ENGINEERING TRIPOS PART IB

Paper 2

STRUCTURES

Crib

1.

a) i). $I = \dfrac{100\times100^3}{12} - \left(\dfrac{96\times96^3}{12}\right) = \underline{1.26\times10^6\ mm^4}$ cf. $\underline{1.33\times10^6\ mm^4}$ if assume $\sum \dfrac{bd^3}{12}$. [2]

$J = \dfrac{4A_e^2}{\oint \frac{ds}{t}} = \dfrac{4(100^2)^2}{400/2} = \underline{2.0\times10^6\ mm^4}$ [2]

(For the area enclosed 100x100 or 98x98 are both fine.)

ii) $M_o = 1.5\times10^3 \times 2.5 = 3750\ Nm$.

$\therefore \sigma_{max} = \dfrac{3750\times50\times10^{-3}}{1.33\times10^{-6}} = \underline{140.98\ MPa}$. [2]

iii). $T = 1.5\times10^3 \times 50\times10^{-3} = 75\ Nm$.

$q = \tau t = \dfrac{T}{2A_e} \Rightarrow \tau_T = \dfrac{75}{2\times(100\times10^{-3})^2\times2\times10^{-3}} = \underline{1.88\ MPa}$. [3]

$200\times2\bar{y} = 2\times50\times2\times25 + 100\times2\times50$

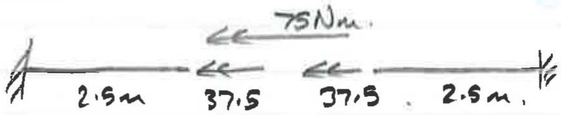
$\Rightarrow \bar{y} = 37.5mm$

$S_o = 1.5\ kN$ $\therefore q = \tau b = \dfrac{SA\bar{y}}{I} \Rightarrow \tau_S = \dfrac{1.5\times10^3\times(50+100+50)\times10^{-3}\times2\times10^{-3}\times37.5\times10^{-3}}{1.33\times10^{-6}\times2\times2\times10^{-3}}$

$= \underline{4.23\ MPa}$. [3]

$\therefore \tau_{tot} = 1.88 + 4.23 = \underline{6.11\ MPa}$.

b). i).



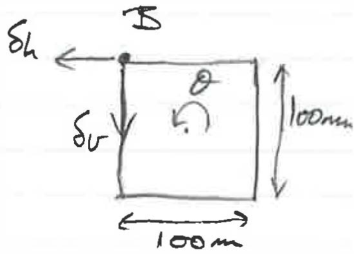
$T = 37.5\ Nm$. $T = GJ\phi$ (warping restrained) $\Rightarrow \phi = \dfrac{37.5}{2\times10^{-6}\,G}$

$= \dfrac{18.75\times10^6}{G}$ rad/m.

$\therefore \theta = \dfrac{18.75\times10^6}{G}\cdot 2.5 = \underline{\dfrac{46.88\times10^6}{G}\ rad}$. [4]

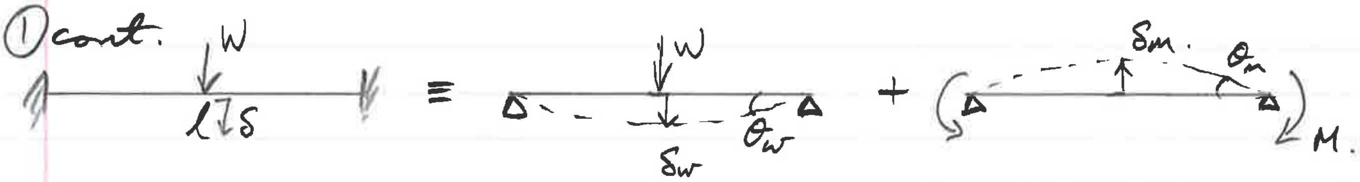
ii). $\theta = \dfrac{46.88\times10^6}{G}$ rad mid-span $\Rightarrow \delta_h = 50\times10^{-3}\times\dfrac{46.88\times10^6}{G}$

$= \underline{\dfrac{2.34\times10^6}{G}\ m}$ [1]



So due to torsion and bending $\therefore$ consider bending displacement mid-span. Using data book...

① cont.



$$\delta_w = \frac{Wl^3}{48EI} \qquad \delta_m = \frac{M(l/2)^2}{2EI} = \frac{Ml^2}{8EI}$$

$$\theta_w = \frac{Wl^2}{16EI} \qquad \theta_m = \frac{M(l/2)}{EI} = \frac{Ml}{2EI}$$

$$\theta = 0 \therefore \theta_w = \theta_m \Rightarrow \frac{Wl^{\cancel{2}}}{16\cancel{EI}} = \frac{M\cancel{l}}{2\cancel{EI}} \Rightarrow M = \frac{Wl}{8}$$

$$\therefore \delta_m = \frac{Wl}{8} \cdot \frac{l^2}{8EI} = \frac{Wl^3}{64EI}$$

$$\therefore \delta = \delta_w - \delta_m = \frac{Wl^3}{EI}\left(\frac{1}{48} - \frac{1}{64}\right) = \frac{Wl^3}{192EI}$$

$$= \frac{1.5\times10^3 \times 5^3}{192E \times 1.33\times10^{-6}}$$

$$= \frac{7.34\times10^8}{E} \text{ m.}$$

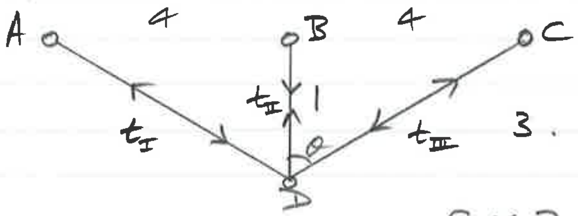
$\therefore$ Total vertical displacement, $\delta_v = \dfrac{7.34\times10^8}{E} + \dfrac{2.39\times10^6}{G}$ m. $\boxed{3}$

