ENGINEERING TRIPOS PART IB JUNE 2012

CHAIRMAN PROF. P DAVIDSON

MONDAY 4TH JUNE 2012 2 TO 4

PAPER 2 SOLUTIONS – STRUCTURES

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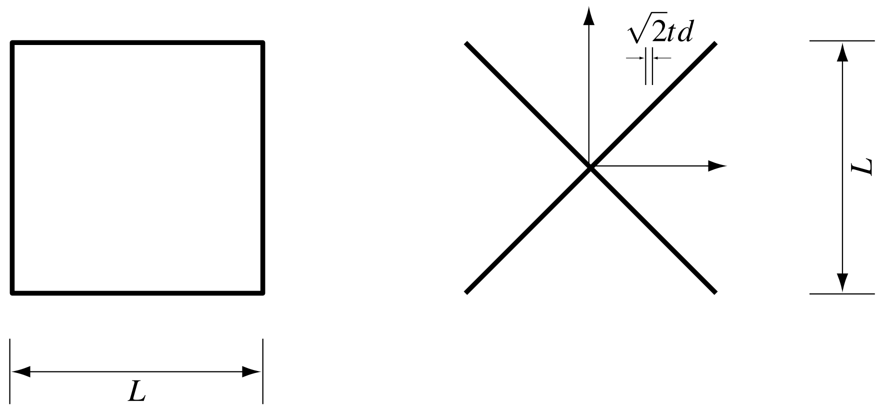
Part IB, Paper 2 – Structures, 2011/12

1. (a)



The second moment of area for the thin-walled square is

$$I_{xx}^{s} = 2\frac{L^3 t}{12} + 2Lt\left(\frac{L}{2}\right)^2 = \frac{L^3 t}{6} + \frac{L^3 t}{2} = \frac{2}{3}L^3 t$$

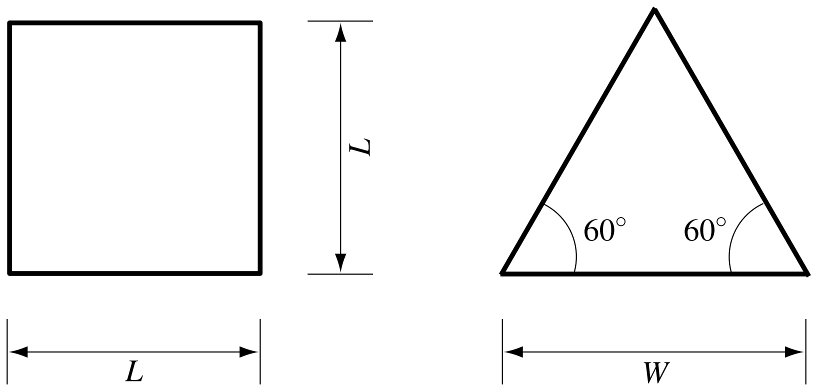
The second moment of area for the cruciform is

$$I_{xx}^{c} = 2\int_{-L/2}^{L/2} \sqrt{2}tx^2 \, \mathrm{d}x = 2\sqrt{2}t\frac{x^3}{3}\Big|_{-L/2}^{L/2} = \frac{\sqrt{2}}{6}L^3 t$$

The ratio of the second moments of both cross-sections has to be equal

$$\frac{I_{xx}^{s}}{I_{xx}^{c}} = \frac{2L^3 t}{3}\frac{6}{\sqrt{2}L^3 t} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

(b)



The torsion constant for a thin-walled closed cross-section is defined as

$$J = \frac{4A_e^2}{\oint \frac{ds}{t}}$$

The torsion constant of the square is

$$J^s = \frac{4L^4 t}{4L} = L^3 t$$

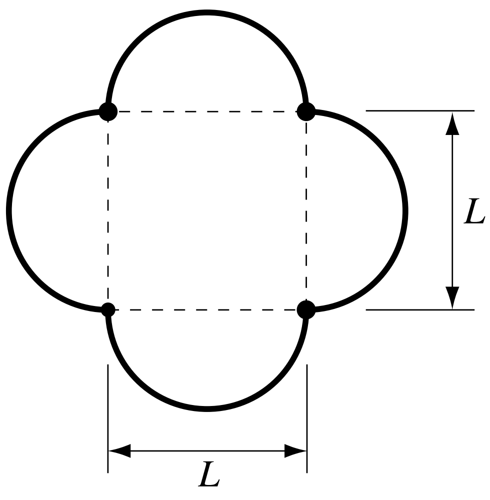
The torsion constant of the equilateral triangle is

