ENGINEERING TRIPOS PART IB JUNE 2012

CHAIRMAN PROF. P DAVIDSON

FRIDAY 8TH JUNE 2012 2.30 TO 4.30

PAPER 7 SOLUTIONS – MATHEMATICAL METHODS

AUTHORS :

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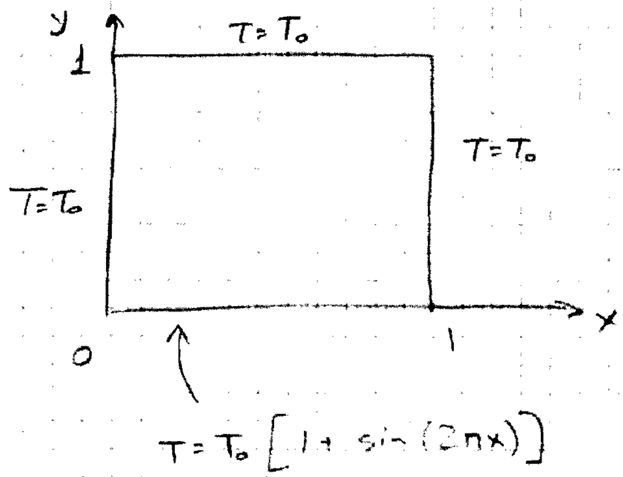
Ib P7 2012

Q.1

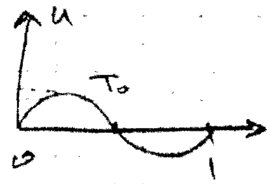
(a)



If $u = T - T_0$, then : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = g(x,y)$

b.c. : $u=0$ at $x=1$, $y=0$, $y=1$

& $u = T_0 \sin(2\pi x)$ at $y=0$



Using separation of variables: we seek a solution $u(x,y) = \mathcal{X}(x)\,Y(y)$

$$\Rightarrow \frac{1}{\mathcal{X}}\frac{d^2\mathcal{X}}{dx^2} = -\frac{1}{Y}\frac{d^2 Y}{dy^2} = \text{constant}$$

Address b.c

$u=0$ at $x=1$ $\Rightarrow$ $u(1,y) = \mathcal{X}(1)\,Y(y) = 0$

If we take $Y(y)=0$, it means $u(x,y)=0$ everywhere $\Rightarrow$ $\mathcal{X}(1)=0$

Similarly, at $x=0$ gives: $\mathcal{X}(0)=0$

At $y=1$, $u(x,1)=0$ $\Rightarrow$ $\mathcal{X}(x)\,Y(1)=0$

To avoid the trivial solution, $Y(1)=0$

At $y=0$, $u(x,0) = T_0 \sin(2\pi x)$

$\Rightarrow$ $\mathcal{X}(x) = T_0 \sin(2\pi x)$ & $Y(0) = 1$

Going back to the equation:

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -k^2 \quad , \quad \frac{1}{Y}\frac{d^2 Y}{dy^2} = k^2 \qquad (k>0)$$

(We write the -ve for the X eqtn since we saw that it must have a sin dependence)

$$\Rightarrow \quad k = 2\pi$$

$$\Rightarrow \quad Y = A e^{2\pi y} + B e^{-2\pi y}$$

$$X = T_0 \sin(2\pi x)$$

At $y=0$, $Y=1$ $\Rightarrow$ $A+B=1$

$y=1$, $Y=0$ $\Rightarrow$ $Ae^{2\pi}+Be^{-2\pi}=0$

$$\Rightarrow \quad B = 1-A$$

$$\Rightarrow \quad Ae^{2\pi} + (1-A)e^{-2\pi} = 0$$

$$\Leftrightarrow \quad A = \frac{-e^{-2\pi}}{e^{2\pi}-e^{-2\pi}} \quad \& \quad B = \frac{e^{2\pi}}{e^{2\pi}-e^{-2\pi}}$$

$$\therefore \quad u(x,y) = T_0 \sin(2\pi x)\cdot\left(Ae^{2\pi y} + Be^{-2\pi y}\right)$$

with A & B from above

(b) It is very difficult to solve the pde for $g \neq 0$. We are seeking $\oint \underline{q} \cdot d\underline{n}$

So a 2-D version of Gauss theorem may be used:

$$\iint \nabla \cdot \underline{q} \, dA = \oint \underline{q} \cdot d\underline{n}$$