2

a) i). Remove one bar to render statically determinate

By Maxwell's eqn.: $s - m = b + r - D_j$

$s - 0 = 3 + 6 - 2\times 4 \Rightarrow s = 1$ [1]

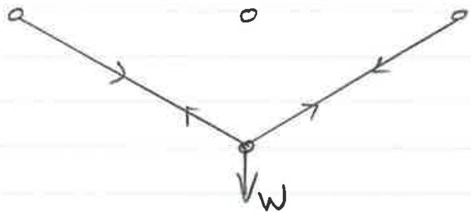


$t_I = t_{III} = t$

$2t\cos\theta = -1$, $2t\cdot\frac{3}{5} = -1 \Rightarrow t = -\frac{5}{6}$

$\therefore$ self-stress state, $\underline{s} = \begin{bmatrix} -5/6 \\ 1 \\ -5/6 \end{bmatrix}$ [3]

ii) Set $t_{II} = 0$, equ. soln then:



$2t\cos\theta = W$, $\frac{6t}{5} = W \Rightarrow t = \frac{5W}{6}$

$\therefore$ general soln $\underline{t} = \underline{t_0} + x\underline{s} = \begin{bmatrix} 5W/6 \\ 0 \\ 5W/6 \end{bmatrix} + x\begin{bmatrix} -5/6 \\ 1 \\ -5/6 \end{bmatrix}$ [2]

$\underline{e} = [F]\underline{t} + \cancel{\underline{e_0}}^{\,0} \Rightarrow \underline{e} = \frac{1}{AE}\begin{bmatrix} 5 & & \\ & 3 & \\ & & 5 \end{bmatrix}\begin{bmatrix} 5W/6 - 5x/6 \\ x \\ 5W/6 - 5x/6 \end{bmatrix} = \frac{1}{AE}\begin{bmatrix} 25W/6 - 25x/6 \\ 3x \\ 25W/6 - 25x/6 \end{bmatrix}$

V.W. $\boxed{\underline{t}\cdot\underline{e} = \underline{f}\cdot\underline{\delta}}$, with $\underline{t} = \underline{s}$ (and $\underline{f} = 0$) $\Rightarrow \underline{s}\cdot\underline{e} = 0$

$\therefore \begin{bmatrix} -5/6 \\ 1 \\ -5/6 \end{bmatrix}\cdot\frac{1}{AE}\begin{bmatrix} 25W/6 - 25x/6 \\ 3x \\ 25W/6 - 25x/6 \end{bmatrix} = 0$, $\begin{bmatrix} -5 \\ 6 \\ -5 \end{bmatrix}\cdot\begin{bmatrix} 25(W-x) \\ 18x \\ 25(W-x) \end{bmatrix} = 0$

$\Rightarrow -125(W-x) + 108x - 125(W-x) = 0$, $-250W + 358x = 0 \Rightarrow x = 0.698W$

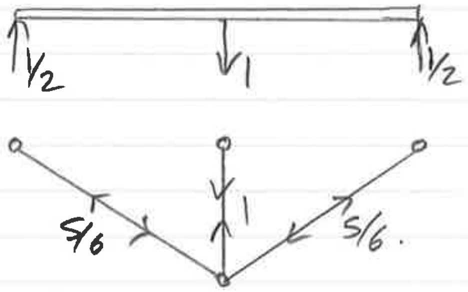
$x = \frac{125}{179}W.$

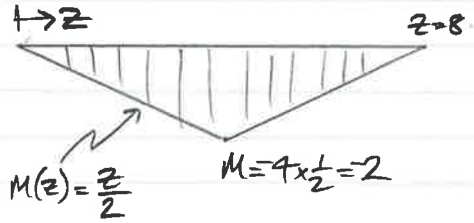
$\therefore \underline{t} = \begin{bmatrix} 0.252 \\ 0.698 \\ 0.252 \end{bmatrix} W.$ [4]

$= \frac{W}{179}\begin{bmatrix} 45 \\ 125 \\ 45 \end{bmatrix}$

(2) cont.

b). i).





[4]

ii). $t_{II} = 0$ gives equ. soln. as before, with zero B.M. in beam.

Hence general soln. $\underline{t} = \begin{bmatrix} 5W/6 \\ 0 \\ 5W/6 \end{bmatrix} + x \begin{bmatrix} -5/6 \\ 1 \\ -5/6 \end{bmatrix}$ in the truss

and $M(z)$ as above, with $|M_{max}| = 2x$.