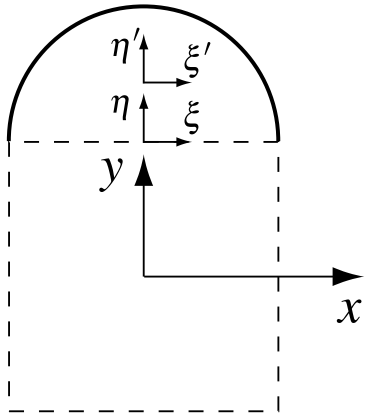
$$A_e = \frac{1}{2} W \frac{\sqrt{3}}{2} W = \frac{\sqrt{3}}{4} W^2$$
$$\Rightarrow J^t = \frac{3W^4 t}{4 \cdot 3S} = \frac{1}{4} W^3 t$$

The ration of the torsion constants is

$$\frac{J^s}{J^t} = \frac{4L^3}{W^3}$$

(c)





Parameters for one (horizontal) semi-circle
Centroid

$$\pi \frac{L}{2} t \eta_s = \int_0^\pi \left(\frac{L}{2}\sin\theta\right)\frac{Lt}{2}\,\mathrm{d}\theta = -\frac{L^2}{4} t\cos\theta\Big|_0^\pi = \frac{L^2}{2}t$$
$$\Rightarrow \eta_s = \frac{L}{\pi}$$

Second moment ofs area

$$I^c_{\xi\xi} = \int_0^\pi \eta^2 \frac{Lt}{2}\,\mathrm{d}\theta = \int_0^\pi \left(\frac{L}{2}\sin\theta\right)^2 \frac{Lt}{2}\,\mathrm{d}\theta = \frac{L^3 t}{8}\int_0^\pi \sin^2\theta\,\mathrm{d}\theta$$
$$\Rightarrow I^c_{\xi\xi} = \frac{L^3 t\pi}{16}$$

$$I^c_{\eta\eta} = \int_0^\pi \xi^2 \frac{Lt}{2}\,\mathrm{d}\theta = \int_0^\pi \left(\frac{L}{2}\cos\theta\right)^2 \frac{Lt}{2}\,\mathrm{d}\theta = \frac{L^3 t}{8}\int_0^\pi \cos^2\theta\,\mathrm{d}\theta$$
$$\Rightarrow I^c_{\eta\eta} = \frac{L^3 t\pi}{16}$$

Second moments of area around the centroid

$$I^c_{\xi'\xi'} = \frac{L^3 t\pi}{16} - \frac{L^3 t}{2\pi}$$

Second moment of area for the entire cross-section

$$\begin{aligned} I_{xx} &= 2\left(I^c_{\xi'\xi'} + \frac{\pi L t}{2}\left(\frac{L}{2}+\frac{L}{\pi}\right)^2\right) + 2I^c_{yy} \\ &= 2\left(\frac{L^3 t\pi}{16} - \frac{L^3 t}{2\pi} + \frac{\pi L^3 t}{8} + \frac{L^3 t}{2} + \frac{L^3 t}{2\pi} + \frac{L^3 t\pi}{16}\right) \\ &= \frac{L^3 t\pi}{2} - \frac{L^3 t}{\pi} + L^3 t + \frac{L^3 t}{\pi} = 2.5708\,L^3 t \end{aligned}$$

Alternatively, the second moments of area for the semi-circle can be determined with the equations for the curved rod given in Mechanics Data Book

$$I^c_{\xi'\xi'} = \frac{\pi L t}{2} k_y^2 \qquad I^c_{\eta'\eta'} = \frac{\pi L t}{2} k_x^2$$

with $a = \frac{L}{2}$ and $\alpha = \frac{\pi}{2}$

$$k_y^2 = \frac{1}{2}\frac{L^2}{4}\left(1 - 2\left(\frac{2}{\pi}\right)^2\right) = \frac{L^2}{8}\left(1 - \frac{8}{\pi^2}\right) \qquad \Rightarrow I^c_{\xi'\xi'} = \frac{\pi L^3 t}{16}\left(1 - \frac{8}{\pi^2}\right)$$

$$k_x^2 = \frac{1}{2}\frac{L^2}{4} = \frac{L^2}{8} \qquad \Rightarrow I^c_{\eta'\eta'} = \frac{\pi L^3 t}{16}$$

