Solutions: IB Paper 6, 2013

ENGINEERING TRIPOS   PART IB

Thursday 6 June 2013   2 to 4

Paper 6

INFORMATION ENGINEERING

Version: jl05

1 (a) We use sinewave testing to find the gain and phase shift of the system at a range of frequencies.

The system *must* be stable, in order that a steady-state can be reached. The system should also be linear and time-invariant (at least approximately). [5]

(b) (i) Extending the low frequency $+20$ dB slope with a straight line intersects the 0 dB line at around 0.35 rad s^{-1}. This corresponds to the *as* term and suggests that $20\log_{10}(0.35\,a) = 0$, so that

$$a \simeq 1/3.5 = 2.9$$

Or, one could also use the gain at 0.01: for low ω, $|G(j\omega) \approx aj\omega$. Therefore

$$20\log|G(j0.01)| = 20\log_{10}(a0.01) = -30$$

$$\implies a \simeq 3.1623$$

The magnitude plot suggests a corner frequency of a pole around 0.1. rad s^{-1}. This agrees with the phase having dropped to +45 degrees at 0.1. rad s^{-1}. Which suggests (with $1 + cs \propto 1 + j$)

$$c = 10$$

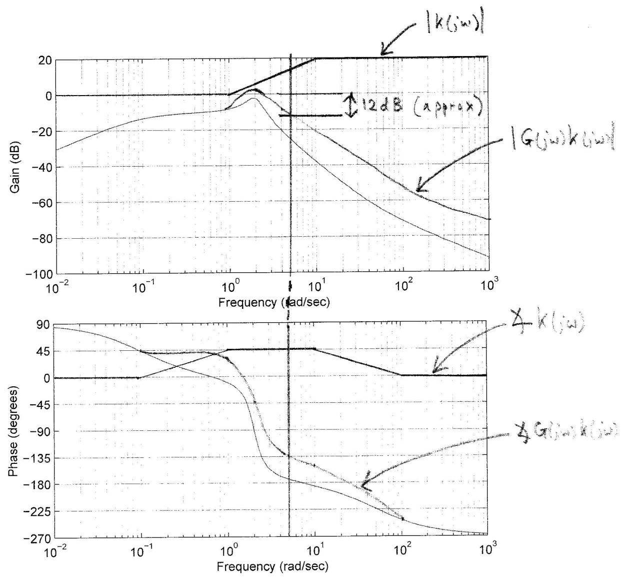
Then we have the complex poles: from the phase intersection at -90 degrees or the peak of the gain, we can read off that

$$w_n = 2$$

We then consider the zeros: look at intersection of phase plot at -225 degrees, which occurs at $\omega = 50$, so that $b = -1/50$. [7]

(ii) The Bode diagram of the compensated system is shown below:

(b) (ii) and (iii)



Copy of Fig. 1. This should be annotated with your constructions and handed in with your answer to question 1.

[5]

(iii) At $\omega = 4$ phase margin is 45 degrees. Need to add around 17 dB, which will give a k of approximately 7. See figure below.

Computer generated plot for (b)(iii)

ENGINEERING TRIPOS PART IB
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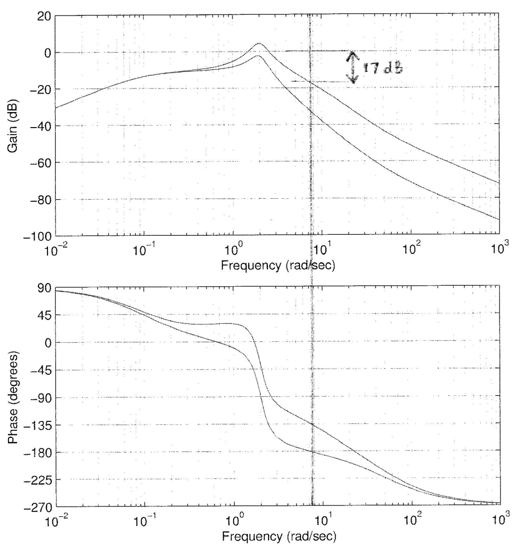


