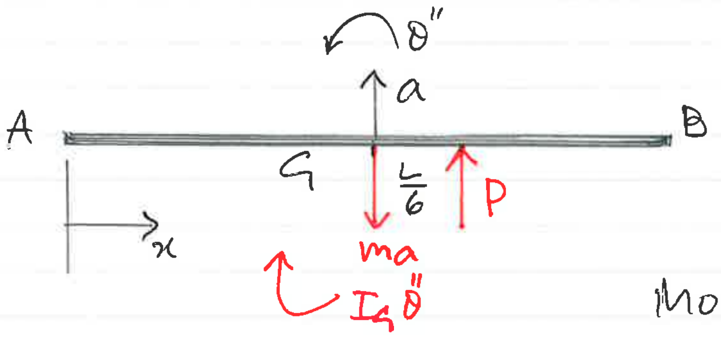
Part IB 2013 Paper 1 Mechanics Solutions

1.



Resolving

$$P - ma = 0$$

Moments about G

Measure x from A

$$\frac{PL}{6} - I_G\,\theta'' = 0 \qquad I_G = \frac{mL^2}{12}$$

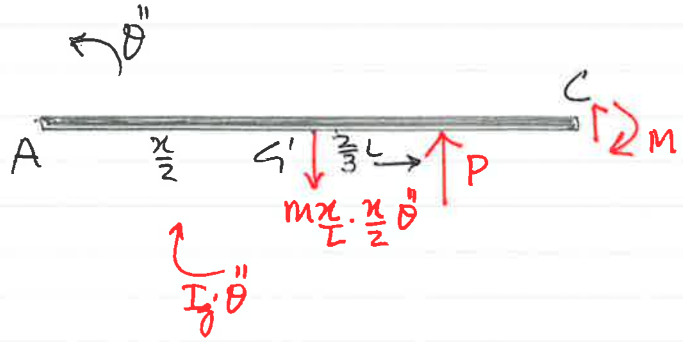
$$\therefore \quad \theta'' = \frac{2P}{mL} = \frac{2a}{L}$$

(a) Zero transverse accln. where $a - \left(\frac{L}{2} - x\right)\theta'' = 0$

$$\text{ie.} \quad a - a + \frac{2ax}{L} = 0$$

$$\text{ie.} \quad x = 0 \text{ at A}$$

(b) Consider section AC of rod length x



$$I_{G'} = \frac{1}{12}\left(\frac{mx}{L}\right)x^2 = \frac{mx^3}{12L}$$

Taking moments about cut at C

$$M + I_{G'}\,\theta'' - \frac{mx}{L}\cdot\frac{x}{2}\,\theta''\cdot\frac{x}{2} + P\left\{x - \tfrac{2}{3}L\right\} = 0$$

where $\left\{x - \frac{2}{3}L\right\} = 0$ if $x < \frac{2}{3}L$

So that for $0 < x < \frac{2}{3}L$

$$M = \frac{mx^3\theta''}{4L} - \frac{mx^3\theta''}{12L} = \frac{mx^3\theta''}{6L}$$

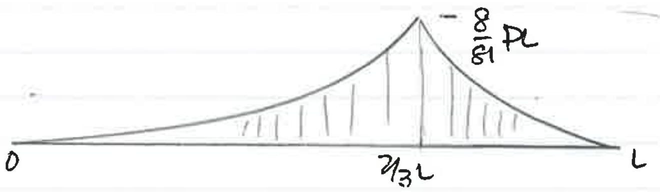
$$\text{ie.} \quad M = \frac{max^3}{3L^2} \quad \text{or} \quad \frac{Px^3}{3L^2}$$

Similarly for $\frac{2}{3}L < x < L$

$$M = \frac{mx^3}{6L}\ddot{\theta} - P\left\{x - \tfrac{2}{3}L\right\}$$

$$M = \frac{Px^3}{3L^2} - P\left\{x - \tfrac{2}{3}L\right\}$$ So $M = 0$ at $x = L$

(c)



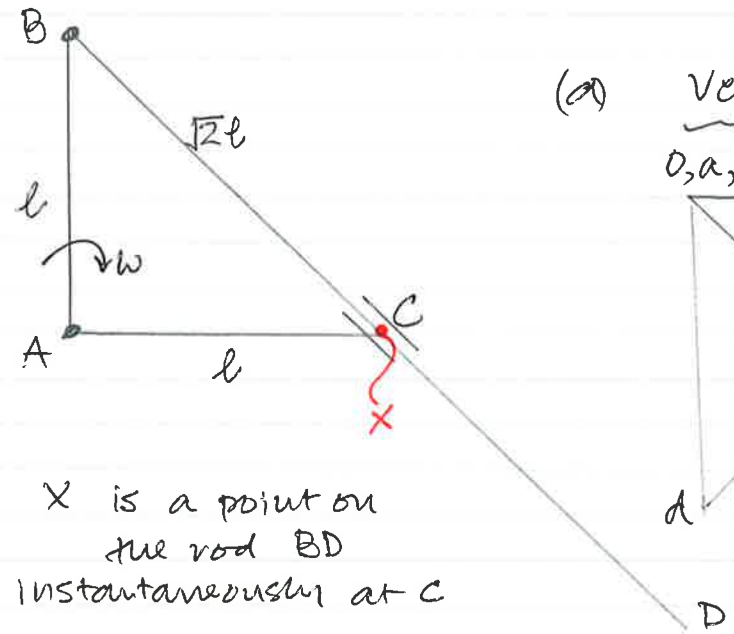
M_{max} when $x = \frac{2}{3}L$, ie at point of application of P

$$M_{max} = \frac{P}{3L^2} \cdot \frac{8L^3}{27} = \frac{8PL}{81}$$

At point of application of P the shear force changes from $\frac{4}{9}P$ to $-\frac{5}{9}P$.

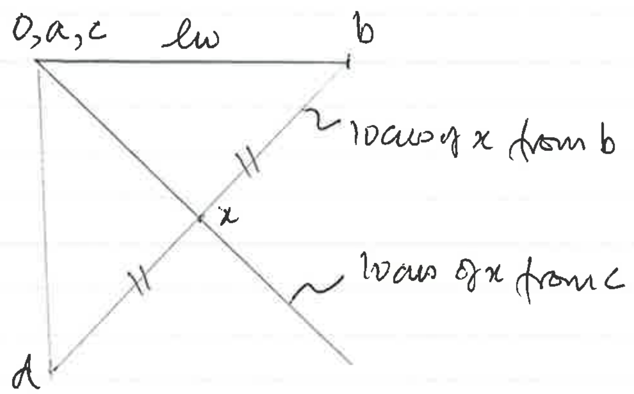
A popular question with a fair number of solutions gaining full marks. The most common error in establishing values of bending moment (or shear force) was to fail to appreciate that the bar will only experience the point force P if x, the length under examination, exceeds $2L/3$. At the point of application of P there will be a step change in shear force and a change in the gradient of the BM distribution. Many candidates wrote down a continuous function for the bending moment over the complete range $0 < x < L$ assuming that P was always present.