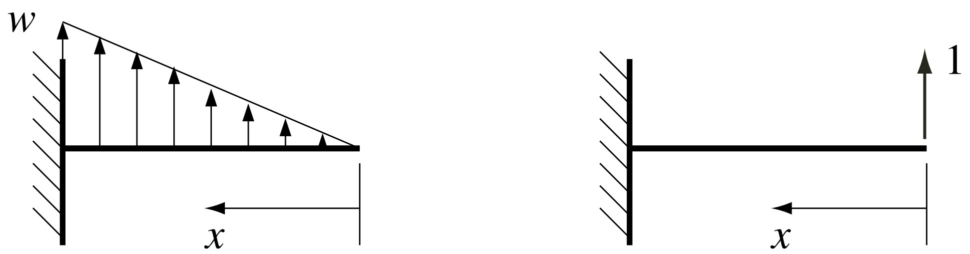
Torsion constant of the entire cross-section

$$A_e = L^2 + 2\pi\left(\frac{L}{2}\right)^2 = L^2 + \frac{\pi L^2}{2}$$

$$\oint \frac{ds}{dt} = \frac{2\pi L}{t}$$

$$\Rightarrow J = \frac{4A_e^2}{\oint \frac{ds}{dt}} = \frac{t}{2\pi L}\left(4L^4 + 4\pi L^4 + \pi^2 L^4\right)$$
$$= \frac{2L^3 t}{\pi} + 2L^3 t + \frac{\pi L^3 t}{2} = 4.21\,L^3 t$$

2. (a) Maximum displacement using virtual work



Virtual moment distribution

$$\overline{M}(x) = 1 \cdot x$$

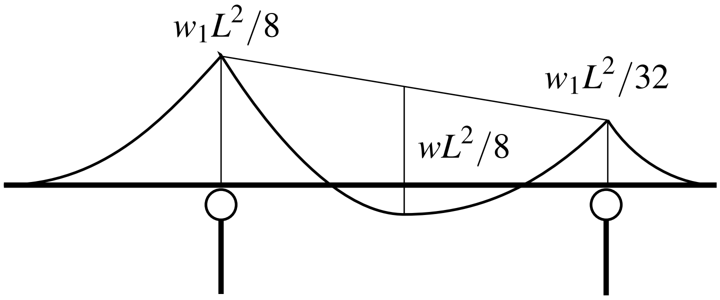
Moment distribution due to w

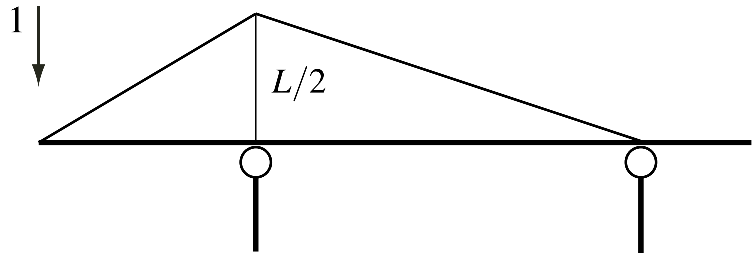
$$M(x) = \frac{1}{2}xw(x)\frac{1}{3}x = \frac{1}{6}x^2\frac{w}{L}x = \frac{1}{6}\frac{x^3}{L}w$$

Principle of virtual work

$$1 \cdot \delta = \frac{1}{EI}\int_0^L x\frac{1}{6}\frac{x^3}{L}w\,\mathrm{d}x = \frac{1}{EI}\frac{w}{6L}\int_0^L x^4\,\mathrm{d}x = \frac{1}{EI}\frac{wL^4}{30}$$

(b) i. Although the system is statically indeterminate the bending moments in the beam can be determined without making use of the force method.
Vertical deflection at Point A





$$\begin{aligned}
1 \cdot \delta_v EI &= \int M(x)\overline{M}(x)\,\mathrm{d}x \\
&= \frac{L}{8}\frac{w_1L^2}{8}\frac{L}{2} + \frac{L}{6}\left(2\frac{w_1L^2}{8} + \frac{w_1L^2}{32}\right)\frac{L}{2} - \frac{L}{3}\frac{L}{2}\frac{w_1L^2}{8} \\
&= \frac{w_1L^4}{128} + \frac{3w_1L^4}{128} - \frac{w_1L^4}{48} = \frac{w_1L^4}{96}
\end{aligned}$$