Copy of Fig. This should be annotated with your constructions and handed in with your answer to question

[3]

2 (a) Bookwork [3]

(b) (i) Using the 'virtual earth' assumption and letting i be the current into the upper output terminal:

$$i = \frac{v_0}{R_1} + C_1 \dot{v}_0 \quad \text{and} \quad i = -\frac{v_i}{R_2}$$

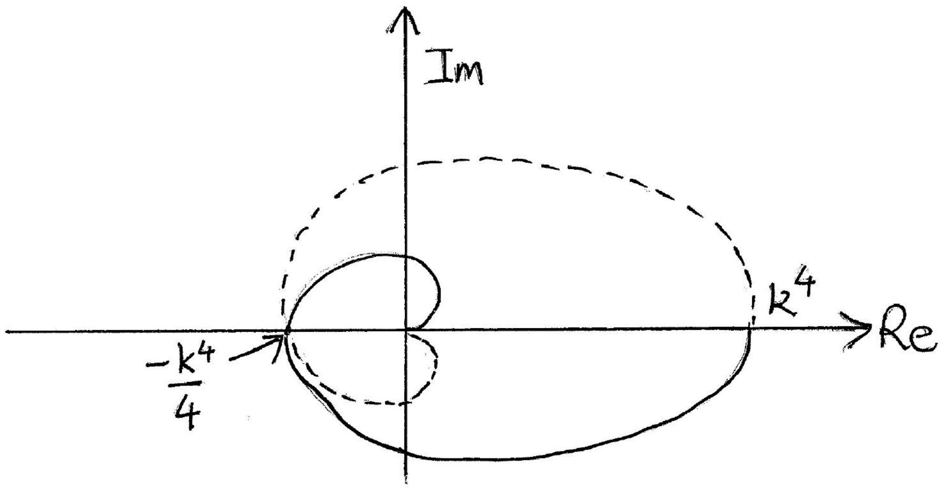
Equating and taking Laplace transforms (with zero initial conditions)

$$\left(\frac{1}{R_1}+C_1 s\right)\bar{v}_0(s)=-\frac{1}{R_2}\bar{v}_i(s)$$

$$\implies \quad \frac{\bar{v}_0(s)}{\bar{v}_i(s)}=-\frac{1}{R_2}\frac{1}{\frac{1}{R_1}+C_1 s}=-\frac{R_1}{R_2}\frac{1}{R_1C_1s+1}$$

[5]

(ii) At $s=j/T$ transfer function is given by $\frac{k^4}{(1+j)^4}=\frac{k^4}{(2j)^2}=-\frac{k^4}{4}$



[7]

(iii) Negative feedback: stable

$\Leftrightarrow \quad 0<k^4/4<1 \quad \Leftrightarrow \quad 0<k<\sqrt{2}$

Positive feedback: stable

$\Leftrightarrow \quad 0<k^4<1 \quad \Leftrightarrow \quad 0<k<1$ [5]

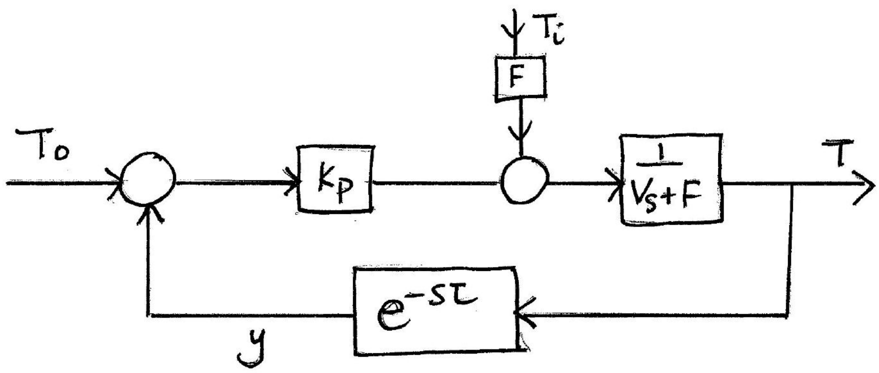
3 (a)

$$Vs\bar{T}=F(\bar{T}_i-T)+\frac{1}{\rho c_\rho}\bar{Q}_{in}$$

$$\implies \quad \bar{T}=\frac{1}{Vs+F}\left(F\bar{T}_i+\frac{1}{\rho c_\rho}\bar{Q}_{in}\right)$$

and

$$\frac{1}{\rho c_\rho}\bar{Q}_{in}=k_p(\bar{T}_0-\mathrm{e}^{-s\tau}\bar{T})$$



[5]