2.



X is a point on the rod BD instantaneously at C

(a) Velocity diagram



velocity image

$$\frac{bx}{xd} = \frac{BX}{XD}$$

So that $|\underline{v}_b| = |\underline{v}_d| = \ell\omega$ and are $\perp^r$

$$\omega_{BCD} = \frac{bd}{BD} = \frac{\sqrt{2}\ell\omega}{2\sqrt{2}\ell} = \frac{\omega}{2}$$

and $cx = \frac{\ell\omega}{\sqrt{2}}$ sliding at c

(b) accelerations

B→A at $\ell\omega^2$

X→B at $\sqrt{2}\ell\left(\frac{\omega}{2}\right)^2 = \frac{\ell\omega^2}{2\sqrt{2}}$

Coriolis accn. at C

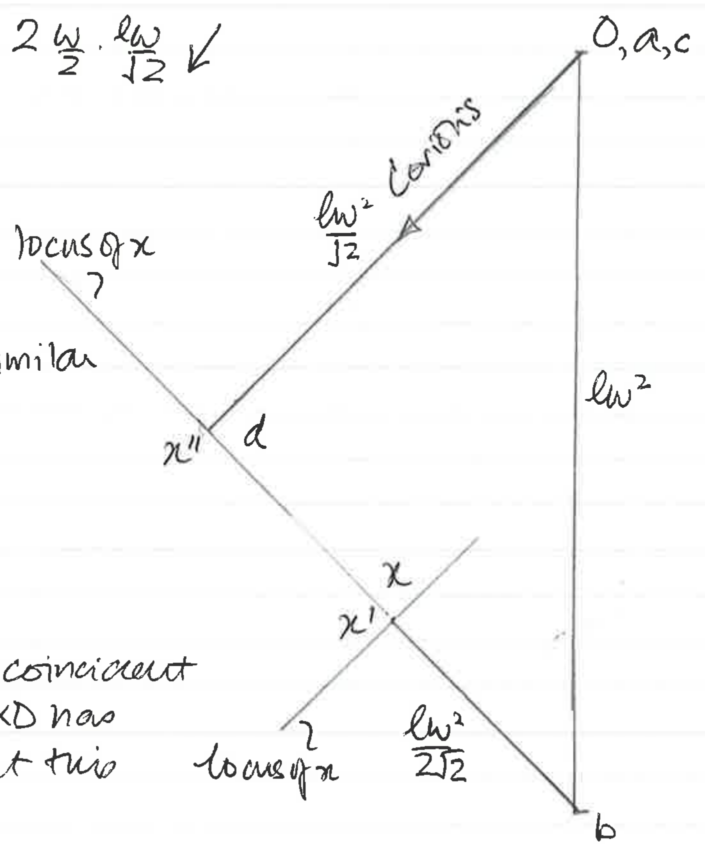
$$= 2\,\underline{\omega}_{BCD} \times \underline{cx} = 2\,\frac{\omega}{2}\cdot\frac{\ell\omega}{\sqrt{2}} \swarrow$$

unknowns are $\dot{\omega}_{BCD}$

and sliding accn at C.

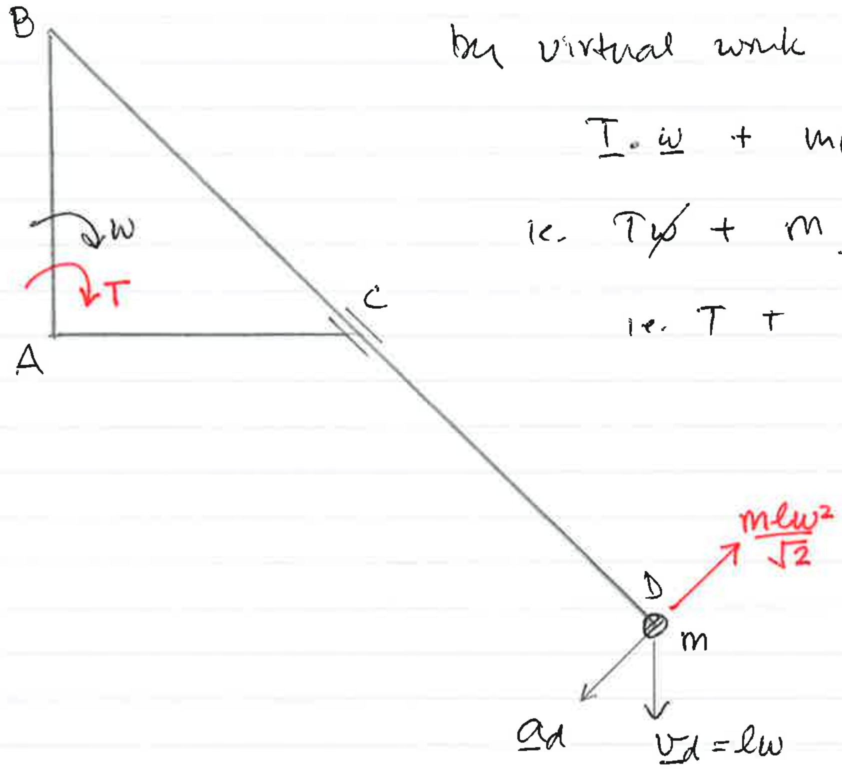
By accn. image bxd similar BXD

hence $\underline{a_d} = \frac{\ell\omega^2}{\sqrt{2}} \swarrow$



Note that since x is coincident with x', the rod BXD has no angular accn. at this instant.

(c) If mass m at D, then since no friction at slide C



by virtual work

$$\underline{T} \cdot \underline{\omega} + m(-\underline{a}_d) \cdot \underline{v}_d = 0$$

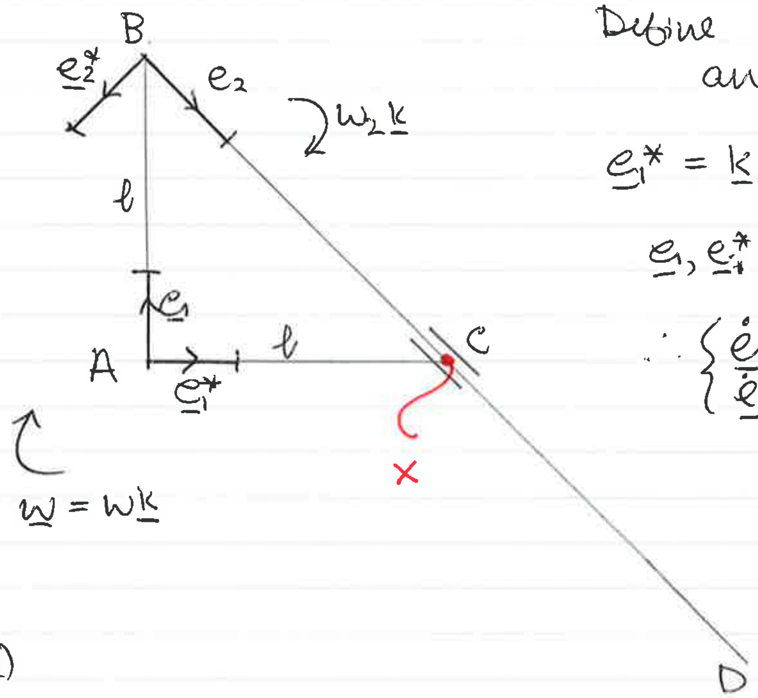
ie. $T\cancel{\omega} + m \frac{l\omega^2}{\sqrt{2}} \cdot l\cancel{\omega} \cdot \cos 135^\circ = 0$