But since $\underline{q} = -\nabla T$, $\nabla \cdot \underline{q} = -\nabla^2 T$

$$\Rightarrow \quad LHS = \int_0^1 \int_0^1 -g(x,y) \, dx \, dy$$

$$= -\int_0^1 \sin(2\pi x) \, dx \int_0^1 \sin(2\pi y) \, dy$$

$$= -\left(-\frac{1}{2\pi}\right)\left[\cos 2\pi x\right]_0^1 \left(-\frac{1}{2\pi}\right)\left[\cos 2\pi y\right]_0^1$$

$$= 0$$

(One could have guessed this by considering symmetry about the line $x = 0.5$)

Q2 (a) $V = \dfrac{Q}{4\pi\varepsilon r}$

$$\underline{E} = -\nabla V = -\nabla\left(\frac{Q}{4\pi\varepsilon r}\right)$$

$$= -\frac{\partial}{\partial r}\left(\frac{Q}{4\pi\varepsilon r}\right)\underline{e}_r$$

$$= \frac{Q}{4\pi\varepsilon r^2}\underline{e}_r$$

Solenoidal if $\nabla\cdot\underline{E} = 0$

In spherical coordinates, $\nabla\cdot\underline{E} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{Q}{4\pi\varepsilon r^2}\right)$

$= 0$ QED

(b) $\underline{E} = \dfrac{k}{r}\underline{e}_r \Rightarrow$ force on q $= qk\dfrac{1}{r}\underline{e}_r$

Work needed to move from R_1 to $2R_1$:

$$\int_{R_1}^{R_2}\frac{1}{r}qk\,dr = qk\ln\frac{R_2}{R_1}$$

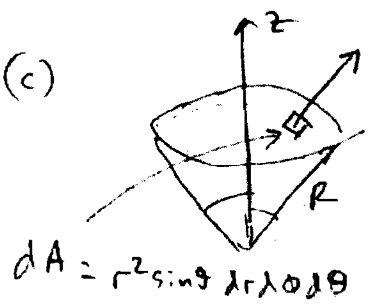
If $R_2 = 2R_1$, work $= qk\ln 2$

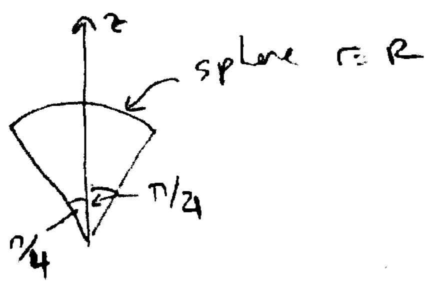
(Alternatively: work done $= [V(R_2) - V(R_1)]q$

since $\dfrac{\partial V}{\partial r} = \dfrac{k}{r} \Rightarrow V = k\ln r + C$

$\Rightarrow V(R_2) - V(R_1) = kq\ln\dfrac{R_2}{R_1}$)

(c)





Flux of $\underline{E}$ out of surface:

- zero from conical surface since $\underline{E}$ aligned with $\underline{r}$

- only through part of sphere at radius R

$$\therefore \text{Flux} = \int_0^{2\pi}\int_0^{\pi/4} R^2 \left(R^3 - R^2\right) \sin\theta \, d\theta \, d\phi$$

$$= \left(R^5 - R^4\right) \int_0^{2\pi} d\phi \int_0^{\pi/4} \sin\theta \, d\theta$$

$$= 2\pi \, R^4 (R-1) \left[\cos\theta\right]_0^{\pi/4}$$

$$= 2\pi \left(1 - \sqrt{2}/2\right) R^4 (R-1)$$

Divergence theorem: $\iiint_V \nabla \cdot \underline{E} \, dV = \int_A \underline{E} \cdot dA$

LHS: $\nabla \cdot \underline{E} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^5 - r^4\right) = 5r^2 - 4r$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^R \left(5r^2 - 4r\right) r^2 \sin\theta \, dr \, d\theta \, d\phi$$