$\Rightarrow \underline{e} = \frac{1}{AE} \begin{bmatrix} 25W/6 - 25x/6 \\ 3x \\ 25W/6 - 25x/6 \end{bmatrix}$ in the truss, and $\kappa(z) = \frac{M(z)}{EI}$ [2]

$\therefore$ V.W. equ. now $\underbrace{\underline{s} \cdot \underline{e}}_{} + \underbrace{\int M\kappa \, dz}_{} = 0$ [

ie. $\begin{bmatrix} -5/6 \\ 1 \\ -5/6 \end{bmatrix} \cdot \frac{1}{AE} \begin{bmatrix} 25/6\,(W - x) \\ 3x \\ 25/6\,(W - x) \end{bmatrix} + \cancel{2} \int_0^4 \frac{z}{\cancel{2}} \cdot \frac{xz}{2EI} \, dz = 0$

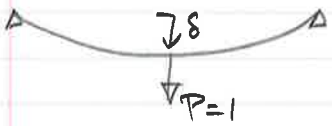
$$-\frac{125}{18AE}(W - x) + \frac{3x}{AE} + \frac{x}{2EI} \int_0^4 z^2 \, dz = 0 .$$

$$-\frac{125W}{18A\cancel{E}} + \frac{179x}{18A\cancel{E}} + \frac{64x}{6\cancel{E}I} = 0 , \quad -125WI + 179xI + 192xA = 0 .$$

$$x(192A + 179I) = 125IW \quad \Rightarrow \quad x = \frac{125IW}{192A + 179I}$$

$$\therefore M_{max} = 2x = \frac{250IW}{192A + 179I}$$ [4]

Alternatively, using data book, state of self stress $[-5/6 \;\; 1 \;\; -5/6]^T$ in truss corresponds to:



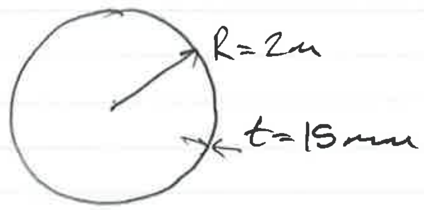
$\delta = \frac{PL^3}{48EI} = \frac{32}{3EI}$ $(L = 8)$.

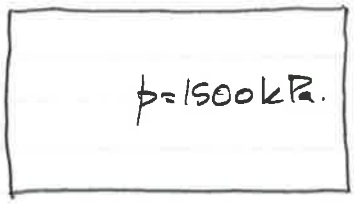
Hence V.W. equ. becomes $\underline{s} \cdot \underline{e} + 1 . \frac{32x}{3EI} = 0$ $\left(P\delta = \int M\kappa\right)$

which leads to the same result.

3.

a).





$E = 70\,GPa$, $\nu = 0.33$.

i) $\sigma_h = \frac{pR}{t} = \frac{1500\times10^3 \times 2}{15\times10^{-3}} = \underline{200\,MPa}$. [2]

$\sigma_\ell = \frac{pR}{2t} = \underline{100\,MPa}$. [2]

ii). $\varepsilon_h = \frac{1}{E}(\sigma_h - \nu\sigma_\ell - \nu\cancel{\sigma_t}^{\,0}) = \frac{1}{70\times10^9}(200\times10^6 - 0.33\times100\times10^6) = \underline{2.39\times10^{-3}}$ [2]

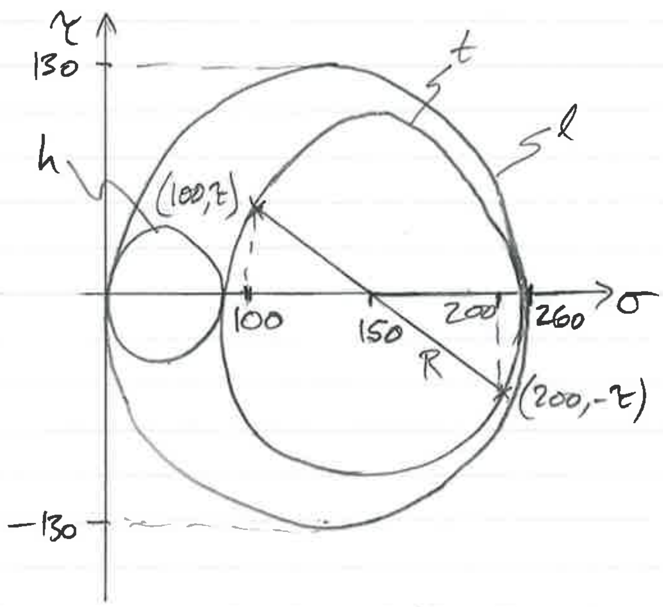
$\varepsilon_\ell = \frac{1}{E}(\sigma_\ell - \nu\sigma_h - \nu\cancel{\sigma_t}^{\,0}) = \frac{1}{70\times10^9}(100\times10^6 - 0.33\times200\times10^5) = \underline{0.49\times10^{-3}}$ [2]

$\varepsilon_t = \frac{1}{E}(\cancel{\sigma_t}^{\,0} - \nu\sigma_h - \nu\sigma_\ell) = \frac{1}{70\times10^9}(-0.33\times200\times10^6 - 0.33\times100\times10^6) = \underline{-1.41\times10^{-3}}$ [2]

b). $q = \tau t = \frac{T}{2A_e} \Rightarrow \tau = \frac{T}{2t.\pi R^2}$, $T = 2\pi R^2 t\tau$. [2]

$Y = 260\,MPa$

i). $p = 1500\,kPa \Rightarrow \sigma_h = 200\,MPa$, $\sigma_\ell = 100\,MPa$, $\sigma_t = 0$



Tresca $\Rightarrow \tau_{max} = \frac{Y}{2} = 130\,MPa$

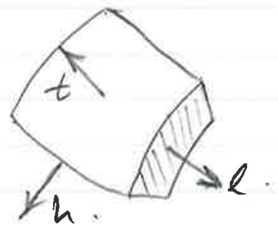
$R = 260 - 150 = 110$

$\therefore (200-150)^2 + \tau^2 = 110^2$

$\Rightarrow \tau = 98\,MPa$. [4].