ie. $T + \frac{ml^2\omega^2}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}} = 0$

$$T = \frac{ml^2\omega^2}{2}$$

The most popular question. The first section is Part I material and completed more or less universally. This is just about the simplest configuration in which Coriolis features in the kinematics and (as usual) it caused difficulties for the majority of candidates. There were only a handful of fully satisfactory treatment of accelerations and thus of inertial effects. A few brave souls used vectors from scratch which really isn't necessary for a plane mechanism and virtually all got lost. Several candidates solved (a) by drawing a nice velocity diagram and then abandoned this method and started again to get to accelerations with vectors. Virtual work was adopted in the final stage and even with incorrect values of acceleration gained some credit. Despite a specific instruction to ignore the effects of gravity 'g' featured in several attempts.

Q2 by vectors



Define unit vectors $\underline{e}_1$ along AB and $\underline{e}_2$ along BXD

$$\underline{e}_1^* = \underline{k} \times \underline{e}_1 \quad \text{and} \quad \underline{e}_2^* = \underline{k} \times \underline{e}_2$$

$\underline{e}_1, \underline{e}_1^*$ rotate at $\omega \underline{k}$

$$\therefore \begin{cases} \dot{\underline{e}}_1 = \omega \underline{k} \times \underline{e}_1 = \omega \underline{e}_1^* \\ \dot{\underline{e}}_1^* = \omega \underline{k} \times \underline{e}_1^* = -\omega \underline{e}_1 \end{cases}$$

$$\begin{cases} \dot{\underline{e}}_2 = \omega_2 \underline{k} \times \underline{e}_2 = \omega_2 \underline{e}_2^* \\ \dot{\underline{e}}_2^* = \omega_2 \underline{k} \times \underline{e}_2^* = -\omega_2 \underline{e}_2 \end{cases}$$

(a)

The point X, a fixed distance $\sqrt{2}\ell$ along BD, can be approached either from B or from C.

viz. $\underline{AX} = \underline{AB} + \underline{BX}$

$$= \ell \underline{e}_1 + \sqrt{2}\ell \underline{e}_2$$

$$\underline{v}_B = \omega \ell \underline{e}_1^*$$

$$\therefore \underline{v}_x = \ell \dot{\underline{e}}_1 + \sqrt{2}\ell \dot{\underline{e}}_2$$

$$\underline{v}_x = \omega \ell \underline{e}_1^* + \sqrt{2}\omega_2 \ell \underline{e}_2^* \qquad (1)$$

But also $\underline{AX} = \underline{AC} + \underline{CX}$

$$= \underline{AC} + r \underline{e}_2$$

although $r = 0, \dot{r} \neq 0$

$$\therefore \underline{v}_x = \underbrace{\dot{\underline{AC}}}_{0} + \dot{r} \underline{e}_2 + r \dot{\underline{e}}_2$$

$$\underline{v}_x = \dot{r} \underline{e}_2 + r \omega_2 \underline{e}_2^* \qquad (2)$$

Now we can note that $\underline{e}_1^* = \frac{1}{\sqrt{2}}(\underline{e}_2 - \underline{e}_2^*)$

$$\therefore \underline{v}_x = \frac{\omega \ell}{\sqrt{2}} \underline{e}_2 - \frac{\omega \ell}{\sqrt{2}} \underline{e}_2^* + \sqrt{2}\omega_2 \ell \underline{e}_2^*$$

hence equating coefficients, and noting that $r \Rightarrow 0$

$$\dot{r} = \frac{\omega \ell}{\sqrt{2}} \quad \text{and} \quad 0 = -\frac{\omega \ell}{\sqrt{2}} + \sqrt{2}\omega_2 \ell \quad \therefore \omega_2 = \frac{\omega}{2}$$

Now $\underline{AD} = \underline{AB} + \underline{BD}$

$$= l\underline{e_1} + 2\sqrt{2}l\underline{e_2}$$

$$\underline{v_D} = l\omega \underline{e_1}^* + \cancel{2}\sqrt{2}l\frac{\omega}{\cancel{2}}\underline{e_2}^*$$

Since $\underline{e_2}^* = -\frac{1}{\sqrt{2}}(\underline{e_1} + \underline{e_1}^*)$

$$\underline{v_D} = l\omega\underline{e_1}^* - l\omega\underline{e_1} - l\omega\underline{e_1}^*$$

$$\underline{v_D} = -l\omega\underline{e_1}$$

illustrating that $\underline{v_B} = l\omega\underline{e_1}^*$ and $\underline{v_D}$ are of same magnitude but $\perp^r$, since $\underline{e_1}\cdot\underline{e_1}^* = 0$

(b) Now differentiate ① & ② to obtain expressions for acceleration of X.

from ① $\underline{a_x} = \omega l\,\underline{\dot{e}_1}^* + \sqrt{2}l\dot{\omega}_2\underline{e_2}^* + \sqrt{2}l\omega_2\underline{\dot{e}_2}^*$

$$= -\omega^2 l\,\underline{e_1} + \sqrt{2}l\dot{\omega}_2\underline{e_2}^* - \sqrt{2}l\omega_2^2\underline{e_2}$$

noting that $\omega_2 = \frac{\omega}{2}$ and $\underline{e_1} = -\frac{1}{\sqrt{2}}(\underline{e_2} + \underline{e_2}^*)$

$$\underline{a_x} = \frac{\omega^2 l}{\sqrt{2}}\underline{e_2} + \frac{\omega^2 l}{\sqrt{2}}\underline{e_2}^* + \sqrt{2}l\dot{\omega}_2\underline{e_2}^* - \frac{\sqrt{2}}{4}\omega^2 l\,\underline{e_2}$$

$$\underline{a_x} = \frac{\omega^2 l}{2\sqrt{2}}\underline{e_2} + \left(\frac{\omega^2 l}{\sqrt{2}} + \sqrt{2}l\dot{\omega}_2\right)\underline{e_2}^*$$

But from ②

$$\underline{a_x} = \ddot{r}\underline{e_2} + \dot{r}\underline{\dot{e}_2} + \dot{r}\omega_2\underline{e_2}^* + r\dot{\omega}_2\underline{e_2}^* + r\omega_2\underline{\dot{e}_2}^*$$

$$= \ddot{r}\underline{e_2} + \dot{r}\omega_2\underline{e_2}^* + \dot{r}\omega_2\underline{e_2}^* + r\dot{\omega}_2\underline{e_2}^* - r\omega_2^2\underline{e_2}$$

$$= (\ddot{r} - r\omega_2^2)\underline{e_2} + (r\dot{\omega}_2 + 2\dot{r}\omega_2)\underline{e_2}^*$$

