ENGINEERING TRIPOS PART IB, 2013
COMMENTARY, PAPER 7 MATHEMATICAL METHODS

Question 1.
The most common candidate error was to ignore the result of b(i), that contours of constant ψ correspond to the flow streamlines, when attempting b(ii). Doing so leaves an extremely difficult integration problem to be solved. The other frequent omission was failure to recognise the differential operator derived in c(i) as the Laplacian, for which the polar form required in c(ii) is a standard result.

Question 2.
Many of those attempting this question were hamstrung by not knowing the bookwork required for part a. Weaker candidates often tried to evaluate the integral in b directly, rather than by taking advantage of Stokes' theorem. The line integral in c(ii) can be done directly without much more effort than the official solution; defining a 'potential' also gives the right answer (because of the form of the vorticity vector), but is cheating, as the field is not irrotational.

Question 3.
The only frequently occurring mistakes here were those associated with mathematical illiteracy, specifically: the inclusion of time derivatives in the Laplacian operator; differentiating the functions $Y(y)$ and $Z(z)$ with respect to variables other than y and z.

Question 4.
This question centres on the use of conditional probability and Bayes' rule. Difficulties with the two initial, more conceptual, questions were more common than problems with the calculations.

Question 5.
This question potentially involves large, and time-consuming, amounts of calculation. The slog can be avoided by utilising knowledge of the fundamental definitions of the relevant vector spaces, rather than relying on their recipes.

Question 6
Like Qu. 4, this question caused more problems in its conceptual aspects than its calculations. Copying the data book formula for the QR factorisation was not regarded as an adequate explanation for how it can be employed.

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July 2nd, 2013

a) $\nabla . \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = \underline{\underline{0}}$

b) (i) $\delta \psi = \frac{\partial \psi}{\partial x} \delta x + \frac{\partial \psi}{\partial y} \delta y$

On a contour of constant ψ, $\delta \psi = 0$, so (substituting for the partial derivatives of ψ):

$$0 = -v \, \delta x + u \, \delta y$$

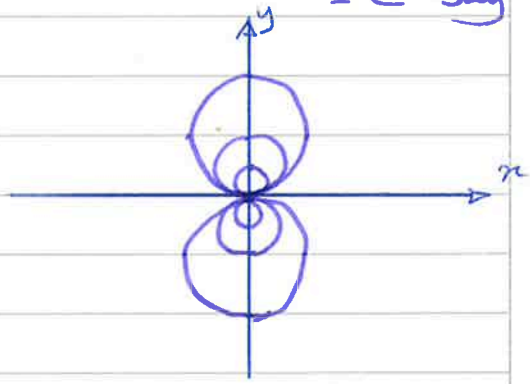
$\underline{\underline{\frac{\delta y}{\delta x} = \frac{v}{u}}}$ Significance: this is the equation that defines the streamlines of the flow.

(ii) Streamlines have been shown to correspond to $\psi = \text{const} = C$ say

$\therefore \quad C(x^2+y^2) = y$

$x^2 + \left(y - \frac{1}{2C}\right)^2 = \left(\frac{1}{2C}\right)^2$

Circles, centred at $(0, 1/2C)$ $\Rightarrow$
radius $|1/2C|$



c)(i) $\nabla \times \underline{u} = 0 \Rightarrow \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\psi/\partial y & -\partial\psi/\partial x & 0 \end{vmatrix} = 0$ ie. $\underline{k}\left[-\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right] = 0$

$$\underline{\underline{\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0}}$$

In vector calculus notation, $\boxed{\nabla^2\psi = 0}$

(ii) Data book : $\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$ in polars (r,θ)

$\psi = r^{-n}f(\theta)$ $\quad \frac{\partial\psi}{\partial r} = -nr^{-(n+1)}f(\theta)$ $\quad \frac{\partial\psi}{\partial\theta} = r^{-n}f'(\theta)$

$\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = +n^2r^{-(n+1)}f(\theta)$ $\quad \frac{\partial^2\psi}{\partial\theta^2} = r^{-n}f''(\theta)$

So $\nabla^2\psi = 0 \Rightarrow n^2f(\theta) + f''(\theta) = 0$

with solutions $f(\theta) = \cos n\theta$

and $f(\theta) = \sin n\theta$

Hence $\psi = \frac{A}{r^n}\cos n\theta + \frac{B}{r^n}\sin n\theta$

with A, B arbitrary constants.