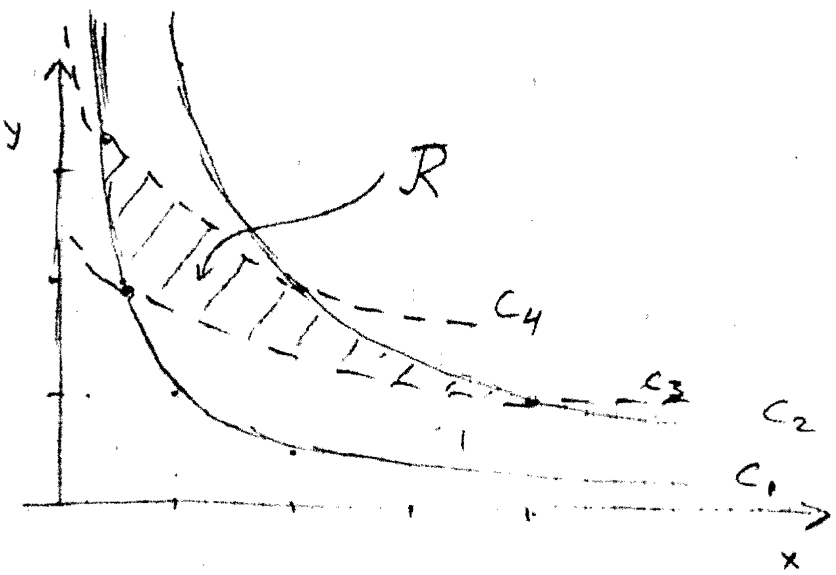
$$= -2\pi \cdot \left[\cos\theta\right]_0^{\pi/4} \int_0^R \left(5r^4 - 4r^3\right) dr$$

$$\left[\frac{5r^5}{5} - \frac{4r^4}{4}\right]_0^R$$

$$= 2\pi \left(1 - \sqrt{2}/2\right) \left(R^5 - R^4\right)$$

same as before.

Q3 (a)



$xy = 1 \rightarrow C_1$

$xy = 4 \rightarrow C_2$

$xy^3 = 4 \rightarrow C_3$

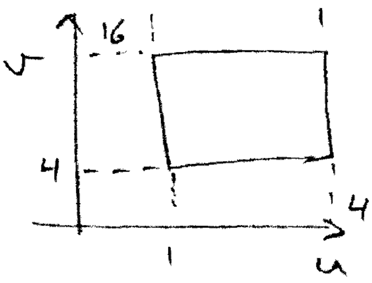
$xy^3 = 16 \rightarrow C_4$

(b) Area $= \iint dx\,dy$ difficult to integrate.

as limits are complicated

Using the transformation $u = xy$
$v = xy^3$

area of integration becomes



$$\int_4^{16}\int_1^4 \frac{1}{J}\,du\,dv$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ y^3 & 3xy^2 \end{vmatrix}$$

$= 2v$

$$\Rightarrow \text{area} = \int_4^{16}\int_1^4 \frac{1}{2v}\,du\,dv = \int_4^{16} \frac{1}{2v}\,dv \cdot [u]_1^4$$

$$= \frac{3}{2}\ln\frac{16}{4} = \frac{3}{2}\ln 4$$

(c) $\oint \underline{F}\cdot d\underline{l} = \iint_R \nabla\times\underline{F}\,dA$ by Stokes theorem

$$\nabla\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{4}xy^4 & x^2y^3 & 0 \end{vmatrix} = (2xy^3 - xy^3)\,\underline{k}$$

$$= xy^3\,\underline{k}$$

$\therefore$ Integration over R of $\nabla\times\underline{F}$ is best done in u-v space

$$\therefore \text{RHS} = \int_4^{16}\int_1^4 \frac{v}{2v}\,du\,dv = \frac{1}{2}\cdot 3\cdot(16-4)$$

$$= 18$$

Q4

(a) Probability is $1 - \Phi\left(\frac{9-6}{\sqrt{2.25}}\right) = 1 - \Phi(2)$

$= 0.0228$

(b) $P(\text{Disease}) = \dfrac{P(\text{positive test}) \cdot P(\text{positive diagnosis})}{P(\text{having disease \& test positive})}$

$= \dfrac{0.6 \times 0.0228}{0.98} = 0.014$

(c) False positive rate = probability of having +ve test, but the patient does not have the disease. Call this $P(+, \bar{D})$

$P(\text{positive test}) = P(+, \bar{D}) \cdot P(\text{negative diagnosis})$

$+ P(\text{having disease \& test positive}) \cdot P(\text{Disease})$

$\Rightarrow P(+, \bar{D}) = \dfrac{0.0228 - 0.98 \times 0.6}{0.986} = 0.0091$

(d) Poisson arrival of +ve test is $\lambda = 200 \times 0.0228 = 4.56$

$\Rightarrow$ Prob of >3 a day:

$1 - \left(1 + 4.56 + \frac{4.56^2}{2} + \frac{4.56^3}{6}\right) e^{-4.56} = 0.668$

5 (a)