Now set $r=0$, $\omega_2 = \omega/2$ and $\dot{r} = \omega l/\sqrt{2}$

then $\underline{a_x} = \ddot{r}\,\underline{e_2} + 2\frac{\omega l}{\sqrt{2}}\cdot\frac{\omega}{2}\underline{e_2}^*$

Now equate coefficients $\ddot{r} = \frac{\omega^2 l}{2\sqrt{2}}$ $\quad \frac{\omega^2 l}{\sqrt{2}} + \sqrt{2}l\dot{\omega}_2 = \frac{\omega^2 l}{\sqrt{2}}$

$$\dot{\omega}_2 = 0$$

(c) we need accn of D.

$$\underline{AD} = \underline{AB} + \underline{BD} = l\underline{e}_1 + 2\sqrt{2}l\,\underline{e}_2$$

$$\underline{v}_D = l\,\dot{\underline{e}}_1 + 2\sqrt{2}l\,\dot{\underline{e}}_2 = l\omega\,\underline{e}_1^* + 2\sqrt{2}l\omega_2\underline{e}_2^*$$

$$\underline{a}_D = l\omega\,\dot{\underline{e}}_1^* + 2\sqrt{2}l\dot{\omega}_2\,\underline{e}_2^* + 2\sqrt{2}l\omega_2\,\dot{\underline{e}}_2^*$$

$$= -l\omega^2\underline{e}_1 - 2\sqrt{2}l\,\omega_2^2\,\underline{e}_2$$

$$\underline{a}_D = -l\omega^2\underline{e}_1 - \frac{2\sqrt{2}l\,\omega^2}{4}\underline{e}_2$$

So D'Alembert force at D $= -m\underline{a}_D$

$$= lm\omega^2\,\underline{e}_1 + \frac{ml\omega^2}{\sqrt{2}}\underline{e}_2$$

So by virtual work

$$\underline{T}\cdot\underline{\omega} + (-m\underline{a}_D)\cdot\underline{v}_D = 0$$

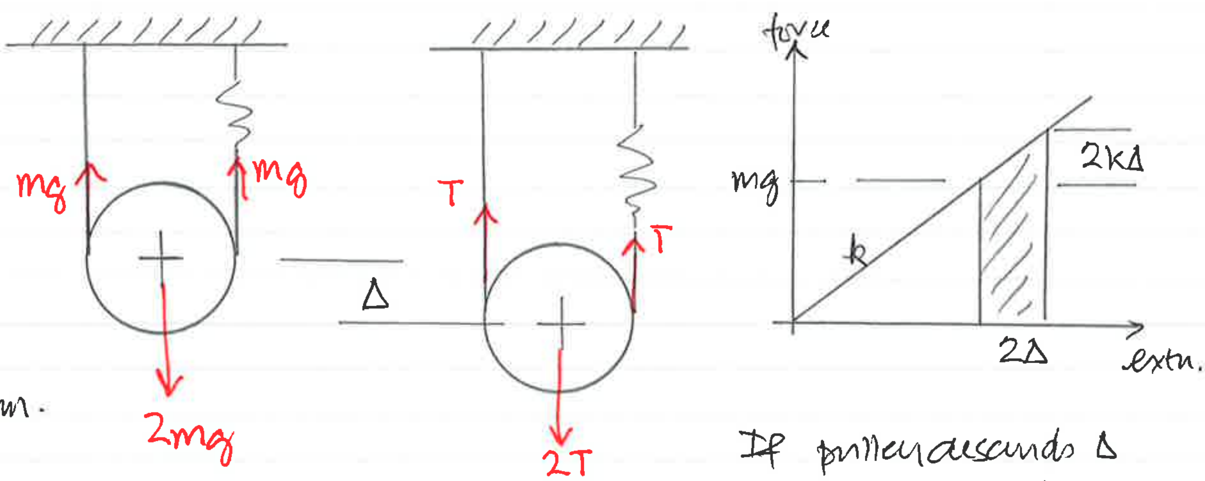
$$T\omega = -ml\omega^2\left(\underline{e}_1 + \frac{\underline{e}_2}{\sqrt{2}}\right)\cdot\left(\underline{e}_1^* + \sqrt{2}\,\underline{e}_2^*\right)l\omega$$

noting that $\begin{cases}\underline{e}_1\cdot\underline{e}_1^* = 0\\ \underline{e}_2\cdot\underline{e}_2^* = 0\end{cases}$ $\begin{cases}\underline{e}_2\cdot\underline{e}_1^* = \frac{1}{\sqrt{2}}\\ \underline{e}_1\cdot\underline{e}_2^* = -\frac{1}{\sqrt{2}}\end{cases}$

$$\therefore\ T\omega = -ml^2\omega^3\left\{-1 + \frac{1}{2}\right\}$$

$$T = \frac{ml^2\omega^2}{2}$$

3.



equilbm.

If pulley descends Δ then spring extends by 2Δ.

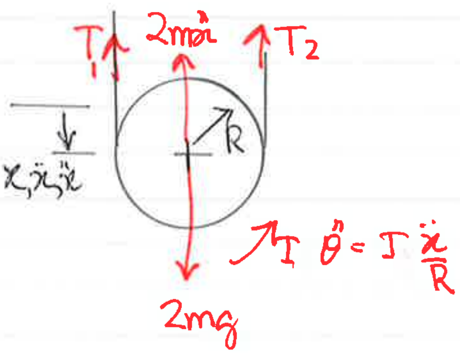
(a)

Change in stored energy, i.e. shaded area,

$$= \frac{1}{2}\left[mg + (mg + 2k\Delta)\right] \times 2\Delta$$

$$= 2(mg + k\Delta)\Delta$$

(b) (i) after release, $x < \Delta$, pulley has rotated x/R



$T_2 = mg + 2kx$ (linear spring)

moments about LH end of diameter

$$T_2 \cdot 2R + 2m\ddot{x} \cdot R - 2mg \cdot R + I\frac{\ddot{x}}{R} = 0$$

i.e. $\cancel{2mgR} + 4kx R + 2m\ddot{x}R - \cancel{2mgR} + I\frac{\ddot{x}}{R} = 0$

$$(2m + I/R^2)\ddot{x} = -4kx$$

$$\ddot{x} = \frac{-4k}{2m + I/R^2}x$$

$$\therefore \omega_n = \sqrt{\frac{4k}{2m + I/R^2}}$$

or using Data Book

potential energy $V(x) = -2mgx + 2(mg + kx)x$

$\therefore V'(x) = -2mg + 2mg + 4kx$

$$V''(x) = 4k$$

kinetic energy $= \frac{1}{2} \cdot 2m \cdot \dot{x}^2 + \frac{1}{2} I \cdot \left(\frac{\dot{x}}{R}\right)^2$

$$= \frac{1}{2}(2m + I/R^2)\dot{x}^2$$

$\therefore \omega_n^2 = \dfrac{4k}{2m + I/R^2}$ as above.