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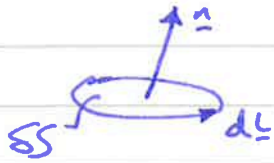
Question number 2 Sheet number 1

a) (i) Let $\underline{u}$ be the vector field, and consider a small surface element in the field:



δS is element area

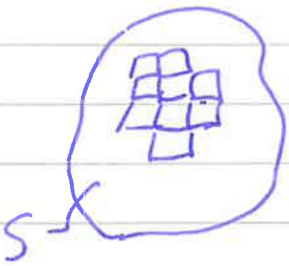
$\underline{n}$ is element normal

$d\underline{l}$ is a line element, in the clockwise direction when the elemental area is viewed in the $\underline{n}$ direction

Then $\nabla\times\underline{u}.\underline{n} = \lim_{\delta S\to 0} \dfrac{\oint \underline{u}.d\underline{l}}{\delta S}$ for arbitrary $\underline{n}$

[$\lim_{\delta S\to 0} \nabla\times\underline{u}.d\underline{S} = \oint \underline{u}.d\underline{l}$ also OK.]

(ii) Consider a large area of finite size, split into many elements:



Define $\delta \underline{S}_j$ as $\delta S\,\underline{n}$ for element j

Then, from the fundamental definition,

$$\lim_{\delta S\to 0} \sum_j (\nabla\times\underline{u})_j . \delta\underline{S}_j = \lim_{\delta S\to 0} \sum_j \oint_j \underline{u}.d\underline{l}$$

- All edges of adjoining cells give RHS contributions that cancel
- LHS becomes an area integral in the limit

Hence $$\int_S \nabla\times\underline{u}.d\underline{S} = \oint_{\text{boundary of } S} \underline{u}.d\underline{l},$$

which is Stokes' theorem.

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b) From Stokes' theorem, $\displaystyle\int_{r\leq R} \nabla\times\underline{u}\,.\,d\underline{S} = \int_{r=R} \underline{u}\,.\,d\underline{l}$

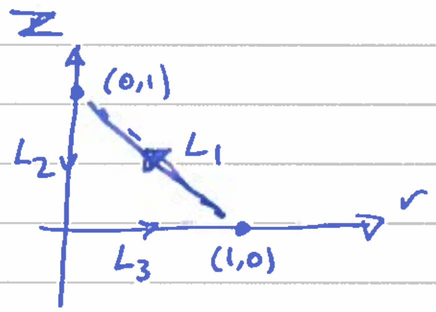
$$= 1 - e^{-1}$$

c) (i) $\nabla\times\underline{v} = \dfrac{1}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{e}_z \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ v_r & rv_\theta & v_z \end{vmatrix}$

N.B. Notation: r used here for ρ

$$= \frac{1}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{e}_z \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ -r & rv_\theta & 2z \end{vmatrix} = -\frac{\partial v_\theta}{\partial z}\underline{e}_r + \left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\right)\underline{e}_z$$

(ii) $\theta = 0$ plane:



Apply Stokes, noting that $\underline{e}.d\underline{S} = 0$ in this plane.

$$\int_{L_1} \underline{v}\,.\,d\underline{l} + \int_{L_2} \underline{v}\,.\,d\underline{l} + \int_{L_3} \underline{v}\,.\,d\underline{l} = 0$$

$$\int_{L_1} \underline{v}\,.\,d\underline{l} = \int_0^1 v_z\,dz - \int_0^1 v_r\,dr$$

$$= \left[z^2\right]_0^1 - \left[\frac{-r^2}{2}\right]_0^1 = 3/2$$

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Question number 3 Sheet number 1

a) (i) Substitute $\phi = \phi_0 e^{i(\omega t - kx)}$ into wave eqn:

$$\left(-k^2 + \frac{\omega^2}{c^2}\right)\phi_0 e^{i(\omega t - kx)} = 0$$

So the proposed form is a solution for $\underline{\underline{k = \pm \omega/c}}$

(ii) The phase of the harmonic solution is constant for

$$\omega t - kx = \text{const}$$

which represents a disturbance travelling at speed ω/k.