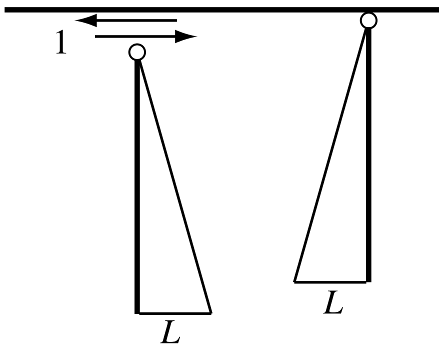
Alternatively, the vertical deflection can be obtained by superposing Data Book cases.
Rotation of the left support

$$\frac{w_1L^3}{24EI} - w_1\frac{L^2}{32}\frac{L}{6EI} - \frac{w_1L^2}{8}\frac{L}{3EI} = -w_1\frac{L^3}{192EI}$$

Vertical deflection

$$\delta_v = w_1\frac{L^4}{128EI} + w_1\frac{L^3}{192EI}\frac{L}{2} = w_1\frac{L^4}{96EI}$$

ii. Horizontal deflection at Point A
To determine the horizontal deflection, first the moments in the columns need to be determined. To this end, consider a statically determinate system where the joint between the left column and beam is released.



The compatibility of the displacements at the joint requires

$$\alpha_1 X + \alpha_0 = 0$$

where X is the yet unknown horizontal shear force at the joint.

Difference in the horizontal displacement between the beam and column caused by the virtual unit load pair is

$$\alpha_1 = \frac{2L^3}{3EI} \quad \text{(computed with principle of virtual work)}$$

Difference in the horizontal displacemen between the beam and column caused by w_2 is

$$\alpha_0 = \frac{w_2 L^4}{30EI} \quad \text{(given in part (a))}$$

Hence, the horizontal shear force at the joint is

$$\Rightarrow X = -\frac{w_2 L}{20}$$

After knowing the shear force at the top of the left column its horizontal deflection can be computed with

$$\delta_h = \frac{1}{EI}\frac{L}{3}L\frac{w_2 L^2}{20} = \frac{1}{EI}\frac{w_2 L^4}{60}$$

Note, this deflection is the same as for a single cantilever beam with twice the EI.

3. (a)

Axial stress

$$\sigma_a = \frac{pR}{2t} = \frac{250 \cdot 10^3 \cdot 1}{2 \cdot 3 \cdot 10^{-3}} = 41.667\,MPa$$