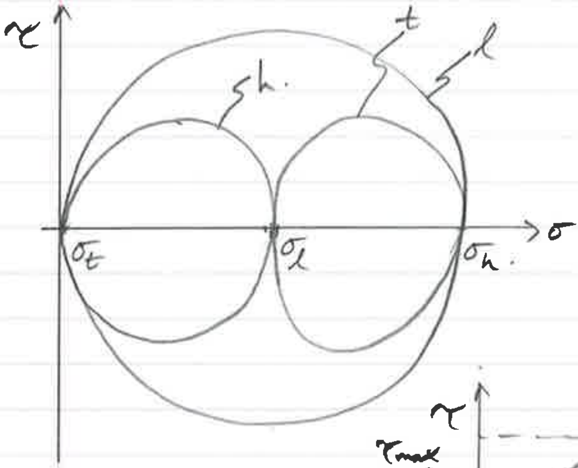
$\therefore T = 2\pi\times2^2\times15\times10^{-3}\times98\times10^6$

$= \underline{36.9\,MNm.}$

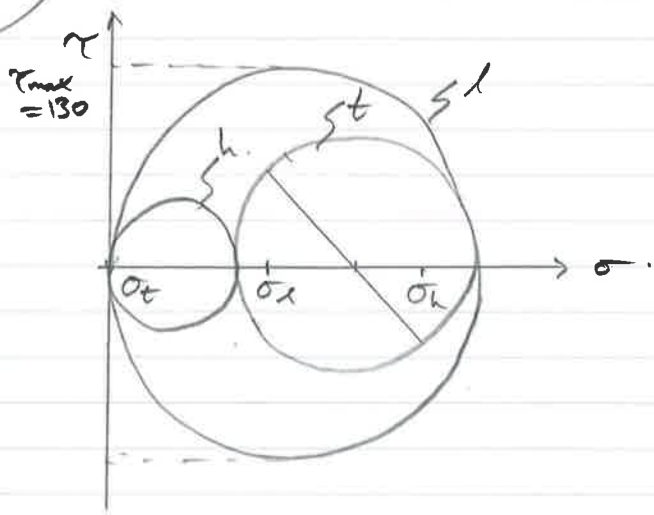


③ cont.

b) ii) $p = 750\,kPa \Rightarrow \sigma_h = 100\,MPa, \ \sigma_l = 50\,MPa, \ \sigma_t = 0.$

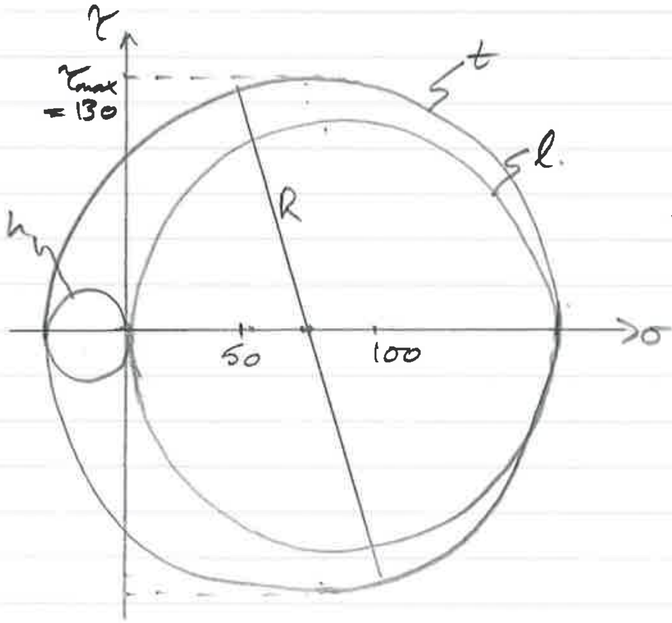


T=0.



T>0.

N.B. The evolution of the circle with increasing T was not required in the question.



T>>0.

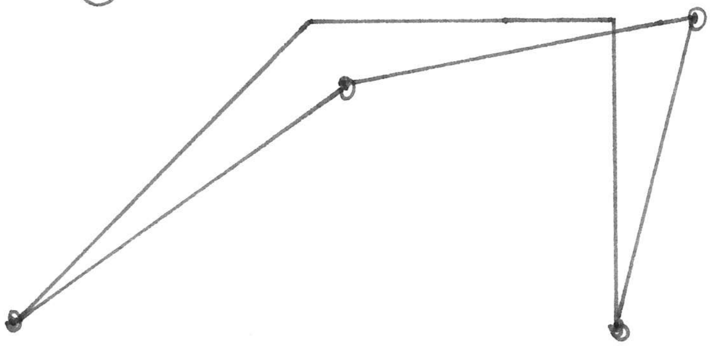
$R = 130 \ \therefore \ \tau = \sqrt{130^2 - 25^2}$
$= 127.6$ MPa.

$\Rightarrow$ $\underline{T = 48.1 \text{ MNm}}$. [4]

Qu 4)

a)

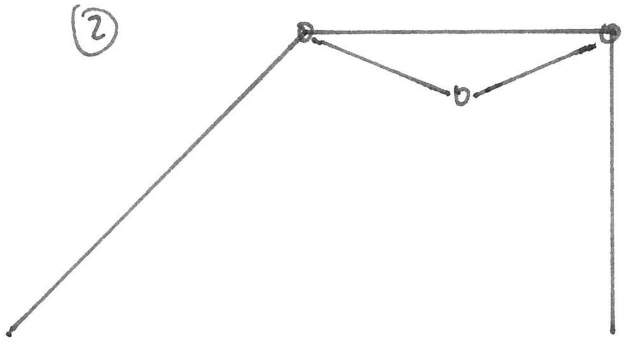
1



Hinges: A, B, D, E

[2]