$k = \omega/c$ $x = ct + \text{const}$; travel in $\underline{+ve}$ x direction

$k = -\omega/c$ $x = -ct + \text{const}$; travel in $\underline{-ve}$ x direction.

b) Substitute $\phi = Y(y)Z(z)e^{i(\omega t - kx)}$ into wave eqn:

$$\left(-k^2 + \frac{Y''}{Y} + \frac{Z''}{Z} + \frac{\omega^2}{c^2}\right)Y(y)Z(z)e^{i(\omega t - kx)} = 0$$

i.e. $k^2 = \frac{\omega^2}{c^2} + \frac{Y''}{Y} + \frac{Z''}{Z}$, or $\underline{\underline{k^2 + \alpha^2 + \beta^2 = \frac{\omega^2}{c^2}}}$

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c) $\frac{Y''}{Y} = -\alpha^2 \Rightarrow Y = A\sin\alpha y + B\cos\alpha y$

$\frac{\partial\phi}{\partial y} = 0$ on $y = 0 \Rightarrow \underline{\underline{A = 0}}$

$\frac{\partial\phi}{\partial y} = 0$ on $y = L_y \Rightarrow -\alpha B \cancel{\sin}\alpha L_y = 0$

Satisfied for all B, but only if $\alpha L_y = n\pi$, n integer

So $Y = B \cancel{\cos} \frac{n\pi y}{L_y}$, $n = 0, 1 \ldots\ldots$

Similarly $Z = C\cos\frac{m\pi z}{L_z}$, $m = 0, 1 \ldots\ldots$

d) x-wise behaviour is determined by k; from (b) ...

$$k = \sqrt{\frac{\omega^2}{c^2} - \alpha^2 - \beta^2}.$$

- Solutions for k are now imaginary, corresponding to exponentials; if $k = iK$, then

$$e^{ikx} = e^{-Kx}$$

- The two available solutions represent either exponential growth or decay
- Exponential growth is unphysical, so exponential decay would be seen.

a) $p(A|B)$ is the probability that A occurs, given the knowledge that B has occurred.

Definition : $p(A|B) = \dfrac{p(A \text{ and } B)}{p(B)}$

b) From the definition of conditional probability, and the fact that $p(A \text{ and } B)$ is identical to $p(B \text{ and } A)$,

$$p(A|B)\,p(B) = p(B|A)\,p(A)$$

Hence : $p(A|B) = \dfrac{p(B|A)\,p(A)}{p(B)}$

c) (i) Lie detector indicates falsehood either if citizen lies and detector is correct, or if citizen tells truth and detector errs.

Let F be this occurrence; $p(F) = 0.001 \times 0.95 + 0.999 \times 0.05$

$= 0.0509$

(ii) Let L be the condition that the person is lying.

Bayes: $p(L|F) = \dfrac{p(F|L)\,p(L)}{p(F)} = \dfrac{0.95 \times 0.001}{0.0509}$

$= 0.0187$

d) Let C represent 'crook'.

We want $p(C|F)$.

Falsehood is ~~now~~ detected if: honest citizen is selected and lie detector indicates falsehood

crook is selected — " —

i.e. $p(F) = 0.9 \times 0.0509 + 0.1 \times 1 \times 0.95 = 0.14081$

from (c)(i)

$$p(C|F) = \frac{p(F|C)\,p(C)}{p(F)}$$

$$= \frac{0.95 \times 0.1}{0.14081}$$

$$= \underline{\underline{0.675}}$$

(a) A is <u>not</u> invertible if $\det(A) = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & t \\ 1 & 4 & t^2 \end{vmatrix} = 0 \qquad 2t^2 - 4t - (t^2 - t) + (4-2) = 0$$

$$t^2 - 3t + 2 = 0$$

$$(t-2)(t-1) = 0 \Rightarrow \underline{\underline{t = 1 \text{ or } 2}}$$

b)(i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{bmatrix}$ | $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 4 & 4 \end{bmatrix}$

Column space: $\underline{\underline{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}}$ and $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ | As left.

Row space $[1\ 1\ 1]$ and $[1\ 2\ 1]$ | Row space $[1\ 1\ 1]$ and $[1\ 2\ 2]$

$\underline{\underline{([1\ 4\ 1] = 3[1\ 2\ 1] - 2[1\ 1\ 1])}}$ | $\underline{\underline{([1\ 4\ 4] = 3[1\ 2\ 2] - 2[1\ 1\ 1])}}$

Null space $\perp$ row space $\Rightarrow$

$$(1,1,1) \times (1,2,1) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -\underline{i} + \underline{k}$$

i.e. $\underline{\underline{[-1\ 0\ 1]}}$

$$(1,1,1) \times (1,2,2) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -\underline{j} + \underline{k}$$

i.e. $\underline{\underline{[0\ -1\ 1]}}$

Left null space $\perp$ col. space $\Rightarrow$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = \underline{\underline{2\underline{i} - 3\underline{j} + \underline{k}}}$$