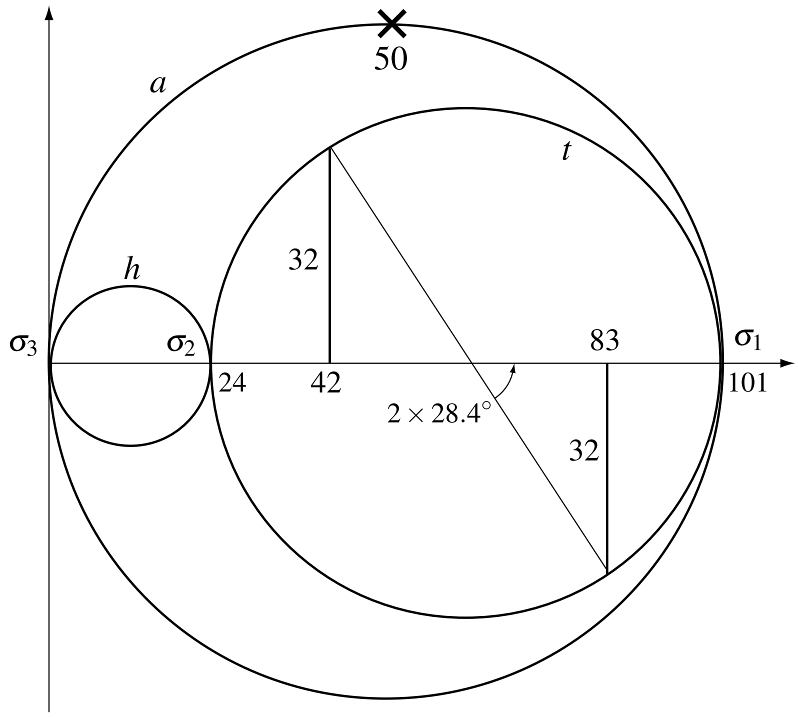
Hoop stress

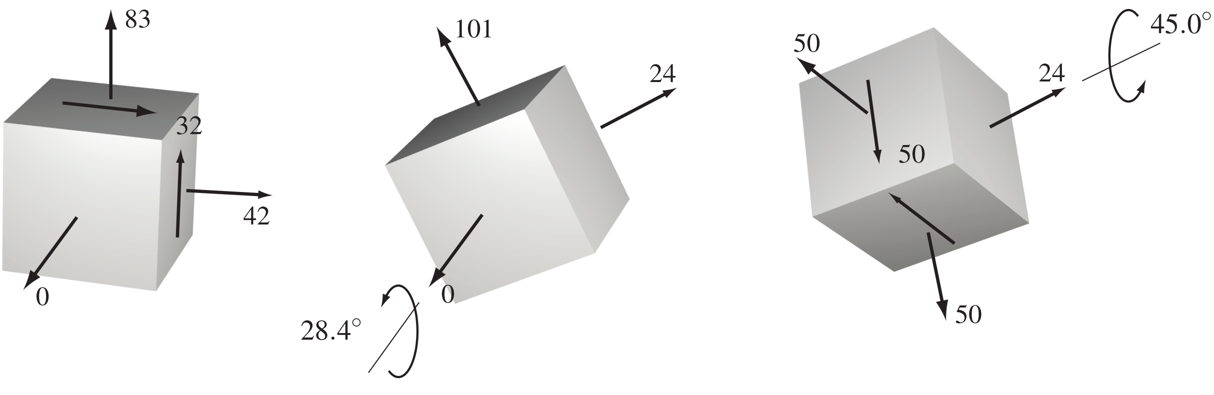
$$\sigma_h = \frac{pR}{t} = 2 \cdot 41.667\,MPa = 83.33\,MPa$$

Shear stress

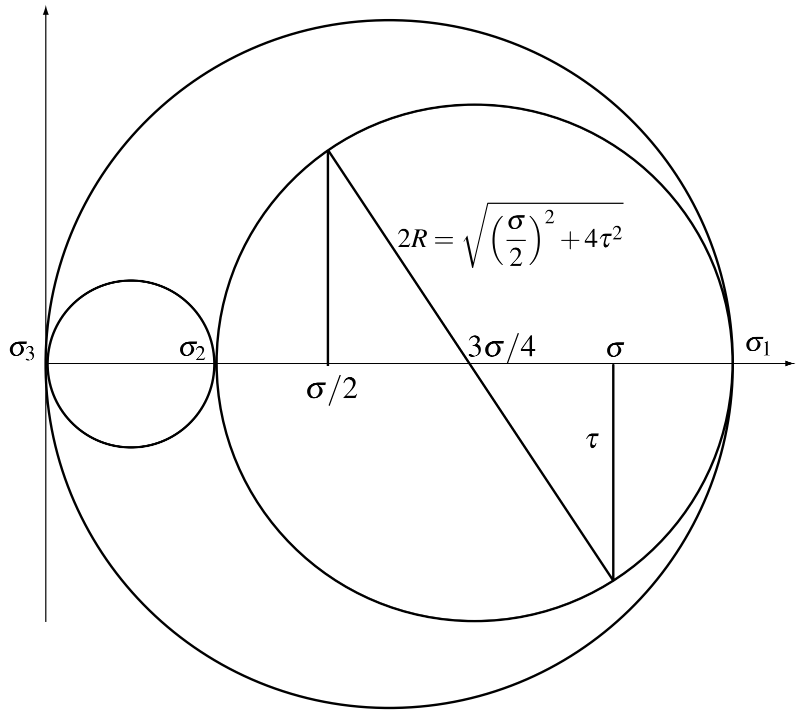
$$\tau = \frac{T}{2\pi R^2 t} = \frac{600 \cdot 10^3}{2\Pi \cdot 1 \cdot 3 \cdot 10^{-3}} = 31.83\,MPa$$

(b)





(c)



Von Mises criterion

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2$$

This gives for plain stress ($\sigma_3 = 0$)

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = Y^2$$

From the Mohr's circle we obtain

$$\sigma_1 = \frac{3}{4}\sigma + \sqrt{\tau^2 + \frac{1}{16}\sigma^2}$$

$$\sigma_2 = \frac{3}{4}\sigma - \sqrt{\tau^2 + \frac{1}{16}\sigma^2}$$

The square root expression in the previous equations is abbreviated with R.

