ENGINEERING TRIPOS PART IB

Friday 10 June 2011 2.30 to 4.30

Paper 7

MATHEMATICAL METHODS

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

Attachments: There are no attachments to this paper.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

EM3

SECTION A

Answer not more than ***two*** *questions from this section.*

1 In one dimension, the heat equation is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

where T is the temperature, $\alpha = \lambda/(\rho c_p)$, λ is the thermal conductivity, ρ is the density, and c_p is the heat capacity at constant pressure. Assume that λ, ρ and c_p are constant and uniform. We seek solutions for $T(x,t)$ in the domain $0 \leq x \leq 1$. The initial condition at $t = 0$ is $T(x,0) = T_0\,[1 + 2\sin(n\pi x)]$ and the boundary conditions are $T(0,t) = T_0$ and $T(1,t) = T_0$ for all t. The constant n is a positive integer.

(a) Use separation of variables to find an analytical expression for $T(x,t)$. [8]

(b) For both $n = 1$ and $n = 2$, sketch $T(x,t)$ as a function of x for a few times t. Sketch also $T(x = 0.5,t)$ for $n = 1$ and $T(x = 0.25,t)$ for $n = 2$ as a function of t. [6]

(c) The heat flow q in the $x-$direction is given by $q = -\lambda \partial T/\partial x$. Derive an expression for the total heat that crosses the boundaries (i.e. at $x = 0$ and $x = 1$) from $t = 0$ to $t = \infty$ for $n = 1$ and for $n = 2$. [6]

2 (a) If the volume V is bounded by the closed surface S, and if $\mathbf{B} = \nabla \times \mathbf{A}$, show that the total flux of $\mathbf{B}$ crossing S is zero. [4]

(b) If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, show that

$$\iint_S \mathbf{r} \cdot \mathbf{n} \, dA = 3V$$

where V is the volume bounded by the closed surface S, $\mathbf{n}$ is the unit normal vector outwards from S, and dA is the element surface. [4]

(c) The fluid velocity $\mathbf{u}$ in a two-dimensional fluid flow around a rotating circular disk of radius R obeys $\mathbf{u} = \nabla\phi$, where ϕ is the potential given in polar coordinates (r, θ) by

$$\phi = Ur\left(1 + \frac{R^2}{r^2}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta$$

where Γ is a constant.

(i) Derive expressions for the two components of the velocity and then show that

$$\oint_C \mathbf{u} \cdot d\mathbf{s} \; = \; \Gamma$$

where the curve C is the disk's perimeter (i.e. $r = R$) and $d\mathbf{s}$ is the line element around the disk. [8]

(ii) Evaluate $\nabla \times \mathbf{u}$. Is your answer consistent with part (c)(i) when Stokes theorem is applied? Explain your answer. [4]

3 Consider the functions $f(x,y) = e^{-\left(x^2+y^2\right)}$ and $g(x,y) = x/\sqrt{x^2+y^2}$.

(a) Evaluate

$$\iint f(x,y)\, g(x,y)\, dx\, dy$$

over the region $\Re$ bounded by the curve $x^2+y^2 \leq R^2$ and the two half-axes $x > 0$ and $y > 0$. [8]

(b) Sketch and calculate the volume enclosed by the surface $z = f(x,y)$ and the two planes $z = 0$ and $z = 1/e$. [12]

SECTION B

*Answer not more than **two** questions from this section.*

4 (a) A manufacturing process is composed of M sequential steps in the production line. Each of these steps involves a probability q that a defect will appear in the product. If two or more defects appear at the end of the production line, the product is discarded. What are the probabilities of obtaining a product with no defects, of obtaining a product with one defect, and of discarding the product? [8]

(b) Empirical evidence from a manufacturing process shows that the cost, x, to replace defective products is a continuous random variable with probability density function

$$P(x) = \lambda e^{-\lambda x}$$

where $0 \leq x < \infty$ and $\lambda > 0$. The overall financial burden to the manufacturer due to a replacement cost x is given by $y = 1 + x + x^2$.

(i) Show that the expectation of x, $E[x]$, is equal to λ^{-1}. [6]

(ii) Show that the expectation of y, $E[y]$, is equal to $1 + \lambda^{-1} + 2\lambda^{-2}$. [6]

5 (a) Explain what is meant by *partial-pivoting* in the context of LU decomposition of an $m \times n$ matrix $\mathbf{A}$ and why it is used. [5]

(b) Using partial-pivoting, find the extended form of the LU decomposition ($\mathbf{PA} = \mathbf{LU}$) for the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 4 \\ 1 & -1 & 1 \end{bmatrix}$$

[8]

(c) Verify that the matrices you have found satisfy $\mathbf{PA} = \mathbf{LU}$. [4]

(d) Describe how the row space and the column space of $\mathbf{A}$ are related to those of $\mathbf{L}$ and $\mathbf{U}$ for the extended form of LU decomposition. [3]

6 (a) For the $m \times n$ matrix $\mathbf{A}$ of rank n (where $m > n$), show that the matrix $\mathbf{P}$ given by

$$\mathbf{P} = \mathbf{A}\left(\mathbf{A}^t\mathbf{A}\right)^{-1}\mathbf{A}^t$$

represents the projection of a vector onto the column space of $\mathbf{A}$. [5]

(b) Derive the form that the matrix $\mathbf{P}$ takes when $\mathbf{A}$ is factorised using the QR decomposition $\mathbf{A} = \mathbf{QR}$. [3]

(c) Find the QR decomposition of the matrix $\mathbf{B}$, where

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 2 & 1 \end{bmatrix}$$

[7]

(d) Using the result of part (c), find a solution of the equation

$$\mathbf{A}\,\mathbf{x} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which has no component in the null space of $\mathbf{A}$. [5]

END OF PAPER