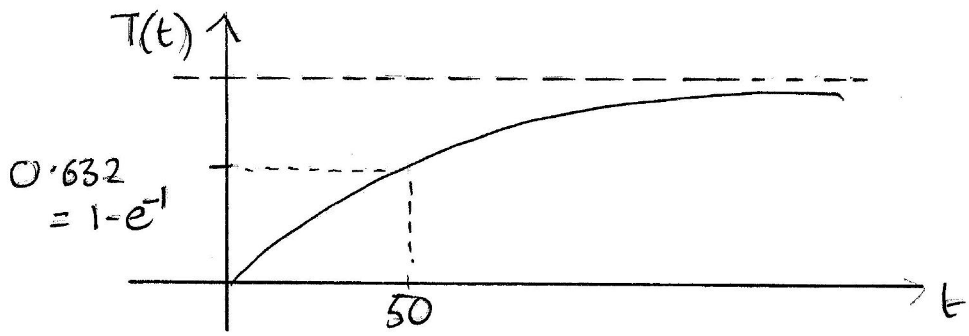
(b)

$$\bar{T} = \frac{1}{Vs+F}\left(F\bar{T}_i + k_p(\bar{T}_0 - \mathrm{e}^{-s\tau}\bar{T})\right)$$

$$\implies \quad \bar{T} = \frac{k_p}{Vs+F+k_p\mathrm{e}^{-s\tau}}\bar{T}_0 + \frac{F}{Vs+F+k_p\mathrm{e}^{-s\tau}}\bar{T}_i$$

[5]

(c) if $k_p = 0$, $\bar{T} = \frac{F}{Vs+F}\bar{T}_i = \frac{1}{50s+1}\bar{T}_i$
So $\bar{T}_i = 1/s \implies \bar{T} = \frac{1}{s} - \frac{1}{s+0.02}$
$\implies \quad T(t) = 1 - \mathrm{e}^{-0.02t} \quad t > 0$
This is sketched below:



[5]

(d) Steady state amplitude of oscillation in T is

$$|H_2(j)| = \frac{0.1}{|5j + 0.1 + 0.05\mathrm{e}^{-2j}|} = 0.0202$$

[5]

SECTION B

4 (a) Part (i) is standard bookwork, part (ii) uses (i) to derive the result:

(i) From notes:
The convolution of two functions $f(t)$ and $g(t)$ is written as $h(t) = f * g$ and defined by

$$h(t) = f * g = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau.$$

Taking the FT of the convolution gives

$$H(\omega) = \int_{t=-\infty}^{\infty} \left\{ \int_{\tau=-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau \right\} \mathrm{e}^{-j\omega t}\,dt.$$

Change the order of integration and substitute $u = t - \tau$, [$\Longrightarrow$ $dt = du$ and no change of limits]:

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} f(\tau)\left[\int_{u=-\infty}^{\infty} g(u)\mathrm{e}^{-j\omega(u+\tau)}du\right]d\tau \\ &= \left\{\int_{-\infty}^{\infty} f(\tau)\mathrm{e}^{-j\omega\tau}d\tau\right\}\left\{\int_{-\infty}^{\infty} g(u)\mathrm{e}^{-j\omega u}du\right\} \\ &= F(\omega)G(\omega). \end{aligned}$$

[3]

