$= 0$ (by careful differentiation)

(Alternatively, since $\underline{u} = \nabla \Phi$, $\nabla \times \underline{u} = 0$).

Stokes theorem: $\displaystyle\iint \nabla \times \underline{u} \, dA = \oint_c \underline{u} \cdot d\underline{s}$

We showed in Part c(i) that RHS $= \Gamma$, while here we say $\nabla \times \underline{u} = 0$. What is going on?

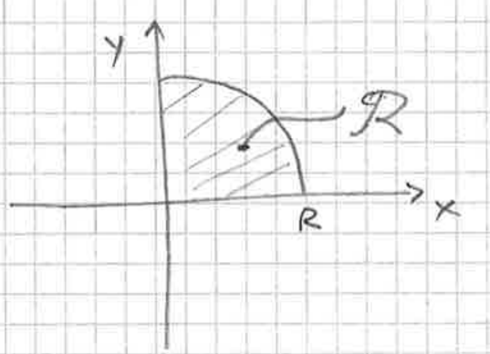
Problem is reconciled by observing that there is a singularity at $r = 0$: $\underline{u}$, Φ, $\nabla \times \underline{u}$ all blow up at $r = 0$.

Q3 (a) $f(x,y) = e^{-(x^2+y^2)}$

$$g(x,y) = x\,(x^2+y^2)^{-1/2}$$

using the transformation $x = r\cos\theta$, $y = r\sin\theta$,

$$f(x,y) = e^{-r^2}, \quad g(x,y) = \cos\theta$$



$$I = \iint f(x,y)\,g(x,y)\,dx\,dy$$

$$= \int_0^R \int_0^{\pi/2} e^{-r^2} \cos\theta \;J\; dr\,d\theta$$

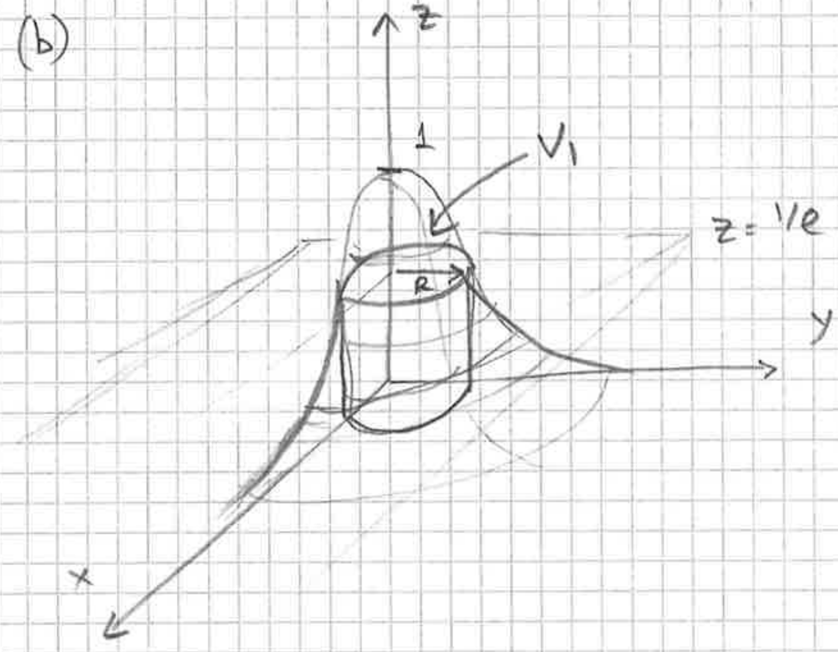
$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

$$\therefore I = \int_0^R \int_0^{\pi/2} e^{-r^2} \cos\theta \; r \; dr\,d\theta = \int_0^R r e^{-r^2} dr \int_0^{\pi/2} \cos\theta\,d\theta$$

$$= \left[-\tfrac{1}{2} e^{-r^2}\right]_0^R \left[\sin\theta\right]_0^{\pi/2}$$

$$= \tfrac{1}{2}\left(1 - e^{-R^2}\right)$$

(b)



Required volume is volume under bell-shaped surface minus volume V_1. To calculate these volumes, go back to Part (a) with

$$g = 1 \Rightarrow V(R) = \int_0^R \int_0^{2\pi} e^{-r^2} r \, dr \, d\theta = \left(1 - e^{-R^2}\right)\pi$$

$z=0$ corresponds to $R \to \infty$, $\Rightarrow$ volume under whole surface $= \pi$

Plane $z = 1/e \Rightarrow R = 1$ $\therefore$ volume until this radius from top is $\pi\left(1 - e^{-1}\right)$

$$\Rightarrow \text{Required volume is } \pi/e + \underbrace{\pi \cdot 1^2 \cdot \frac{1}{e}}_{\text{volume of cylinder radius 1 \& height } \frac{1}{e}} = \frac{2\pi}{e}$$

Alternatively: required volume is $\displaystyle\int_{r=1}^{\infty}\int_0^{2\pi} e^{-r^2} r \, dr \, d\theta = 2\frac{\pi}{e}$

$\left(\text{Yet an easier method: volume} = \displaystyle\int_0^{1/e} \pi(x^2+y^2)\, dz = -\pi \int_0^{1/e} \ln z \, dz = \frac{2\pi}{e}\right)$

$\left(\int \ln x = x \ln x - x + c\right)$

Q4 (a) The probability of defects follows a binomial.