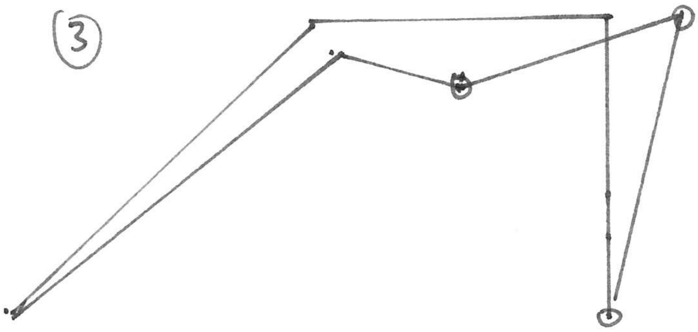
2



Hinges : B, C, D

[2]

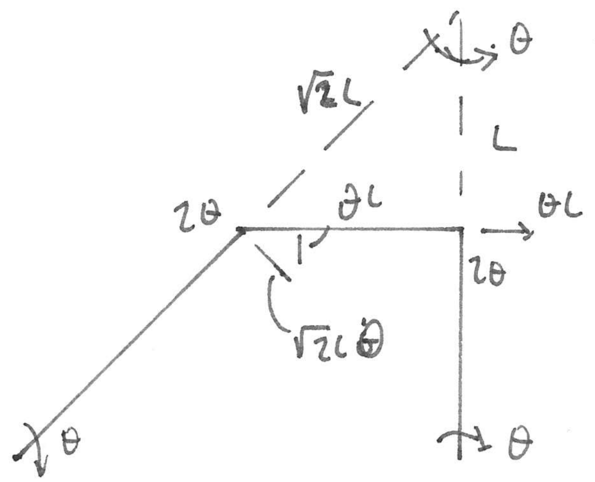
3



Hinges: C, D, E and A

[2]

Mechanism 1:

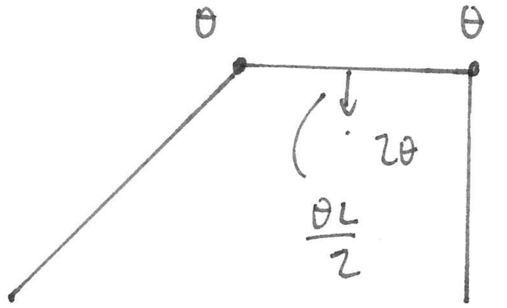


External work: $H \cdot \theta L + V \cdot \frac{\theta L}{2}$

Internal work: $M_p \theta + 2M_p\theta + 3\frac{M_p}{2}\theta = \frac{9}{2}M_p\theta$

$$\Rightarrow HL + \frac{VL}{2} = \frac{9}{2}M_p \qquad [3]$$

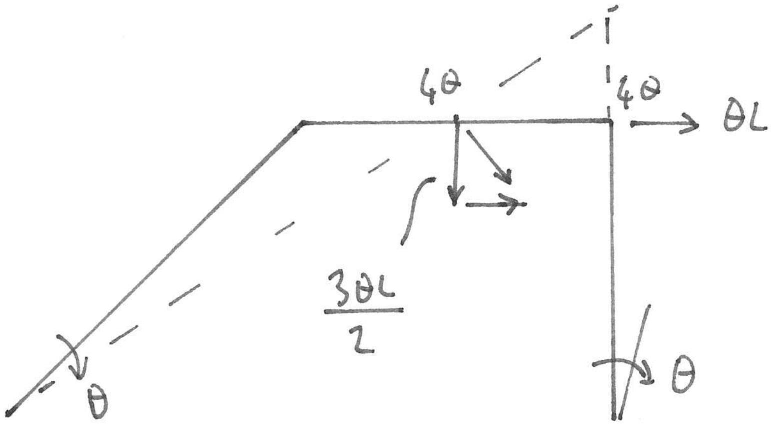
Mechanism 2:



$$V \cdot \frac{\theta L}{2} = 3M_p\theta + \frac{M_p}{2}\theta$$

$$\Rightarrow VL = 7M_p \qquad [2]$$

Mechanism 3:



External work: $H \cdot \theta L + V \frac{3\theta}{2}$

Internal work: $M_p \theta + M_p 4\theta + \frac{M_p}{2} 4\theta + \frac{M_p \theta}{2}$