$$\frac{9}{16}\sigma^2 + \frac{3}{2}\sigma R + R^2 + \frac{9}{16}\sigma^2 - \frac{3}{2}\sigma R + R^2 - \frac{9}{16}\sigma^2 + R^2 = Y^2$$

$$\frac{9}{16}\sigma^2 + 3R^2 = Y^2$$

$$\frac{9}{16}\sigma^2 + 3\tau^2 + \frac{3}{16}\sigma^2 = Y^2$$

$$\frac{3}{4}\sigma^2 + 3\tau^2 = Y^2$$

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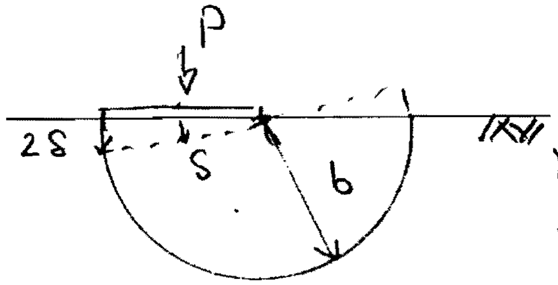
4 (a)

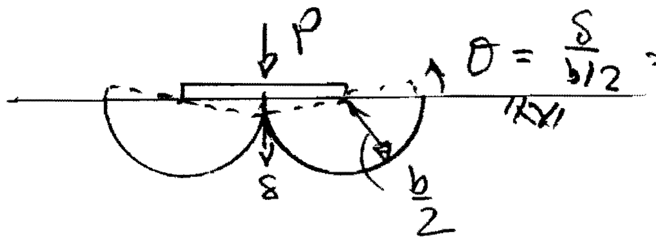


NOTE: All calculations per unit length into page.

Work: $P_a \delta = \pi b \times 2\delta \times k$

$\therefore \quad \underline{\underline{P_a = 2\pi b k}}$

(b)



$\theta = \frac{\delta}{b/2} = \frac{2\delta}{b}$

Work: $P_b \delta = 2\left(\pi \frac{b}{2} \times \delta \times k\right) + 2\theta \times m$

$$\underline{\underline{P_b = \pi b k + 4\frac{m}{b}}}$$

(c) To prevent mode (b), $P_a \leq P_b$

$\therefore \quad 2\pi b k \leq \pi b k + 4\frac{m}{b}$

or $\quad m \geq \frac{\pi b^2 k}{4}$

$m = Z_p \sigma_y$, where $Z_p = \frac{t^2}{4}$

4(c) Contd

and $\sigma_y = 2ck$ (see Materials Data Book)

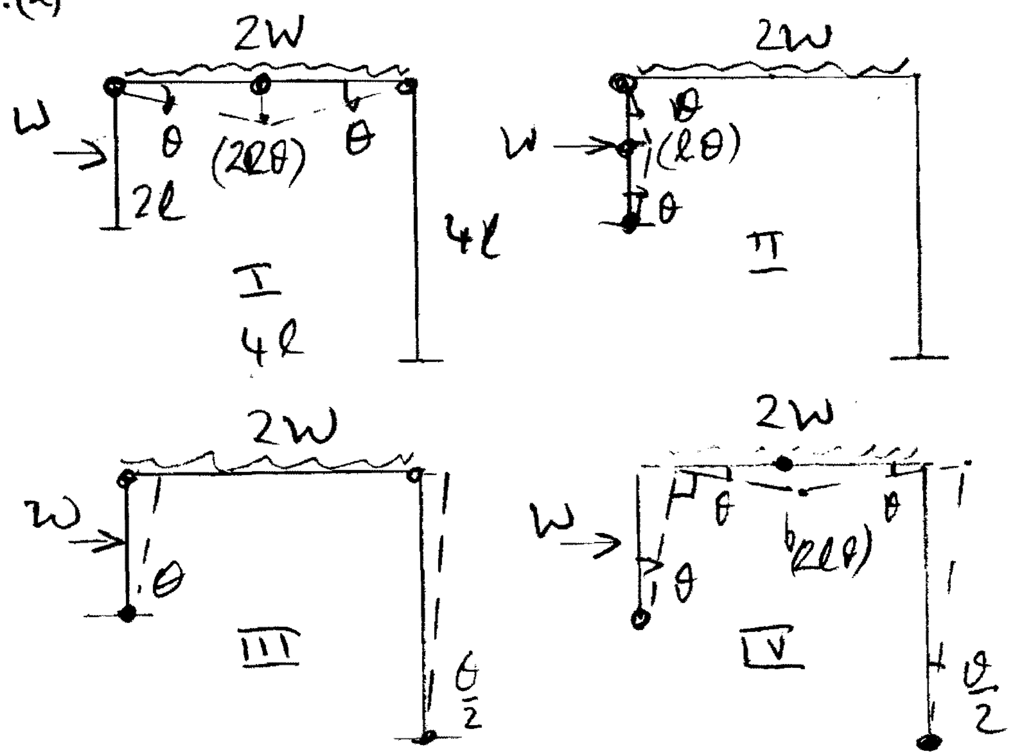
$\therefore \quad m = \frac{t^2 ck}{2} \geqslant \frac{\pi b^2 k}{4}$