- prob. of zero defects : $(1-q)^M$
- prob. of one defect $= \binom{M}{1} q (1-q)^{M-1}$

$$= M q (1-q)^{M-1}$$

- prob. of discarding

$$= 1 - (1-q)^M - Mq(1-q)^{M-1}$$

(b) x obeys the PDF $\lambda e^{-\lambda x}$

$\Rightarrow$ expectation of $x = \int_0^\infty x \lambda e^{-\lambda x} dx$

$$= \left[-x e^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x} dx = \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^\infty = \lambda^{-1}$$

$y = 1 + x + x^2$

$$\Rightarrow E(y) = \int_0^\infty (1 + x + x^2) \lambda e^{-\lambda x} dx$$

$$= \underbrace{\int_0^\infty \lambda e^{-\lambda x} dx}_{\Downarrow\ 1} + \underbrace{\int_0^\infty x \lambda e^{-\lambda x} dx}_{\Downarrow\ \lambda^{-1}} + \underbrace{\int_0^\infty x^2 \lambda e^{-\lambda x} dx}_{\Downarrow\ 2\lambda^{-2}}$$

(integration by parts, twice)

$$\Rightarrow E(y) = 1 + \lambda^{-1} + 2\lambda^{-2}$$

Q5(a) When performing LU decomposition at each stage we select a pivot element in a column of the reduced matrix. Partial pivoting involves selecting the largest (in magnitude) element in the column as the pivot.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & i \end{bmatrix}$$

pivot using non-partial pivoting LU

largest of a, d, g is pivot when using partial pivoting.

Conceptually partial pivoting is equivalent to re-ordering the rows of the original matrix

It is used to make sure that pivots are large & multiplies small so that problems involving LU remain well-conditioned.

(b)

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 4 \\ 1 & -1 & 1 \end{bmatrix} = \begin{array}{c c} & \begin{matrix} 2 & 6 & 4 \end{matrix} \\ \begin{matrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{matrix} & \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix} \end{array} + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -4 & -1 \end{bmatrix}$$

$$= \begin{array}{c c} & \begin{matrix} 0 & -4 & -1 \end{matrix} \\ \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & -1 \end{bmatrix} \end{array} + \begin{array}{c c} & \begin{matrix} 0 & 0 & -1 \end{matrix} \\ \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 4 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 \\ 0 & -4 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

Now $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P} \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \end{bmatrix}$

$$\therefore \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 \\ 0 & -4 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

(c)

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 4 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 1 & -1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 \\ 0 & -4 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 1 & -1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

(d) Row space of A = row space of U

= non-zero rows of U (There are r of them $r = \text{rank}(A)$)

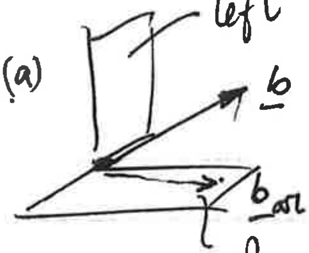
(unaffected by presence of P)

$$PA = LU \Rightarrow A = (P^T L)\, U$$

So column space of A = 1st r cols of $P^T L$

i.e. elements of cols of L have to be re-ordered using P.

Q6 (a)



Any vector $\underline{b} = \underline{b}_{col} + \underline{b}_{left\text{-}null}$

& $\underline{b}_{col} \perp \underline{b}_{left\text{-}null}$

i.e. $\underline{b}_{col}$ = projection of $\underline{b}$

Now left-null space of A = null space of A^t

$$\therefore \quad A^t \underline{b} = A^t \underline{b}_{col} + \cancel{A^t \underline{b}_{LN}} \rightarrow 0$$

Let $\underline{x}$ be such that $A\underline{x} = \underline{b}_{col}$

($\underline{b}_{col} \in$ col. space $\Rightarrow \underline{x}$ exists)

then $\quad A^t A \underline{x} = A^t \underline{b}_{col} = A^t \underline{b}$

$\Rightarrow \quad \underline{x} = (A^t A)^{-1} A^t \underline{b} \quad$ since $(A^t A)^{-1}$ exists

$\Rightarrow \quad \underline{b}_{col} = A\underline{x} = A(A^t A)^{-1} A^t \underline{b} = P\underline{b}$

where P = projection

(b) If $\quad A = QR \qquad R^{-1}$ exists since rank$(A) = n$

$$P = QR\,(R^t \underbrace{Q^t Q}_{I} R)^{-1} R^t Q^t$$

$$= Q R R^{-1} (R^t)^{-1} R^t Q^t = Q Q^t$$

(c) $\quad B = \begin{bmatrix} \uparrow & \uparrow \\ \underline{b}_1 & \underline{b}_2 \\ \downarrow & \downarrow \end{bmatrix} \qquad \underline{q}_1 = \dfrac{\underline{b}_1}{|\underline{b}_1|} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$

using Gram-Schmidt process

$\underline{b}_2' = \underline{b}_2 - \underline{q}_1 . \underline{b}_2 \, \underline{q}_1$

$$= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

This is a unit vector, so no need to normalise

$$\therefore \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{3} & -2/3 \\ \frac{2}{3} & 2/3 \\ \frac{2}{3} & -1/3 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}}_{R}$$

(d) Need to find a solution and then project onto row space of A = column space of $A^t = B$

(Null space $\perp$ row space)

$$P = Q\,Q^t = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & \frac{-2}{9} & \frac{4}{9} \\ -\frac{2}{9} & \frac{8}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{5}{9} \end{bmatrix}$$

A solution

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Take $z = 0 \Rightarrow y = \frac{1}{2}$

$\Rightarrow x = 0$

ie $\underline{x} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} = \underline{x}_{row} + \underline{x}_{null}$

$\underline{x}_{row}$ = solution, since $A\underline{x}_{null} = 0$

$\Rightarrow$ solution in row-space = $\begin{bmatrix} -\frac{1}{9} \\ \frac{4}{9} \\ \frac{1}{9} \end{bmatrix}$