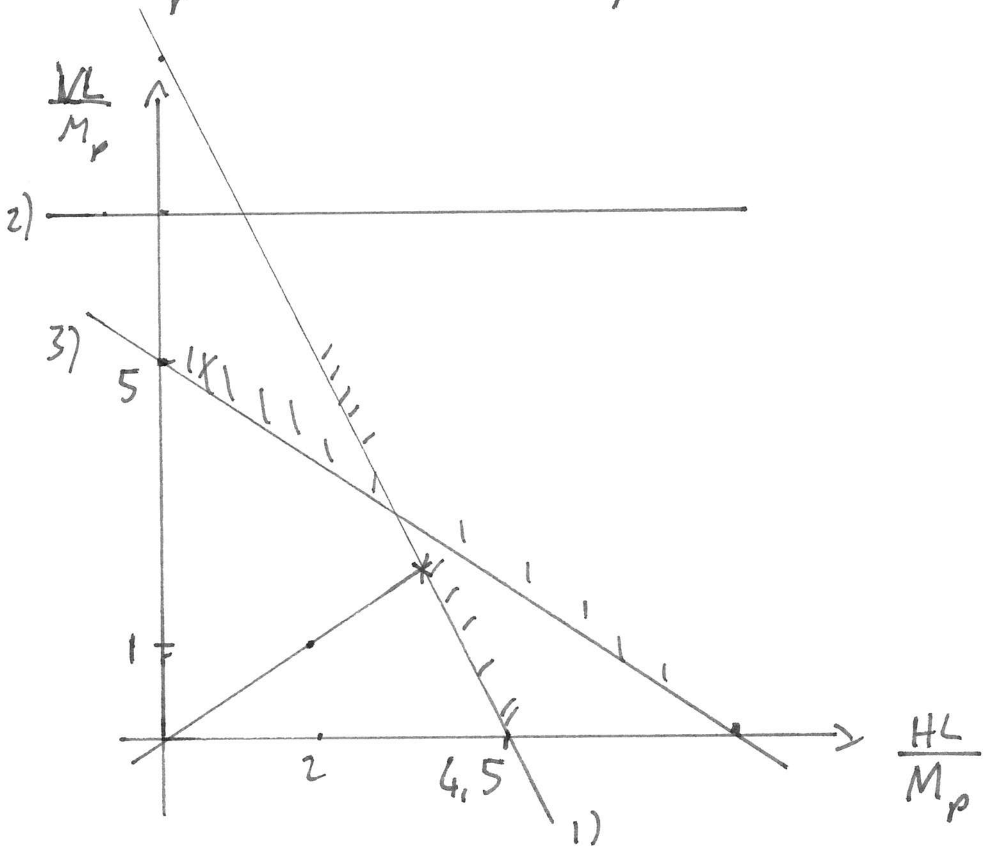
$$HL + \frac{3}{2} VL = \frac{15}{2} M_p$$ [3]

c)

1) $\frac{VL}{M_p} = 9 - \frac{2HL}{M_p}$

2) $\frac{VL}{M_p} = 7$

3) $\frac{VL}{M_p} = 5 - \frac{2}{3} \frac{HL}{M_p}$



[6]

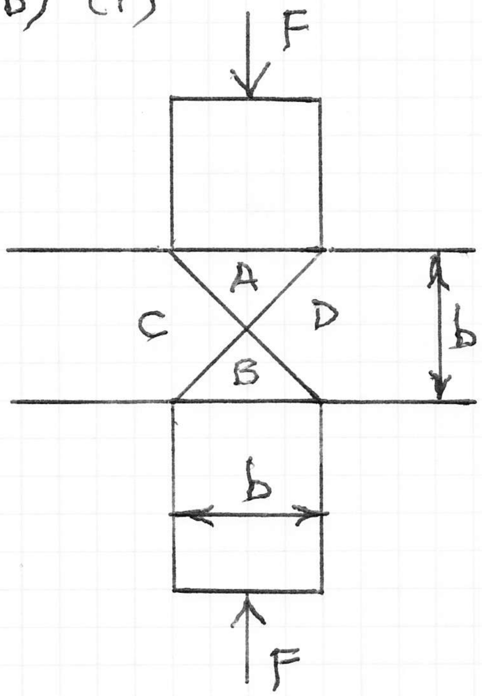
$$g - \frac{2HL}{M_p} = \frac{HL}{2M_p} \implies \frac{HL}{M_p} = 3{,}6$$

$$\frac{HL}{M_p} = 2\lambda = 3{,}6 \implies \lambda = 1{,}8$$

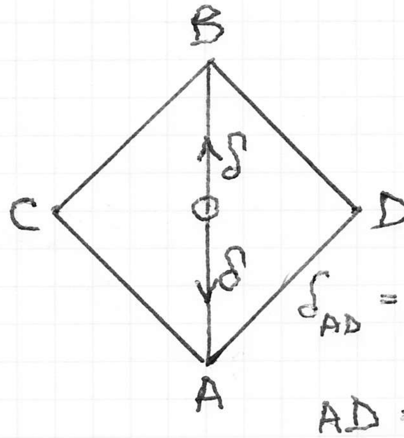
Qu 5)

a) An upper bound collapse calculation requires a kinematically admissible mechanism that respects <u>compatibility</u>. The collapse load is then estimated from a balance of work and energy, for which the <u>material laws</u> of plastic resistance are required. It is not necessary, and usually is not possible, to demonstrate equilibrium at every point for an upper bound calculation. [3]

b) (i)



displacement diagram



$\delta_{AD} = \delta\sqrt{2}$

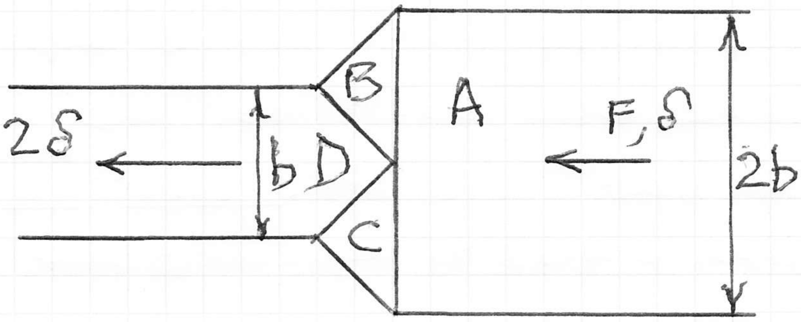
$AD = b/\sqrt{2}$

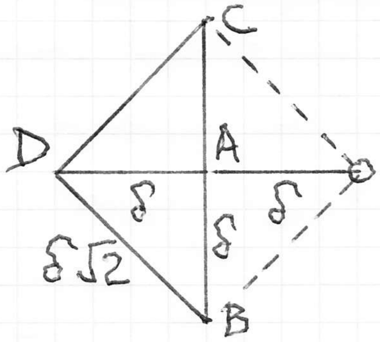
External work : $2F\delta$

Internal work : $4(\delta\sqrt{2})(b/\sqrt{2})k$

$\Longrightarrow$ $\underline{F = 2bk}$ [5]