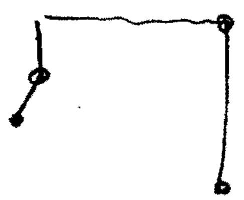
so for (a) to control,

require $t^2 \geqslant \frac{\pi b^2}{2c}$; $t \geqslant b\sqrt{\frac{\pi}{2c}}$

(Most common error was to set $\sigma_y = ck$)

5.(a)





(Very unlikely because of large span (4l).)

5(b) Work equations

I: $2W_1 \times 2\frac{l}{2}\theta = 4m_p\theta$; $W_1 = 2\frac{m_p}{l}$

II: $W_2 \times l\theta = 4m_p\theta$; $W_2 = 4\frac{m_p}{l}$

III: $W_3 \times l\theta = m_p(2\theta + 2\frac{\theta}{2})$

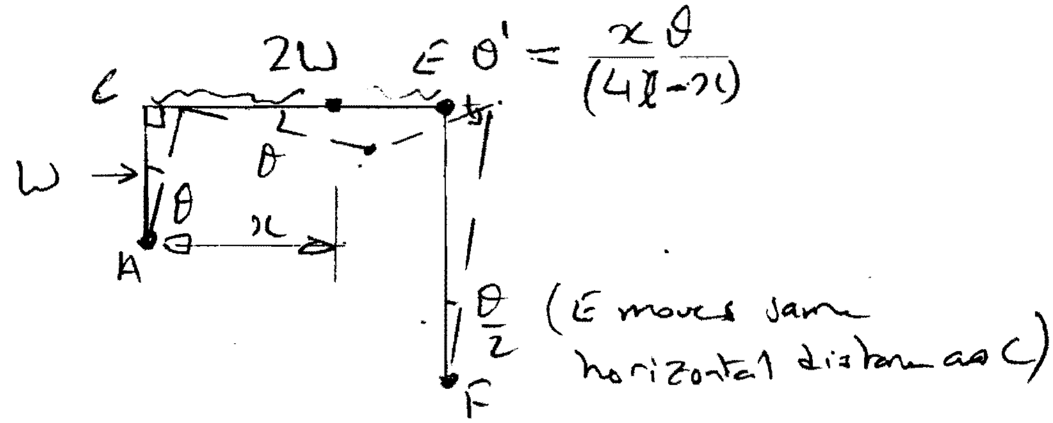
$\therefore W_3 = 3\frac{m_p}{l}$

IV: $W_4 \times l\theta + 2W_4 \times 2\frac{l}{2}\theta = m_p(4\theta + 2\frac{\theta}{2})$

$\therefore W_4 = \frac{5 m_p}{3 l}$

5(c) Mode IV is critical, so optimise this (Modes II and III have no plastic pin in beam, so no improvement possible. In Mode I, the beam mechanism is symmetrical, so considering a variable pin position will confirm the critical form of Mode I indeed has the pin in the middle! Many candidates "proved" this for little reward.)

5(c) cont^n



Work: $W \times l\theta + 2W \times \frac{x\theta}{2} = M_p\left(2\theta + \frac{2x\theta}{(4l-x)} + \frac{2\theta}{2}\right)$

(load × average drop)

$\therefore \quad W(l+x) = M_p\left(3 + \frac{2x}{(4l-x)}\right)$

so $\quad W = M_p \frac{(12l-x)}{(4l^2+3lx-x^2)}$

Extreme value $\left/ \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} = 0 \right)$