(ii)

$$FT(R_{fg}) = \int_{t=-\infty}^{\infty} \left\{ \int_{\tau=-\infty}^{\infty} f(\tau)g(\tau+t)\,d\tau \right\} \mathrm{e}^{-j\omega t}\,dt$$

as in part (i), substitute $u = \tau + t$ to give

$$\begin{aligned} FT(R_{fg}) &= \int_{-\infty}^{\infty} f(\tau)\left[\int_{u=-\infty}^{\infty} g(u)\mathrm{e}^{-j\omega(u-\tau)}du\right]d\tau \\ &= \left\{\int_{-\infty}^{\infty} f(\tau)\mathrm{e}^{j\omega\tau}d\tau\right\}\left\{\int_{-\infty}^{\infty} g(u)\mathrm{e}^{-j\omega u}du\right\} \\ &= F(-\omega)G(\omega). \end{aligned}$$

[Also OK to put $F^*(\omega)G(\omega)$]. [2]

(iii) Because $FT(f(t)*g(t)) = F(\omega)G(\omega)$, we know that $FT(g(t)*f(t)) = G(\omega)F(\omega) = F(\omega)G(\omega) = FT(f(t)*g(t))$, so without resorting to the integral form we know that $f*g = g*f$, ie convolution is commutative. [Also fine to write down the integrals and show this].

Similarly $FT(R_{fg}(t)) = F(-\omega)G(\omega)$, so that $FT(R_{gf}(t)) = G(-\omega)F(\omega) \neq FT(R_{fg})$, so $R_{fg}(t) \neq R_{gf}(t)$, ie cross-correlation is not commutative. Since $f(-t)$ will have a FT of $F(-\omega)$, we see that

$$FT(R_{gf}(t)) = FT(R_{fg}(-t)), \quad \text{so that} \quad R_{gf}(t) = R_{fg}(-t)$$

[Note, also OK to do this via the original integral – from which it is obvious.]

[2]

(b) (i) Again, this is bookwork. From the notes:

Inverse FT gives:

$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} q(\omega)e^{j\omega t}\,d\omega$$

Now, if we replace t in the above by $-\omega'$ and ω by t' (ω is simply an integration variable), we have

$$p(-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} q(t')e^{-j\omega' t'}\,dt'$$

Rearranging this gives:

$$2\pi p(-\omega') = \int_{-\infty}^{\infty} q(t')e^{-j\omega' t'}\,dt' = FT(q(t'))$$

where the RHS is the exact form of the FT of $q(t')$.

Therefore, if we have one Fourier transform pair:

$$p(t) \overset{FT}{\longleftrightarrow} q(\omega)$$

then we automatically have (without any integration) the *dual* Fourier transform pair:

$$q(t) \overset{FT}{\longleftrightarrow} 2\pi p(-\omega)$$

[4]

(ii) From the databook we know that a rectangular pulse, $s(t)$, of width b and height c, centred on the origin, has a FT given by $S(\omega) = bc\ \text{sinc}\frac{\omega b}{2}$.

From duality we know then that $S(t) \leftrightarrow 2\pi s(-\omega)$.

If we take $b = 2$ and $c = 1/2$, we have that the FT of a pulse, $s(t)$, centred on the origin with width 2 and height 1/2 is $S(\omega) = \ \text{sinc}\omega$.

Therefore, by duality, the FT of $\text{sinc}\, t$ is given by $2\pi s(-\omega)$, ie the FT of $\text{sinc} t$ is a rectangular pulse centred on the origin, of height π and width 2. [4]

(iii) Recall that Parseval's theorem tells us that

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Therefore

$$\int_{-\infty}^{\infty} \text{sinc}^2 t dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |s(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-1}^{1} \pi^2 d\omega = \frac{\pi^2}{2\pi} [\omega]_{-1}^{1} = \pi$$

[5]

5 (a) The ideal frequency response for perfect reconstruction is:

$$H_r(\omega) = \begin{cases} T, & -\omega_{max} < \omega < +\omega_{max} \\ 0 & \text{otherwise} \end{cases}$$

where $T = 2\pi/\omega_s$,(where $\omega_s = 2\pi f_s$), since this filter is designed to encompass the complete FT of the original signal as we know we are sampling at or above the Nyquist frequency.