(ii) Resolving $T_1 + T_2 + 2m\ddot{x} - 2mg = 0$

But $T_2 = mg + 2kx$ and $\ddot{x} = \dfrac{-4k}{2m + I/R^2} \cdot x$

If pulley is simple disc $I = \dfrac{mR^2}{2}$

$\therefore \ddot{x} = -\dfrac{8k}{5m} x$

and if $T_1 \to 0$

$$mg + 2kx - \frac{16kx}{5} - 2mg = 0$$

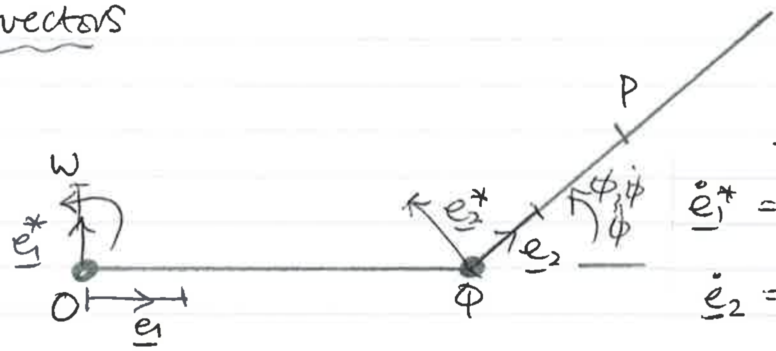
i.e. $-\frac{6}{5}kx = mg$

$$x = -\frac{5}{6}\frac{mg}{k}$$

So $\Delta = |x| = \dfrac{5}{6}\dfrac{mg}{k}$

Very similar to an examples paper question and so perhaps surprisingly unpopular. Mention of energy in the first part was a hint to a suitable approach and this was taken by the majority. But the problem is barely any more difficult if tackled from first principles.

4. (a) By vectors



$\underline{e_1^*} = \underline{k} \times \underline{e_1}$; $\underline{e_2^*} = \underline{k} \times \underline{e_2}$

$\dot{\underline{e}}_1 = \omega \underline{k} \times \underline{e_1} = \omega \underline{e_1^*}$

$\dot{\underline{e}}_1^* = \omega \underline{k} \times \underline{e_1^*} = -\omega \underline{e_1}$

$\dot{\underline{e}}_2 = (\omega + \dot\phi)\underline{k} \times \underline{e_2} = (\omega + \dot\phi)\underline{e_2^*}$

$\dot{\underline{e}}_2^* = (\omega + \dot\phi)\underline{k} \times \underline{e_2^*} = -(\omega + \dot\phi)\underline{e_2}$

$$\underline{OP} = \underline{r_p} = l\underline{e_1} + \frac{l}{2}\underline{e_2}$$

$$\dot{\underline{OP}} = \underline{v_p} = l\dot{\underline{e}}_1 + \frac{l}{2}\dot{\underline{e}}_2 = l\omega \underline{e_1^*} + \frac{l}{2}(\omega + \dot\phi)\underline{e_2^*}$$

$$= l\omega\underline{e_1^*} + \frac{l}{2}\omega\underline{e_2^*} + \frac{l}{2}\dot\phi\underline{e_2^*}$$

$$\ddot{\underline{OP}} = \underline{a_p} = l\omega\dot{\underline{e}}_1^* + \frac{l}{2}\omega\dot{\underline{e}}_2^* + \frac{l}{2}\ddot\phi\underline{e_2^*} + \frac{l}{2}\dot\phi\dot{\underline{e}}_2^* \checkmark$$

$$= -l\omega^2\underline{e_1} \checkmark - \frac{l}{2}\omega(\omega + \dot\phi)\underline{e_2} + \frac{l}{2}\ddot\phi\underline{e_2^*} - \frac{l}{2}\dot\phi(\omega + \dot\phi)\underline{e_2}$$

$$\therefore \underline{a_p} = -l\omega^2\underline{e_1} - \frac{l}{2}(\omega + \dot\phi)^2\underline{e_2} + \frac{l}{2}\ddot\phi\underline{e_2^*}$$

OR

Using Data Book p2 $\qquad \underline{r} = l/2\,\underline{e_2} \qquad \underline{\omega} = \omega\underline{k}$

$$\underline{a_p} = \underline{a_Q} + \left[\frac{d^2\underline{r}}{dt^2}\right]_R + \underbrace{\frac{d\underline{\omega}}{dt}}_{\Downarrow 0 \ \underline{\omega}\text{ const}} \times \underline{r} + 2\underline{\omega} \times \left[\frac{d\underline{r}}{dt}\right]_R + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

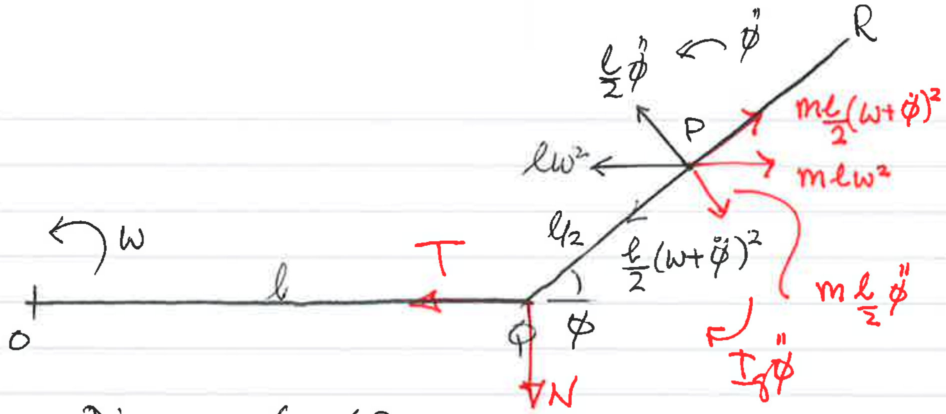
$$\therefore \underline{a_p} = -l\omega^2\underline{e_1} - \frac{l}{2}\dot\phi^2\underline{e_2} + \frac{l}{2}\ddot\phi\underline{e_2^*} + 2\omega\underline{k} \times \frac{l}{2}\dot\phi\underline{e_2^*}$$

$$+ \frac{l}{2}\omega^2\,\underline{k} \times (\underline{k} \times \underline{e_2})$$

$$\Rightarrow -l\omega^2\underline{e_1} - \frac{l}{2}\dot\phi^2\underline{e_2} + \frac{l}{2}\ddot\phi\underline{e_2^*} - l\omega\dot\phi\underline{e_2} - \frac{l}{2}\omega^2\underline{e_2}$$

$$= -l\omega^2\underline{e_1} - \frac{l}{2}(\dot\phi + \omega)^2\underline{e_2} + \frac{l}{2}\ddot\phi\underline{e_2^*}$$