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & -2 & 0 \\ 2 & 3 & 1 & 2 \\ 2 & 2 & 0 & a \end{pmatrix} = \begin{array}{cc} & \begin{matrix} 4 & 2 & -2 & 0 \end{matrix} \\ \begin{matrix} 1 \\ 0.5 \\ 0.5 \end{matrix} & \begin{pmatrix} 4 & 2 & -2 & 0 \\ 2 & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 \end{pmatrix} \end{array} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & 1 & a \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & 1 & a \end{pmatrix} = \begin{array}{cc} & \begin{matrix} 0 & 2 & 2 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 0.5 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a-1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a-1 \end{pmatrix} = \begin{array}{cc} & \begin{matrix} 0 & 0 & 0 & a-1 \end{matrix} \\ \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a-1 \end{pmatrix} \end{array} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & 0.5 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & -2 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & a-1 \end{pmatrix} = \mathbf{L}\,\mathbf{U} \quad \rightarrow \quad \text{For rank 2, we need } a = 1. \qquad [6]$$

(b) We have $\mathbf{Ax} = \mathbf{LUx} = \mathbf{Lc}$ where $\mathbf{c} = (c_1, c_2, c_3)^{\mathrm{T}} = \mathbf{Ux}$. Thus,

$$\mathbf{Lc} = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & 0.5 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1.25 \end{pmatrix} \quad \rightarrow \quad \begin{aligned} c_1 &= 1 \\ 0.5c_1 + c_2 &= 2 \\ 0.5c_1 + 0.5c_2 + c_3 &= 1.25 \end{aligned} \quad \rightarrow \quad \begin{aligned} c_1 &= 1 \\ c_2 &= 1.5 \\ c_3 &= 0 \end{aligned}$$

Hence, with $a = 1$ and writing $\mathbf{x} = (x, y, z, t)^{\mathrm{T}}$, we have,

$$\begin{pmatrix} 4 & 2 & -2 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1.5 \\ 0 \end{pmatrix}$$

The free variables are z and t (because of the zeros in row 3, columns 3 and 4).

The particular solution $\mathbf{x}_0$ is found by setting $z = 0$ and $t = 0$. Thus,

$$\begin{aligned} 2y + 2z + 2t &= 1.5 \quad \rightarrow \quad y = 0.75 \\ 4x + 2y - 2z &= 1 \quad \rightarrow \quad 4x = 1 - 1.5 \quad \rightarrow \quad x = -0.125 \end{aligned}$$

Thus, $\mathbf{x}_0 = (-0.125, 0.75, 0, 0)^{\mathrm{T}}$ [3]

The general solution is, $\mathbf{x} = \mathbf{x}_0 + \lambda\,\mathbf{u}_1 + \mu\,\mathbf{u}_2$ where $\mathbf{u}_1$ and $\mathbf{u}_2$ are in the null space and λ and μ are arbitrary scalar constants. To find $\mathbf{u}_1$ and $\mathbf{u}_2$ we solve $\mathbf{Uu} = \mathbf{0}$. Thus,

$$\begin{pmatrix} 4 & 2 & -2 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{aligned} 2y + 2z + 2t &= 0 \\ 4x + 2y - 2z &= 0 \end{aligned}$$

Setting $z = 0$ and $t = 1$, we obtain $y = -1$ and $x = 0.5$. Thus, $\mathbf{u}_1 = (0.5, -1, 0, 1)^{\mathrm{T}}$ [2]

Setting $z = 1$ and $t = 0$, we obtain $y = -1$ and $x = 1$. Thus, $\mathbf{u}_2 = (1, -1, 1, 0)^{\mathrm{T}}$ [2]

Hence, the general solution is,

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -0.125 \\ 0.75 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 0.5 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad [1]$$

(c)

$$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} 1 \\ 0.5 \\ 0.5 \end{pmatrix}\begin{pmatrix} 4 & 2 & -2 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0.5 \end{pmatrix}\begin{pmatrix} 0 & 2 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0.5 \\ 0.5 \end{pmatrix}\begin{pmatrix} 4 & 2 & -2 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0.5 \end{pmatrix}\begin{pmatrix} 0 & 2 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0.5 & 1 \\ 0.5 & 0.5 \end{pmatrix}\begin{pmatrix} 4 & 2 & -2 & 0 \\ 0 & 2 & 2 & 2 \end{pmatrix} = \mathbf{BC} \qquad [3]$$

(d) The column space of **B** is in the column space of **L** but does not span it. This is because it does not include the independent vector $(0, 0, 1)^{\mathrm{T}}$.

