ENGINEERING TRIPOS PART IB

Friday 8 June 2012 2.30 to 4.30

Paper 7

MATHEMATICAL METHODS

Answer not more than ***four*** *questions.*

Answer not more than ***two*** *questions from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

Attachments: There are no attachments to this paper.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

VEM3

SECTION A

Answer not more than **two** *questions from this section.*

1 In two dimensions and with a source, the steady-state heat equation for the temperature T is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = g(x,y)$$

We seek solutions for $T(x,y)$ in the domain $0 \leq x \leq 1, 0 \leq y \leq 1$. The boundary conditions are $T(0,y) = T_0$, $T(1,y) = T_0$, $T(x,1) = T_0$, and $T(x,0) = T_0\,[1 + \sin(2\pi x)]$.

(a) Use separation of variables to find an analytical expression for $T(x,y)$ if $g(x,y) = 0$. [12]

(b) The heat flow is defined as $\mathbf{q} = -\nabla T$. If $g(x,y) = \sin(2\pi x)\sin(2\pi y)$, find the total heat flow $\oint \mathbf{q} \cdot d\mathbf{n}$, where the integral is taken along the whole boundary of the domain and $\mathbf{n}$ is the unit vector normal to domain's boundary. [8]

2 The electric field $\mathbf{E}$ can be written as $\mathbf{E} = -\nabla V$, where V is the potential. The force on a charge q is given by $q\mathbf{E}$.

(a) The potential around a point charge Q is given by $V = Q/(4\pi\varepsilon r)$, where r is the distance from the point charge and ε is a constant. Find $\mathbf{E}$ in the region $r > 0$ and show that the field is solenoidal. [5]

(b) If $\mathbf{E} = (K/r)\hat{\mathbf{e}}_r$, with K a positive constant and $\hat{\mathbf{e}}_r$ the unit vector in the radial direction, show that the work needed to move a point charge q from r to $2r$ is independent of r. [5]

(c) Consider the volume Ψ enclosed by the sphere $r = R$ and the cone whose apex is the origin, its axis is aligned with the $z-$axis, and its contained angle is $\pi/2$. If $\mathbf{E} = (r^3 - r^2)\hat{\mathbf{e}}_r$, with $\hat{\mathbf{e}}_r$ the unit vector in the radial direction, evaluate directly the flux of $\mathbf{E}$ through the surface bounding the volume Ψ and then show that the Divergence Theorem is satisfied. [10]

3 Consider the region $\Re$ bounded by the curves $xy = 1$, $xy = 4$, $xy^3 = 4$, and $xy^3 = 16$.

(a) Sketch $\Re$. [4]

(b) Evaluate the area of $\Re$. [8]

(c) If $d\mathbf{l}$ is the infinitesimal line element along the boundary of $\Re$, find $\oint \mathbf{F} \cdot d\mathbf{l}$ if $\mathbf{F} = \frac{1}{4}xy^4\,\mathbf{i} + x^2y^3\,\mathbf{j}$. [8]

SECTION B

*Answer not more than **two** questions from this section.*

4 Over a population, a blood test gives a score Y, which obeys a normal distribution with mean $\mu = 6$ and variance $\sigma^2 = 2.25$. The test is used to diagnose a disease, with the diagnosis deemed positive if $Y > 9$. For a patient with the disease, the test is positive with probability 0.98. The probability of having the disease given that the test is positive is 0.6.

(a) What is the probability that Y exceeds 9? [4]

(b) What is the proportion of the population that has the disease? [5]

(c) What is the false positive rate of the test? [5]

(d) A lab receives on average 200 blood samples per day. What is the probability that it gets more than three positive test results in a day? [6]

5 (a) By performing LU decomposition on the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & -2 & 0 \\ 2 & 3 & 1 & 2 \\ 2 & 2 & 0 & a \end{bmatrix}$$

find the value of a that makes the rank of $\mathbf{A}$ equal to 2. [8]

(b) For this value of a, find the general solution of the equation

$$\mathbf{Ax} = \begin{bmatrix} 1 \\ 2 \\ 1.25 \end{bmatrix}$$

[4]

(c) By expressing $\mathbf{A}$ as an outer product of the columns and rows of $\mathbf{L}$ and $\mathbf{U}$, express $\mathbf{A}$ as the product of two matrices, $\mathbf{A} = \mathbf{BC}$, where $\mathbf{B}$ is a 3×2 matrix and $\mathbf{C}$ is a 2×4 matrix. [4]

(d) How are the column spaces of $\mathbf{B}$ and $\mathbf{C}$ related to those of $\mathbf{L}$ and $\mathbf{U}$? [4]

6 (a) Two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be *similar* if there is a non-singular $n \times n$ matrix $\mathbf{X}$, such that

$$\mathbf{A} = \mathbf{X}\mathbf{B}\mathbf{X}^{-1}.$$

Show that similar matrices have the same eigenvalues and find the relationship between their eigenvectors that correspond to the same eigenvalue. [4]

(b) Find the eigenvalues and the *normalised* eigenvectors of the matrix $\mathbf{B}$, where

$$\mathbf{B} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Hence, express $\mathbf{B}$ in the form $\mathbf{B} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$, where $\mathbf{D}$ is a diagonal matrix. [10]

(c) Find the eigenvectors of $\mathbf{A}$, expressing your answer in terms of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, the vectors representing the columns of $\mathbf{X}$. [6]

END OF PAPER