(b) (i)



Free Body Diagram for QR

Moments about Q; $m\frac{l}{2}\ddot{\phi}\cdot\frac{l}{2} + mlw^2\frac{l}{2}\sin\phi + I_g\ddot{\phi} = 0$

But $I_g = \frac{ml^2}{12}$ $\therefore$ $\frac{ml^2}{4}\ddot{\phi} + \frac{ml^2w^2\sin\phi}{2} + \frac{ml^2}{12}\ddot{\phi} = 0$

But $\ddot{\phi} = \dot{\phi}\frac{d\dot{\phi}}{d\phi}$ $\qquad \frac{\ddot{\phi}}{3} = -\frac{w^2\sin\phi}{2}$

$$\therefore \dot{\phi}\frac{d\dot{\phi}}{d\phi} = -\frac{3}{2}w^2\sin\phi$$

$$\frac{\dot{\phi}^2}{2} = \frac{3}{2}w^2\cos\phi + A$$

But $\dot{\phi} = 0$ when $\phi = \pi/2$ $\therefore$ $A = 0$

So that $\dot{\phi}^2 = 3w^2\cos\phi$

Now resolving $T = mlw^2 + \frac{ml}{2}\ddot{\phi}\sin\phi + \frac{ml}{2}(w+\dot{\phi})^2\cos\phi$

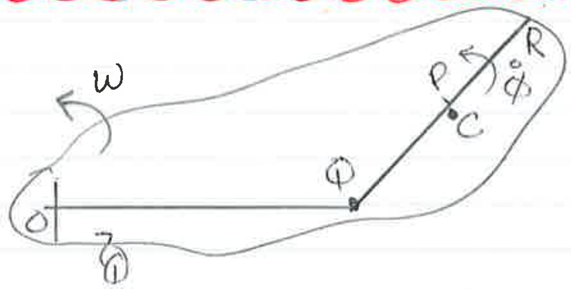
ie. $T = mlw^2 - \frac{3mlw^2}{4}\sin^2\phi + \frac{mlw^2}{2}\cos\phi + mlw\dot{\phi}\cos\phi + \frac{ml}{2}\dot{\phi}^2\cos\phi$

When ϕ becomes 0, $\sin\phi \to 0$, $\cos\phi = 1$ and $\dot{\phi} = -\sqrt{3}w$

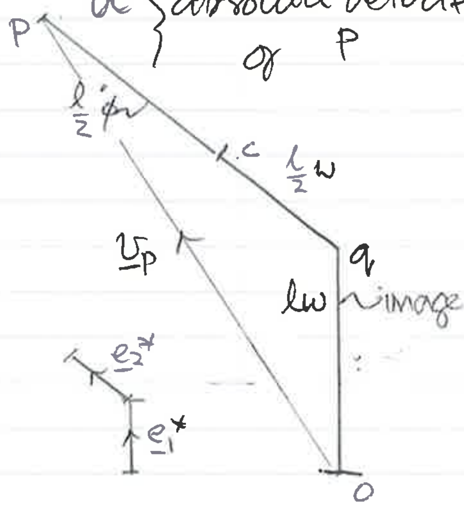
$$\therefore T = mlw^2\left(1 + \tfrac{1}{2} - \sqrt{3} + \tfrac{3}{2}\right) = (3-\sqrt{3})mlw^2$$

A very popular question. A surprisingly large number of students failed to differentiate properly the velocity vector to get the acceleration vector in part (a). Part (b) was approached by many students as a purely kinematic question with many attempts to solve it using velocity and acceleration diagrams. Part (b-ii) was generally well approached, although very few students realised that ϕ was negative when $\phi = 0$ is reached.

By velocity & accn. diagrams



Let C be the point on OQ, body ①, extended, that is instantaneously coincident with P.



velocity diagram

We $\begin{Bmatrix}\text{absolute velocity} \\ \text{of } P\end{Bmatrix} = \begin{Bmatrix}\text{velocity of} \\ C\end{Bmatrix} + \begin{Bmatrix}\text{vely of } P \text{ rel.} \\ \text{to } C\end{Bmatrix}$

ie. $\underline{v}_P = l\omega\, \underline{e}_1^* + \frac{l}{2}\omega\, \underline{e}_2^* + \frac{l}{2}\dot\phi\, \underline{e}_2^*$

But

$\begin{Bmatrix}\text{absolute accln} \\ \text{of } P\end{Bmatrix} = \begin{Bmatrix}\text{accln of} \\ C\end{Bmatrix} + \begin{Bmatrix}\text{accln of } P \\ \text{relative to } C\end{Bmatrix} + \begin{Bmatrix}\text{Coriolis} \\ \text{accln}\end{Bmatrix}$

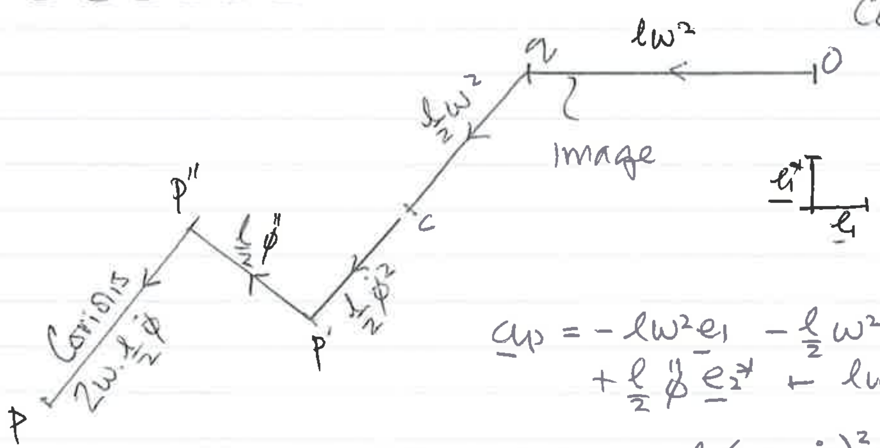
$= \left\{-l\omega^2 \underline{e}_1 - \frac{l}{2}\omega^2 \underline{e}_2\right\}$

$+ \left\{-\frac{l}{2}\dot\phi^2 \underline{e}_2 + \frac{l}{2}\ddot\phi\, \underline{e}_2^*\right\}$

$+ \left\{2\,\omega . \frac{l}{2}\dot\phi \swarrow\right\}; -\underline{e}_2$

Coriolis — from rely diagram.