As left, i.e. $\underline{\underline{\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}}}$

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Question number 5 Sheet number 2

(ii) $A\underline{x} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Hence one possible $\underline{x}$ is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ (both possible matrices)

General solutions include arbitrary contribution from null space, i.e.

for $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{bmatrix}$ $\quad \underline{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \kappa \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

for $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 4 & 4 \end{bmatrix}$ $\quad \underline{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

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Question number 6 Sheet number 1

a) Least-squares sol'n satisfies:

$$A^t A \underline{x} = A^t \underline{b}$$

$A = QR$: $\quad R^t Q^t Q R \underline{x} = R^t Q^t \underline{b}$

$Q^t Q = I$: $\quad R^t R \underline{x} = R^t Q^t \underline{b}$

R^t is invertible : $\quad R\underline{x} = Q^t \underline{b}$

Since R is upper triangular, $\underline{x}$ can now be evaluated row-by-row, starting with the ~~first~~ last.

b) (i) By inspection, $\underline{q}_1 = \underline{a}_1 = [1\ 0\ 0\ 0]^t$

$\underline{a}_2 - (\underline{a}_2 \cdot \underline{q}_1)\underline{q}_1 = [0\ 1\ 2\ 2]^t \Rightarrow \underline{q}_2 = \frac{1}{3}[0\ 1\ 2\ 2]^t$

$$\underline{a}_3 - (\underline{a}_3 \cdot \underline{q}_1)\underline{q}_1 - (\underline{a}_3 \cdot \underline{q}_2)\underline{q}_2 = \begin{bmatrix}1\\3\\1\\2\end{bmatrix} - \begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \frac{9}{9}\begin{bmatrix}0\\1\\2\\2\end{bmatrix} = \begin{bmatrix}0\\2\\-1\\0\end{bmatrix}$$

Hence $\underline{q}_3 = \dfrac{1}{\sqrt{5}}[0\ 2\ -1\ 0]^t$

This completes the Gram-Schmidt orthogonalization of the columns of A.

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Question number 6 Sheet number 2

Now we seek R such that $[\underline{a}_1\ \underline{a}_2\ \underline{a}_3] = [\underline{q}_1\ \underline{q}_2\ \underline{q}_3]R$

$\underline{a}_1 = \underline{q}_1 \Rightarrow$ first column of R is $\begin{bmatrix}1\\0\\0\end{bmatrix}$.

$\underline{a}_2$ can be expressed in terms of $\underline{q}_1$ and $\underline{q}_2$: $\underline{a}_2 = 2\underline{q}_1 + 3\underline{q}_2$

$\Rightarrow$ 2nd column of R is $\begin{bmatrix}2\\3\\0\end{bmatrix}$

$\underline{a}_3$ expressed in terms of $\underline{q}_1, \underline{q}_2$ and $\underline{q}_3$: $\underline{a}_3 = \underline{q}_1 + 3\underline{q}_2 + \sqrt{5}\,\underline{q}_3$

$\Rightarrow$ 3rd column of R is $\begin{bmatrix}1\\3\\\sqrt{5}\end{bmatrix}$

So $A = QR$ with $Q = \begin{bmatrix}1 & 0 & 0\\ 0 & 1/3 & 2/\sqrt{5}\\ 0 & 2/3 & -1/\sqrt{5}\\ 0 & 2/3 & 0\end{bmatrix}$ $R = \begin{bmatrix}1 & 2 & 1\\ 0 & 3 & 3\\ 0 & 0 & \sqrt{5}\end{bmatrix}$

(ii) $Q^t \underline{b} = \begin{bmatrix}1\\ 10/3\\ -3/\sqrt{5}\end{bmatrix}$ $\cancel{R^t = \begin{bmatrix}1 & 0 & 0\\ 2 & 3 & 0\\ 1 & 3 & \sqrt{5}\end{bmatrix}}$

$$\begin{bmatrix}1 & 2 & 1\\ 0 & 3 & 3\\ 0 & 0 & \sqrt{5}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\ 10/3\\ -3/\sqrt{5}\end{bmatrix} \Rightarrow$$

$x_3 = -3/5$

$x_2 = \frac{10}{9} - x_3 = 77/45$

$x_1 = 1 - 2x_2 - x_3 = -82/45$

$$\underline{x} = \frac{1}{45}\begin{bmatrix}-82\\77\\-27\end{bmatrix} = \begin{bmatrix}-1.82\\1.71\\-0.6\end{bmatrix}$$