(ii)





$AB = b$

$\delta_{AB} = \delta$

$BD = b/\sqrt{2}$

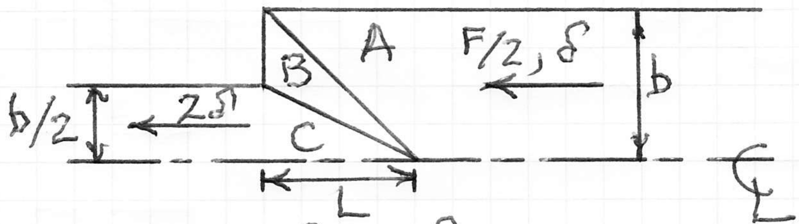
$\delta_{BD} = \delta\sqrt{2}$

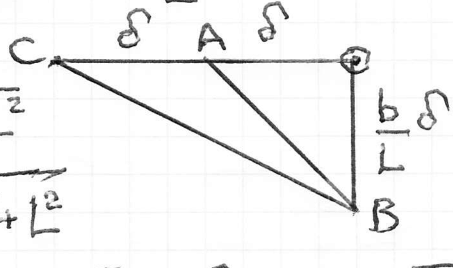
External work : $F\delta$

Internal work : $2\left[b\delta + \frac{b}{\sqrt{2}}\delta\sqrt{2}\right]k$

$\Longrightarrow \underline{F = 4bk}$ [6]

(iii)





$AB = \sqrt{b^2 + L^2}$

$BC = \sqrt{\left(\frac{b}{2}\right)^2 + L^2}$

$\delta_{AB} = \delta\sqrt{1 + \frac{b^2}{L^2}}$ $\delta_{BC} = \delta\sqrt{4 + \frac{b^2}{L^2}}$

External work : $F\delta/2$

Internal work : $\left[\sqrt{b^2+L^2}\sqrt{1+\frac{b^2}{L^2}} + \sqrt{\frac{b^2}{4}+L^2}\sqrt{4+\frac{b^2}{L^2}}\right]k\delta$

$\Longrightarrow F = \frac{2k}{L}\left[L^2 + b^2 + \frac{1}{2}\left(4L^2 + b^2\right)\right]$

$$\text{So } F = 6kL\left(1 + \frac{b^2}{2L^2}\right)$$

- Optimal F is F_{min} at $\frac{dF}{dL} = 0$

Now $\frac{F}{6k} = L + \frac{b^2}{2L}$

So $\frac{dF}{dL} \propto 1 - \frac{b^2}{2L^2}$

$\rightarrow 0$ when $L = b/\sqrt{2}$

Furthermore $\frac{d^2F}{dL^2} > 0$ so F_{min}

$$F_{min} = 6k\frac{b}{\sqrt{2}}\left(1 + \frac{1}{2}\cdot 2\right)$$

$$F_{min} = 6\sqrt{2}\,kb$$

[6]

Qu 6)

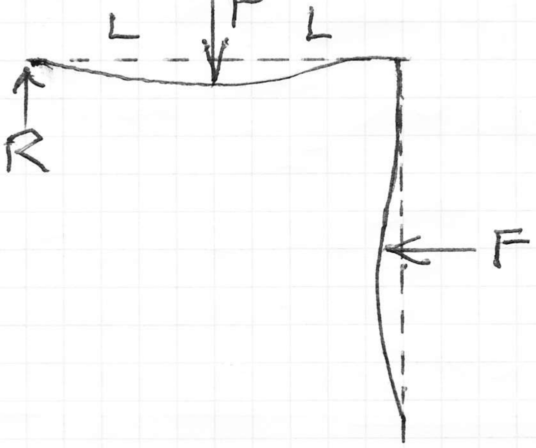
a) $I = \dfrac{100.100^3}{12} - \dfrac{80.80^3}{12} = 4.92.10^6 \text{ mm}^4$

$Z_e = \dfrac{4.92\ 10^6}{50} = 98.4.10^3 \text{ mm}^3$

$Z_p = 2(100.10.45 + 2.40.10.20)$

$= 122.10^3 \text{ mm}^3$ [7]

b) By symmetry, the knee joint can not rotate:



$L = 2m$

Find R to give $\delta_R = 0$

Due to R : $\delta_R \uparrow = \dfrac{R(2L)^3}{3EI} = \dfrac{8}{3}\dfrac{RL^3}{EI}$

Due to F : $\delta_F = \dfrac{FL^3}{3EI}$

$\theta_F = \dfrac{FL^2}{2EI}$

So $\delta_R \downarrow = \dfrac{FL^3}{3EI} + \dfrac{FL^3}{2EI} = \dfrac{5}{6}\dfrac{FL^3}{EI}$

So for $\delta_R = 0$, $\dfrac{8}{3}R = \dfrac{5}{6}F$ so $\dfrac{R}{F} = \dfrac{5}{16}$

Now find bending moments.

$$M_F = \frac{5}{16} FL$$

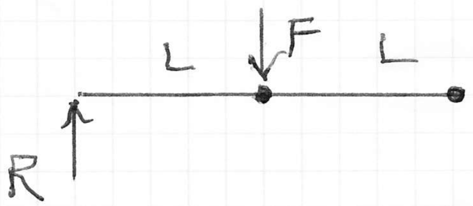
$$M_{knee} = \frac{5}{16} F\,2L - FL = -\frac{6}{16} FL$$