The column space of **C** is 2-dimensional whereas that of **U** is 3-dimensional. Hence, strictly, the two column spaces are not comparable. However, because the final row of **U** is composed of zeros, the column spaces are effectively identical. [3]

6 (a) Let the eigenvalues and eigenvectors of the matrix $\mathbf{B}$ be λ and $\mathbf{u}_B$ respectively. Thus,

$$\mathbf{B}\mathbf{u}_B = \lambda \mathbf{u}_B$$

Premultiplying by the matrix $\mathbf{X}$ gives,

$$\mathbf{X}\mathbf{B}\mathbf{u}_B = \lambda \mathbf{X}\mathbf{u}_B$$

From $\mathbf{A} = \mathbf{X}\mathbf{B}\mathbf{X}^{-1}$ (given) we have $\mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{B}$. Thus,

$$\mathbf{A}(\mathbf{X}\mathbf{u}_B) = \lambda(\mathbf{X}\mathbf{u}_B)$$

Hence, λ are the eigenvalues and $\mathbf{X}\mathbf{u}_B$ are the eigenvectors of the matrix $\mathbf{A}$. [4]

(b) $\mathbf{B}$ is an upper triangular matrix and so the eigenvalues are the elements on the diagonal,

$$\lambda_1 = 2, \qquad \lambda_2 = 1, \qquad \lambda_3 = 3.$$ [2]

The eigenvector corresponding to $\lambda_1 = 2$ is given by,

$$\begin{pmatrix} 2-\lambda_1 & 0 & 1 \\ 0 & 1-\lambda_1 & 1 \\ 0 & 0 & 3-\lambda_1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus, $e_3 = 0$ and $-e_2 + e_3 = 0 \rightarrow e_2 = 0$.
Hence, the normalised eigenvector is $\mathbf{u}_{B,1} = (1, 0, 0)^T$. [2]

The eigenvector corresponding to $\lambda_2 = 1$ is given by,

$$\begin{pmatrix} 2-\lambda_2 & 0 & 1 \\ 0 & 1-\lambda_2 & 1 \\ 0 & 0 & 3-\lambda_2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus, $e_3 = 0$ and $e_1 + e_3 = 0 \rightarrow e_1 = 0$.
Hence, the normalised eigenvector is $\mathbf{u}_{B,2} = (0, 1, 0)^T$. [2]

The eigenvector corresponding to $\lambda_3 = 3$ is given by,

$$\begin{pmatrix} 2-\lambda_3 & 0 & 1 \\ 0 & 1-\lambda_3 & 1 \\ 0 & 0 & 3-\lambda_3 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus, $e_1 = e_3$ and $-2e_2 + e_3 = 0 \rightarrow e_2 = e_3/2$.
Hence, the normalised eigenvector is $\mathbf{u}_{B,3} = (1, 1/2, 1)^T / (9/4)^{1/2} = (2/3, 1/3, 2/3)^T$. [2]

The eigenvalues of $\mathbf{B}$ are distinct and so $\mathbf{B}$ can be diagonalised. Writing $\mathbf{B} = \mathbf{U}_B\mathbf{D}\mathbf{U}_B^{-1}$ we construct $\mathbf{U}_B$ from the eigenvectors and $\mathbf{D}$ from the eigenvalues and obtain,

$$\mathbf{U}_B = \begin{pmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 2/3 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Note that $\mathbf{U}_B$ is not an orthogonal matrix ($\mathbf{B}$ is not symmetric) and hence $\mathbf{U}_B^{-1} \neq \mathbf{U}_B^T$. To find $\mathbf{U}_B^{-1}$ we can use the Gauss-Jordan elimination method:

$$\begin{pmatrix} 1 & 0 & 2/3 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & 0 & 1 & 0 \\ 0 & 0 & 2/3 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2/3 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3/2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 3/2 \end{pmatrix}$$

$$\mathbf{U}_B^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 3/2 \end{pmatrix}$$

Thus,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 2/3 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 3/2 \end{pmatrix}$$ [4]

(c) As shown in Part (a) the eigenvectors $\mathbf{u}_A$ of $\mathbf{A}$ are the column vectors $\mathbf{X}\mathbf{u}_B$. Thus,

For $\lambda_1 = 2$, $$\mathbf{u}_{A,1} = \mathbf{X}\mathbf{u}_{B,1} = \mathbf{X}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{x}_1$$

For $\lambda_2 = 1$, $$\mathbf{u}_{A,2} = \mathbf{X}\mathbf{u}_{B,2} = \mathbf{X}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \mathbf{x}_2$$

For $\lambda_3 = 3$, $$\mathbf{u}_{A,3} = \mathbf{X}\mathbf{u}_{B,3} = \mathbf{X}\begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} = \frac{2}{3}\mathbf{x}_1 + \frac{1}{3}\mathbf{x}_2 + \frac{2}{3}\mathbf{x}_3$$

where $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are the column vectors of $\mathbf{X}$. [4]