The impulse response of the filter is the inverse Fourier transform of $H_r(\omega)$. We know that the inverse Fourier transform of a rectangular pulse is a sinc function, and it can easily be shown (use the databook) that the IFT of $H_r(\omega)$ is

$$h_r(t) = \frac{\omega_{max} T}{\pi} \text{sinc}(\omega_{max} t)$$

If we are sampling **at** the Nyquist frequency, $\omega_{max} = \omega_s/2$, the above becomes

$$h_r(t) = \text{sinc}(\omega_s t/2)$$

Since multiplication in the frequency domain implies convolution in the time domain, we are therefore able to completely recover the signal via the following convolution

$$f(t) = h_r(t) * f_s(t)$$

By substituting $f_s(t) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t-nT)$ and performing the convolution, the above equation becomes

$$f(t) = \int_{-\infty}^{\infty} f_s(\tau) h_r(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(nT)\delta(\tau - nT) h_r(t-\tau) d\tau$$

Rearranging integral and summation signs gives

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT) \left\{ \int_{-\infty}^{\infty} \delta(\tau - nT) h_r(t-\tau) d\tau \right\}$$

which can be evaluated as

$$f(t) = \sum_{-\infty}^{\infty} f(nT) h_r(t - nT) \tag{1}$$

Which can be viewed as interpolation of the samples with a sinc function. Note that the above holds as a reconstruction filter even if we do not sample exactly at the Nyquist frequency (as long as we sample at or above the Nyquist frequency). Full marks could be gained without explicitly putting down equation 1, but there had to be some reference to convolution with the reconstruction filter. [6]

(b) If we sample at 0.25s intervals we will have $N = 8$ samples at $t = 0, 0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75$. Thus our discrete set of samples is

$$y_n = \{1, 1, 1, 1, 0, 0, 0, 0\}$$

Since our DFT coefficients are given by

$$Y_k = \sum_{n=0}^{N-1} y_n \mathrm{e}^{-jkn\frac{2\pi}{N}}$$

we can rewrite this as

$$Y_k = \sum_{n=0}^{3} y_n \mathrm{e}^{-jkn\frac{\pi}{4}}$$

since only the first 4 components of y_n are non-zero, and $N = 8$.

We can then evaluate the first 3 DFT coefficients as follows:

$$Y_0 = \{1 + 1 + 1 + 1\} = 4$$

$$\begin{aligned} Y_1 &= \{1 + 1.\mathrm{e}^{-j.1.1.\pi/4} + 1.\mathrm{e}^{-j.1.2.\pi/4} + 1.\mathrm{e}^{-j.1.3.\pi/4}\} \\ &= \{1 + \mathrm{e}^{-j\pi/4} + \mathrm{e}^{-j\pi/2} + \mathrm{e}^{-j3\pi/4}\} = 1 - (1 + \sqrt{2})j \end{aligned}$$

$$\begin{aligned} Y_2 &= \{1 + 1.\mathrm{e}^{-j.2.1.\pi/4} + 1.\mathrm{e}^{-j.2.2.\pi/4} + 1.\mathrm{e}^{-j.2.3.\pi/4}\} \\ &= \{1 + \mathrm{e}^{-j\pi/2} + \mathrm{e}^{-j\pi} + \mathrm{e}^{-j3\pi/2}\} = 1 - j - 1 + j = 0 \end{aligned}$$

So that $Y_k = [4, 1 - j(1 + \sqrt{2}), 0, ...]$.

We know that the FT of a rectangular pulse centred on the origin is a sinc, thus if we shift the pulse ($y(t)$ is such a shifted pulse) we multiply by a complex exponential – which means the magnitude remains unchanged. For a pulse of width b and height c centred on the origin, the FT is $\mathrm{sinc}\,(\omega b/2)$. Thus, for $y(t)$ where $b = 1$ and $c = 1$, our continuous FT takes the form $\mathrm{sinc}(\omega/2)$. Thus, the first null will occur when $\omega = 2\pi$, or $f = \omega/(2\pi) = 1$. We know that Y_k corresponds to a frequency, f, of $\frac{k}{NT}$, and in this case $T = 0.25s$ so $\frac{k}{NT} = k/2$.