$\Rightarrow \quad (4l^2+3lx-x^2) \times -1 - (12l-x)(3l-2x) = 0$

$\Rightarrow \quad x^2 - 24lx + 40l^2 = 0$

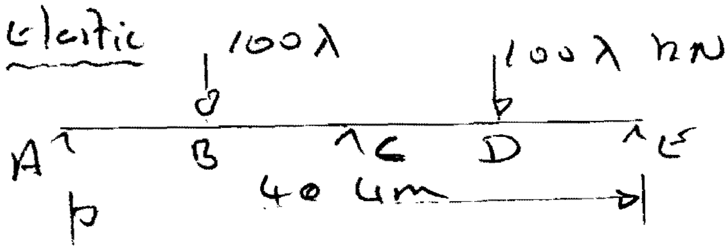
$\Rightarrow \quad x = 12l \pm 10.198l$

$\therefore \quad x = 1.802l$

$\Rightarrow \quad W = \frac{M_p}{l} \frac{(12-1.802)}{(4+5.41-3.25)} = 1.66 \frac{M_p}{l}$

(cf $5/3 = 1.667$ for central pin)

6 (a) <u>Elastic</u>

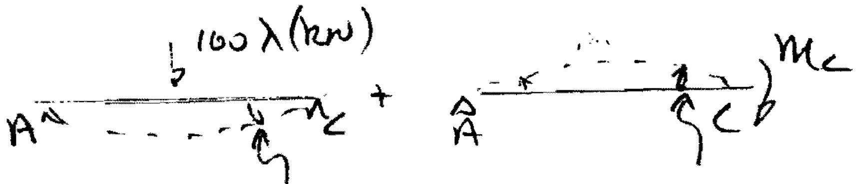


UB356×127×39

$\sigma_y = 355$ N/mm²

First yield at $M_{max} = M_y = Z_e \sigma_y$

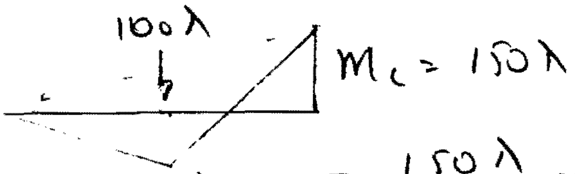
Need elastic analysis for M_{max}



Compatibility $\dfrac{100\lambda \times (8^2)}{16EI} = \dfrac{M_c \times 8}{3EI}$

$\Rightarrow M_c = \frac{3}{2} \times 100\lambda = 150\lambda$ kNm

(Other elastic methods give same result)



$M_B = \dfrac{150\lambda}{2} - \dfrac{100\lambda \times 8}{4} = -125\lambda$ kNm

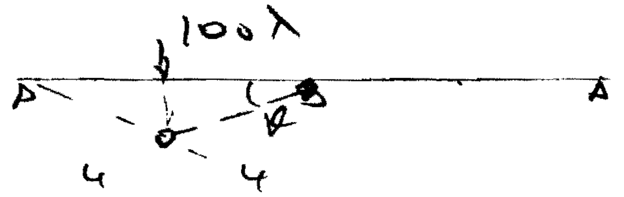
$\therefore M_{max} = 150\lambda$ kNm

Now $M_y = Z_e \sigma_y = 576 \times 10^3 \times 355$ Nmm
(cm³)
$= 204$ kNm

So $M_{max} = M_y \Rightarrow 150\lambda = 204$, $\underline{\lambda = 1.36}$

(NOTE. Ans for major axis bending; full marks were also given for minor axis).

6(b) Upper Bound (UB)



(symmetric mode gives same ans)

Work: $100\lambda \times 4\theta = 3M_p\theta$.

$\therefore \quad \lambda = \frac{3}{400} M_p$

$M_p = Z_p \sigma_y = \frac{659 \times 355}{10^3} = 234$ kNm

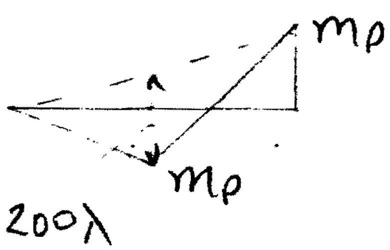
$\therefore \lambda = \frac{3 \times 234}{400} = \underline{\underline{1.76}}$

6(c) Lower Bound (LB). Q asked for "a" LB, not best, so any BMD in equilibrium with load, and M_p to match M_{max} is OK.

eg  $M_{max} = "\frac{PL}{4}" = 200\lambda$ kNm

$\Rightarrow \lambda = \frac{234}{200} = \underline{\underline{1.17}}$ OK eh

Most candidates looked for best, i.e.



so $\frac{M_p}{2} - 200\lambda = -M_p$

$\therefore \quad 200\lambda = \frac{3}{2} M_p, \Rightarrow \underline{\underline{\lambda = 1.76}}$

($\equiv$ UB)

RDM 20/6/12.