accn. diagram



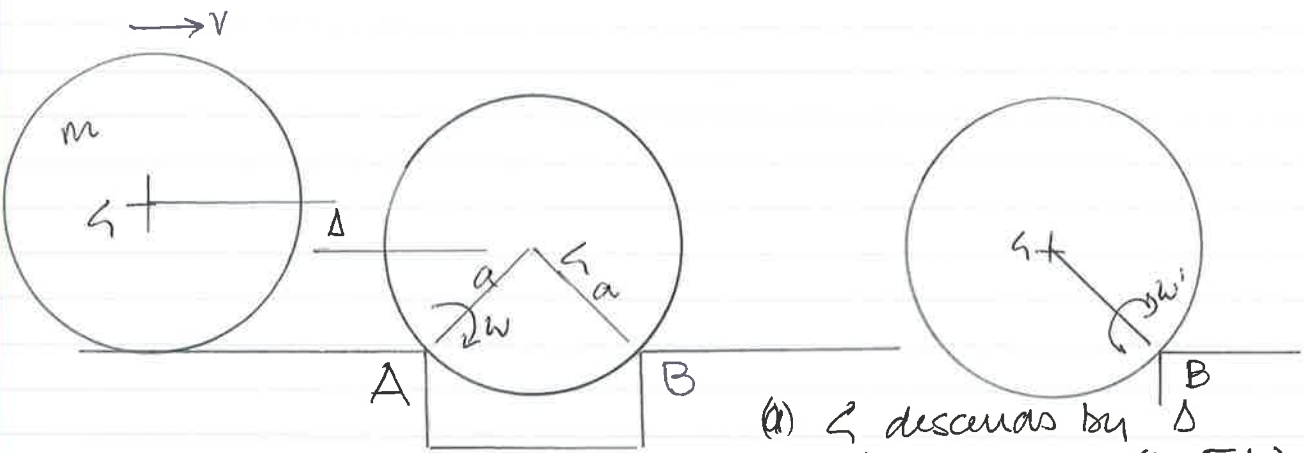
$a_p = -l\omega^2 e_1 - \frac{l}{2}\omega^2 \underline{e}_2 - \frac{l}{2}\dot\phi^2 \underline{e}_2 + \frac{l}{2}\ddot\phi\, \underline{e}_2^* - l\omega\dot\phi\, \underline{e}_2$

$\Rightarrow -l\omega^2 \underline{e}_1 - \frac{l}{2}(\omega+\dot\phi)^2 \underline{e}_2 + \frac{l}{2}\ddot\phi\, \underline{e}_2^*$

etc

Lecture notes for conservation: mechanical = k.e. + p.e.

5.



(a) G descends by Δ

$\Delta = a - \frac{a}{\sqrt{2}} = (1-\sqrt{2}/2)a$

∴ gain of k.e. = loss of p.e.

$$\frac{1}{2}m(a\omega)^2 + \frac{1}{2}I_G\omega^2 - \underbrace{\left(\frac{1}{2}mV^2 + \frac{1}{2}I_G\left(\frac{V}{a}\right)^2\right)}_{\text{initial k.e.} = \frac{3}{4}mV^2} = mg\Delta$$

But $I_G = \frac{ma^2}{2}$ 'disc'

$$\therefore \frac{3}{4}\cancel{m}a^2\omega^2 = \frac{3}{4}\cancel{m}V^2 + \cancel{m}g\Delta$$

i.e. $(a\omega)^2 = V^2 + \frac{4g\Delta}{3}$ QED

(b) $V = 2\sqrt{ag}$ $\therefore (a\omega)^2 = 4ag + \frac{4}{3}ag(1-\sqrt{2}/2)$

After impact, suppose angular velocity of cylinder is ω'.

By conservation of moment of mtm. about B

$$I_G\omega = ma^2\omega' + I_G\omega'$$

i.e. $\frac{ma^2}{2}\omega = ma^2\omega' + \frac{ma^2}{2}\omega'$

$$\therefore \omega' = \omega/3$$

∴ ke after impact $= \frac{3}{4}ma^2.\omega'^2 = \frac{3}{4}\frac{ma^2\omega^2}{9}$

$$= \frac{ma^2\omega^2}{12}$$

ie. final ke $= \frac{m}{12}\left(v^2 + \frac{4g\Delta}{3}\right)$

But initial ke $= \frac{3}{4} m v^2$ (see (a))

$\therefore \frac{\text{final}}{\text{Initial}} = \frac{\frac{1}{12}(v^2 + 4g\Delta/3)}{\frac{3}{4}v^2}$

$= \frac{4}{3} \cdot \frac{1}{12}\left(1 + \frac{4g\Delta}{3v^2}\right)$

$= \frac{1}{9}\left(1 + \frac{\cancel{4g}}{3 \cdot \cancel{4g}} \frac{\Delta}{a}\right)$

$= \frac{1}{9}\left(1 + \frac{1 - \sqrt{2}/2}{3}\right)$

$= \frac{4 - \sqrt{2}/2}{27} = .0122$

ie. .878 ie. 87.8% lost in impact

(c) ke after impact at B $= \frac{ma^2\omega^2}{12}$ from (b)

$\therefore$ In limit $\frac{ma^2\omega^2}{12} = mg\Delta$

ie. $a^2\omega^2 = 12g\Delta$

ie. $v^2 + \frac{4g\Delta}{3} = 12g\Delta$

or $v^2 = \frac{32}{3} g\Delta$

$= \frac{32}{3} g(1 - \sqrt{2}/2)a$

So $v^2 \geq 3.12ag$ or $v \geq 1.77\sqrt{ag}$

A popular question. Too many students confused *mechanical* and *kinetic* energies in part (a). Part (b) was generally well addressed. The most common issues were: (i) some students ignored the rotational kinetic energy in the initial state, and (ii) students assumed that the velocity of the disk at the impact with B was horizontal. Part (b-iii) was generally well understood.

6(a) On basis of conservation of moment of momentum about AB after impact:

$$mVd = J\omega_0 + md^2\omega_0$$

$$\therefore \quad \omega_0 = \frac{mVd}{J + md^2}$$

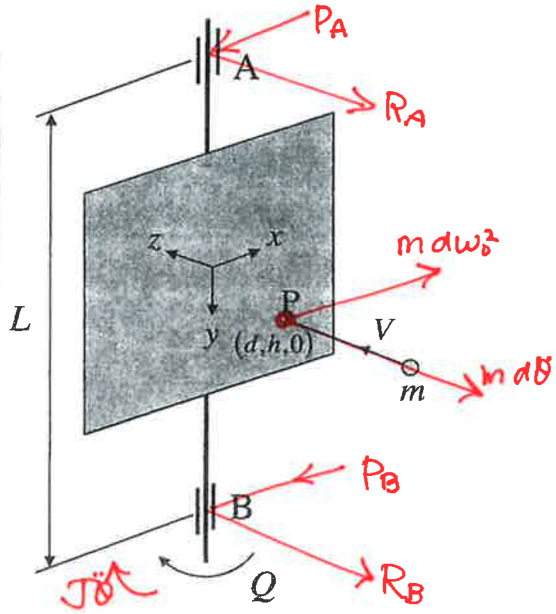
For max ω_0 $\quad \frac{d\omega_0}{dd} = \frac{(J+md^2)\cancel{m}V - \cancel{m}Vd \cdot 2md}{(J+md^2)^2} = 0$

$$\therefore \quad J + md^2 - 2md^2 = 0$$

$$md^2 = J \qquad d = \sqrt{J/m}$$

$$\omega_0 = \frac{mVd}{2J} = \frac{V}{2}\sqrt{\frac{m}{J}}$$

(b)



Let bearing reactions be P_A and R_A at A and P_B and R_B at B.