Therefore, Y_o corresponds to $f = 0$, Y_1 corresponds to $f = 1/2$, Y_2 corresponds to $f = 1$. At $f = 1$ we expect the coefficient to be zero (first null of the sinc) in the absence of aliasing – the answer needs to take into account aliasing and to show that the aliased components also have nulls at $f = 1$, or at least to mention aliasing. [Ideally we would like something which states that the spectrum repeats at intervals of the sampling frequency, in this case 8Hz, and from this we can see that all of the aliased components will also have null values at $\omega = 2\pi$.] [8]

(c) If we are to avoid aliasing, we need to sample at twice the maximum frequency, ie 40kHz – one might also include a 10% roll off, which would mean sampling at 44kHz. So if we have a total of 1.5×10^6 bits, then the number of bits per sample is N, where

$$N = \frac{1.5 \times 10^6}{2 \times 44 \times 10^3} = 3000/4 \times 44 \approx 17 \text{ bits}$$

If the signals can be approximated by sinusoidal components, we can take the signal power as $V^2/2$ (the square of the RMS signal), if we assume our voltage range is $-V$ to $+V$. We also know that the RMS quantisation noise is given by $\Delta/\sqrt{12}$ if the step size is Δ and we assume that the real signal is equally likely to occur anywhere in the interval.

Thus the SNR is given by the ratio of the square of the RMS signal and noise values;

$$\text{SNR} = \frac{V^2/2}{\Delta^2/12} = \frac{6V^2}{\Delta^2}$$

If our interval is of length $2V$ and we have N bits (2^N levels), our Δ is given by

$$\Delta = \frac{2V}{2^N} = 2^{1-N}V$$

So that SNR = $3 \times 2^{2N-1}$.

Therefore, in dB, our SNR is given by

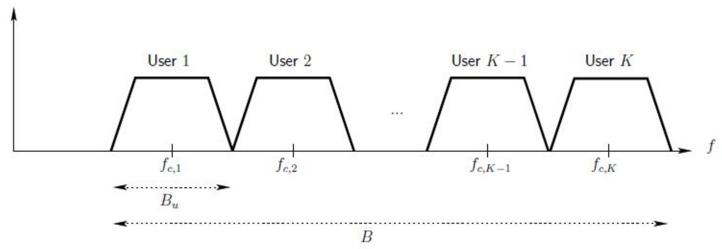
$$10\log_{10}(3 \times 2^{2N-1}) = 1.76 + 6.02N$$

which is in the Databook! So, if N=17, SNR$\approx$ 104dB. [6]

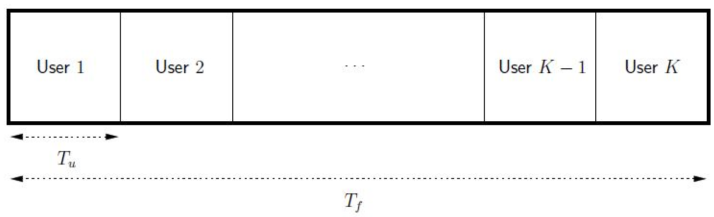
6 (a) This is bookwork, so we expect them to reproduce the explanations in the Comms notes.

The 3 main methods are FDMA (Frequency Division Multiple Access), TDMA (Time Division Multiple Access) and CDMA (Code Division Multiple Access).

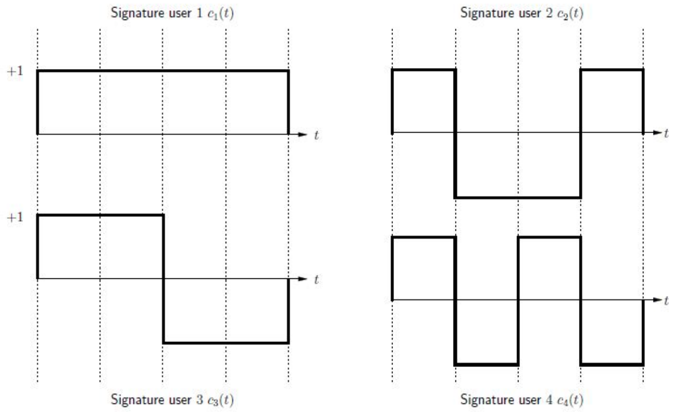
FDMA: Multiple users are multiplexed in the frequency domain, such that they do not interfere with each other, using a fraction of the total bandwidth. DSB-SC modulation can be used for each user so that the spectrums do not overlap.