So the maximum bending stress is encountered at the knee joint, where first yield will occur.

$$\frac{3}{8} F.2 = 98.4 . 10^{-6} . 355 . 10^{6}$$

$$\therefore \quad \underline{F = 46.6 \text{ kN}} \qquad [7]$$

c)



$L = 2m$

Now specify plastic hinges carrying M_p at the point of loading, and at the knee. Find the new value of R for equilibrium.

$$R.L = M_p$$

$$R.2L - F.L = -M_p$$

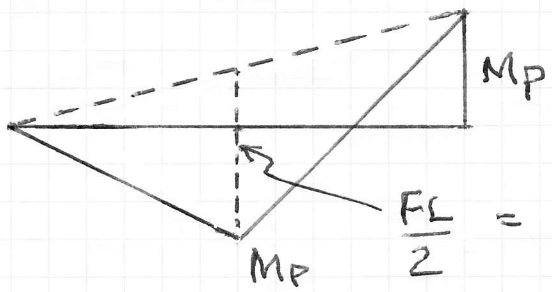
$$\therefore R = F/3$$

$$\text{and } M_p = FL/3 = \frac{2}{3}F$$

$$\text{So} \quad F = \frac{3}{2} . 122 . 10^{-6} . 355 . 10^{6}$$

$$\underline{F = 65.0 \text{ kN}} \qquad [6]$$

NB: A more elegant solution can be obtained by considering two bending moment cases, a triangle with $\frac{FL}{2}$ due to the central load F on a simply supported beam of length $2L$, and a particular self-stressing solution with M_p at the knee.



$$\frac{FL}{2} = \frac{M_p}{2} + M_p$$

$$\text{so } F = \frac{3}{2}\frac{M_p}{2} \quad \text{etc}$$

This clearly displays strict Lower Bound principles.

ENGINEERING TRIPOS PART IB 2013

EXAMINER'S REPORT: PAPER 2 – STRUCTURES

The examination was taken by 299 candidates. The average mark was 64.3% (62.0% from 616 attempts in Section A and 66.8% from 580 attempts in Section B).

Question 1 Attempts 165, average 11.8/20, maximum 20/20, 5 candidates scored 19 or 20.

Part (a) was generally answered well, although units were often missed off the section constants, especially the torsion constant. Occasionally, rotation per unit length was not converted to rotation. A few candidates incorrectly attempted to calculate the torsion constant from the second moments of area by applying the perpendicular axis theorem. In general, the torsion analysis was not as well done as the bending, with some confusion between shear flow and shear stress and the appropriate thickness to use. In Part (b), the specification of E and G in the question (rather than numerical values) led to unnecessary confusion over units. Many struggled to calculate the central bending deflection of the built-in beam, despite reference to the data book. A few candidates confused the rotation of the cross-section in torsion with rotation due to bending.

Question 2 Attempts186, average 10.8/20, maximum 20/20, 3 candidates scored 19 or 20.

Part (a) was answered well, although a few candidates leapt in assuming a 45-degree truss. Most struggled with Part (b), where virtual work was applied poorly.

Question 3 Attempts 265, average 13.9/20, maximum 20/20, 24 candidates scored 19 or 20.

Part (a) was answered well, although a few candidates attempted to rearrange the stress-strain relations, rather than using the strain-stress relations directly, and one or two reported strain in millimetres. In Part (b), all used the correct torsion-shear relation, and all adopted to draw Mohr's Circles. These were usually successful for (i) (with some measuring off scale diagrams) but few obtained correct results for (ii).

Question 4 Attempts 288, average 13.8/20, maximum 20/20, 26 candidates scored 19 or 20.

In this question on the plastic collapse of an unsymmetrical portal frame, candidates found it easy to place plastic hinges so as to form alternative mechanisms but many failed to recognise downward displacement of the vertical force during sway. The beam and the combined mechanisms were clearly familiar to them and well-handled. Almost all could create an interaction diagram but a significant proportion could not see how to use it to solve the question that was posed, getting tangled over the use of the load factor λ.

Question 5 Attempts 171, average 14.2/20, maximum 20/20, 25 candidates scored 19 or 20.

This question on plastic metal-forming attracted the highest marks. They would have been a few marks higher if more candidates had learnt how to answer the first, supposedly easy, part (a) on the application of fundamental structural principles in upper bound analysis. And they would have been even higher still if candidates had taken the time to label all zones of deformation, state clearly where symmetry was being invoked, and produce a proper displacement diagram. Nevertheless, the capacity of candidates to perform upper bound plastic calculations was clearly very good.

Question 6 Attempts 121, average 11.1/20, maximum 20/20, 16 candidates scored 19 or 20.

This question combined the calculation of elastic and plastic section properties with their application in elastic and lower bound plastic calculations for first yield and then collapse of a redundant frame. Although the average mark of the attempts fell just within the required range, the response to this question was rather disappointing and difficult to assess. Credit given for the calculation of section moduli rescued the majority from failure, though there was a proportion of highly inefficient calculations with attendant slips. Although 11 candidates scored full marks on the question, a more typical response in part (b) was to write down a variety of elastic displacement components and bending moment diagram fragments, apparently at random, in an attempt to glean credit. It was particularly depressing that candidates in general could not simply state that symmetry required the knee not to rotate. Nevertheless, the use of a “particular solution” (or a reactant line) did permit many to solve for plastic collapse in part (c).

Examiners and Assessors: Setting – Dr Fehmi Cirak, Marking – Dr James Talbot (Q1, 2 and 3); Prof Malcolm Bolton (Q4, 5 and 6).

Malcolm Bolton 19 June 2013