Taking moments about AB,

$$\Phi + J\ddot{\theta} + md^2\ddot{\theta} = 0$$

$$\text{i.e.} \quad \ddot{\theta} = \frac{-\Phi}{J+md^2} = \frac{-\Phi}{2J}$$

$$\therefore \quad \dot{\theta} = \omega_0 - \frac{\Phi}{2J} \cdot t$$

$\therefore$ if $\dot{\theta} \to 0$ when $t = \Delta t$

$$\frac{\Phi}{2J}\Delta t = \omega_0 = \frac{V}{2}\sqrt{\frac{m}{J}}$$

$$\therefore \quad \Delta t = \frac{\cancel{2}J}{\Phi} \cdot \frac{V}{\cancel{2}}\sqrt{\frac{m}{J}} = \frac{V}{\Phi}\sqrt{Jm}$$

To obtain P_A and R_A take moments about B

$$\begin{cases} P_A . L = m d \omega_0^2 (L/2 - h) \\ R_A . L = - m d \ddot{\theta} (L/2 - h) \end{cases}$$

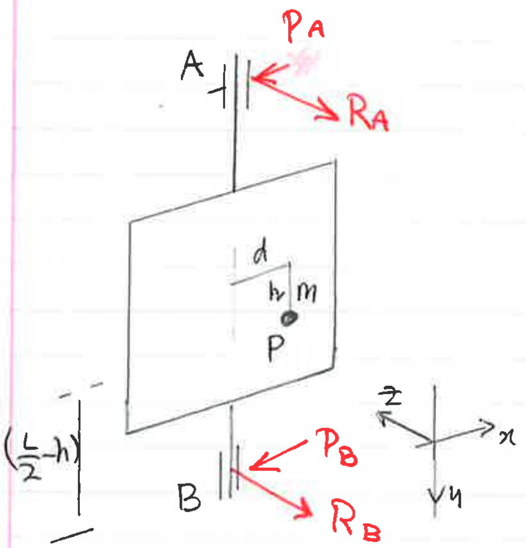
$$\therefore \begin{cases} P_A = m \sqrt{\frac{J}{m}} \frac{V^2}{4} \frac{m}{J} (1/2 - h/L) \\ R_A = + m \sqrt{\frac{J}{m}} \frac{\Phi}{2J} (1/2 - h/L) \end{cases}$$

$$\therefore \begin{cases} P_A = \frac{mV^2}{4} \sqrt{\frac{m}{J}} (1/2 - h/L) \\ R_A = \frac{\Phi}{2} \sqrt{\frac{m}{J}} (1/2 - h/L) \end{cases}$$

then either by resolving or moments about A

$$\begin{cases} P_B = \frac{mV^2}{4} \sqrt{\frac{m}{J}} (\frac{1}{2} + h/L) \\ R_B = \frac{\Phi}{2} \sqrt{\frac{m}{J}} (1/2 + h/L) \end{cases}$$

OR use $\underline{T} = \dot{\underline{H}}$



$\dot{\underline{\theta}} = - \dot{\theta} \underline{j} \qquad \dot{\theta} > 0$

from $\underline{v} = \dot{\underline{\theta}} \times \underline{BP}$

$$\underline{v} = - \dot{\theta} \underline{j} \times [-(\tfrac{L}{2} - h) \underline{j} + d \underline{i}]$$

$$= - d \dot{\theta} \underline{j} \times \underline{i} = d \dot{\theta} \underline{k}$$

So about B

$$\underline{H} = J \dot{\underline{\theta}} + \underline{BP} \times \underline{v} m$$

$$= - J \dot{\theta} \underline{j} + [-(\tfrac{L}{2} - h) \underline{j} + d \underline{i}] \times m d \dot{\theta} \underline{k}$$

$$= - J \dot{\theta} \underline{j} - (\tfrac{L}{2} - h) m d \dot{\theta} \underline{j} \times \underline{k} + m d^2 \dot{\theta} \underline{i} \times \underline{k}$$

$$= - (J + m d^2) \dot{\theta} \underline{j} - (\tfrac{L}{2} - h) m d \dot{\theta} \underline{i}$$

$$\therefore \dot{\underline{H}} = -(J+md^2)\ddot{\theta}\,\underline{j} - (J+md^2)\dot{\theta}\,\dot{\underline{j}} - \left(\tfrac{L}{2}-h\right)md\ddot{\theta}\,\underline{i} - \left(\tfrac{L}{2}-h\right)md\dot{\theta}\,\dot{\underline{i}}$$

Now $(\underline{i}, \underline{j}\, \underline{k})$ rotate with $-\dot{\theta}\,\underline{j}$

$$\therefore \quad \dot{\underline{i}} = -\dot{\theta}\,\underline{j}\times\underline{i} = \dot{\theta}\,\underline{k}\ ; \quad \dot{\underline{j}} = 0$$

$$\therefore \dot{\underline{H}} = -(J+ma^2)\ddot{\theta}\,\underline{j} - \left(\tfrac{L}{2}-h\right)md\ddot{\theta}\,\underline{i} - \left(\tfrac{L}{2}-h\right)md\dot{\theta}^2\,\underline{k}$$

But about B, $\underline{T} = -\Phi\underline{j} + (-L\underline{j})\times(-P_A\underline{i} - R_A\underline{k})$

$$\therefore \underline{T} = -\Phi\underline{j} - LP_A\underline{k} + LR_A\underline{i}$$

So equating terms $\quad \Phi = (J+md^2)\ddot{\theta} \qquad \ddot{\theta} = \dfrac{\Phi}{J+md^2}$

So if $J = md^2 \quad \ddot{\theta} = \dfrac{\Phi}{2J}$

$$LR_A = -\left(\tfrac{L}{2}-h\right)md\ddot{\theta} \Rightarrow R_A = \left(\tfrac{1}{2}-\tfrac{h}{L}\right)\frac{md\Phi}{2md^2} = \left(\tfrac{1}{2}-\tfrac{h}{L}\right)\frac{\Phi}{2d}$$

$$LP_A = \left(\tfrac{L}{2}-h\right)md\dot{\theta}^2 \Rightarrow P_A = \left(\tfrac{1}{2}-\tfrac{h}{L}\right)\frac{mdV^2}{4d^2} = \left(\tfrac{1}{2}-\tfrac{h}{L}\right)\frac{mV^2}{4d}$$

as before

An unpopular question, probably due to the combination of the concepts of impact and balancing. However, most students attempting it did fairly well. Many students used energy considerations to try to calculate the time required for the plate to stop, instead of directly considering moments acting on the axis to establish a simple dynamics equation. Only a handful of students realised that since the plate rotation is slowing down there will be a component of force normal to the plate in part (b-ii).

J A Williams
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