TDMA: Multiple users are multiplexed in time, so that they transmit one after the other, each using the whole bandwidth B. We divide the *frame duration*, T_f, into K slots of duration $T_u = T_f/K$ for K users.



CDMA: Multiple users are multiplexed in code or signature, and transmit using the whole bandwidth B over the whole time frame duration T_f. a signature is a signal characteristic to each user, and known to the receiver. There would be K different signatures for K users.



[5]

(b) For a channel with additive white Gaussian noise (noise PSD is N_0), the bandwidth, B, capacity, C and transmitted power, P are related by

$$C = B\log_2(1+\frac{P}{N_0B})$$

Thus

$$2^{C/B} = 1+\frac{P}{N_0B}$$

which leads to

$$P = BN_0(2^{C/B}-1)$$

[3]

(c) For FDMA we have

$$C^{FDMA} = \frac{B}{K}\log_2\left(1+\frac{P}{N_0\frac{B}{K}}\right)$$

and for TDMA we have

$$C^{TDMA} = \frac{1}{K}\left[B\log_2\left(1+\frac{PK}{N_0B}\right)\right]$$

Putting $B/K = B_u$, the bandwidth per user we see that both of these reduce to

$$C = B_u \log_2 \left(1 + \frac{P}{N_0 B_u} \right)$$

[3]

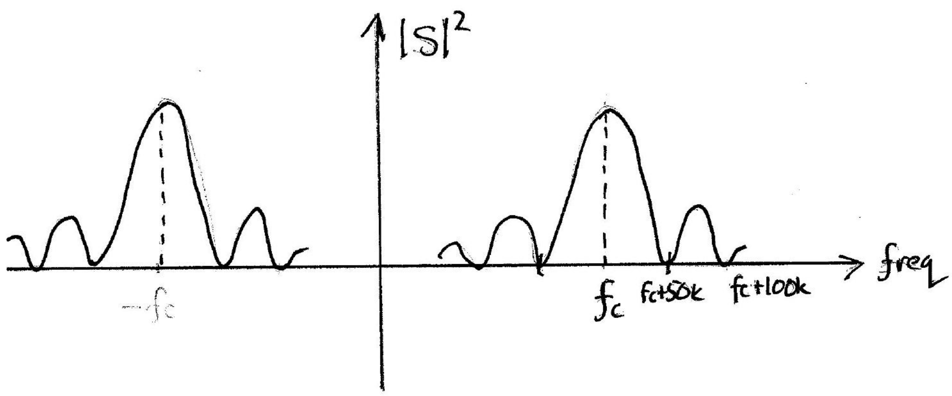
(d) (i) The bandwidth available is $250 - 50 = 200$MHz. We know that the BPSK spectrum is given by

$$S_{BPSK}(f) = \frac{1}{2}[X(f - f_c) + X(f + f_c)]$$

where f_c is the carrier frequency. For a rectangular pulse of unit amplitude and duration T we know that X is given by $T \text{ sinc}(\pi f T)$, so that the spectrum would look like the case below, where the first null occurs when $\pi f T = \pi$, ie $f = 1/T$. In our case $R = 50 \times 10^3$ bit/sec, so

$$T = 1/R = 1/(5 \times 10^4)\text{sec}$$

Giving the first null at $\Delta f = 50$kHz and the second null at 100kHz.



[5]

(ii) If any overlaps are to occur beyond the first sidelobe, each spectrum will take up 2×100kHz, so the bandwidth of each user will be 200kHz. The maximum number of users is therefore

$$\frac{200 \times 10^6}{200 \times 10^3} = 1000$$

We can therefore accommodate $M = 1000$ users. [4]

END OF SOLUTIONS