ENGINEERING TRIPOS PART IB

Friday 7 June 2013 2 to 4

Paper 7

MATHEMATICAL METHODS

Answer not more than ***four*** *questions.*

Answer not more than **two** *questions from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

wrg/02

SECTION A

*Answer not more than **two** questions from this section*

1 A two-dimensional fluid flow field with velocity vector $u\mathbf{i}+v\mathbf{j}$ has the property that u and v can be expressed in terms of a single function $\psi(x,y)$, as follows:

$$u=\frac{\partial\psi}{\partial y}, \qquad v=-\frac{\partial\psi}{\partial x}.$$

(a) What is the divergence of the flow field? [2]

(b) (i) Write down an expression for the change $\delta\psi$ in ψ due to changes δx in x and δy in y. Hence obtain the general equation that defines the contours of constant ψ in terms of u and v. Comment on its physical significance. [6]

(ii) For

$$\psi=\frac{y}{x^2+y^2}$$

derive the equations of the flow streamlines, and sketch them. (You need not indicate the field direction.) [4]

(c) If the flow is now also required to be irrotational,

(i) find the equation that ψ must satisfy; [2]

(ii) find the solutions that have $\psi=\rho^{-n}f(\theta)$ in polar coordinates (ρ,θ), with n a positive integer. [6]

2 (a) (i) State the coordinate-free definition of the curl of a vector field. [3]

(ii) Derive Stokes' theorem. [3]

(b) A two-dimensional vector field **u**, expressed in polar coordinates (ρ, θ), has $u_\rho = 0$ and

$$u_\theta = \frac{1 - e^{-\rho^2/R^2}}{2\pi\rho},$$

where R is a constant. Find the area integral

$$\int_{\rho \le R} \nabla \times \mathbf{u} \cdot d\mathbf{S}.$$

[4]

(c) A three-dimensional vector field **v**, expressed in cylindrical polar coordinates (ρ, θ, z), has

$$v_\rho = -\rho, \quad v_z = 2z.$$

The tangential component, v_θ, is unspecified.

(i) Derive an expression for $\nabla \times \mathbf{v}$. [4]

(ii) Evaluate the line integral of **v** along the straight line from $\rho = 1$, $\theta = 0$, $z = 0$ to $\rho = 0$, $\theta = 0$, $z = 1$. [6]

3 Disturbances in a wave-bearing medium are characterised by the variable ϕ, which obeys the equation

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0.$$

The parameter c is a positive constant, and represents the wave speed.

(a) (i) Show that $\phi = \phi_0 e^{i(\omega t - kx)}$, with ϕ_0, ω and k constant, is a possible solution of the equation, and find the two specific values of k for which it is valid. [4]

(ii) Explain why your two valid solutions represent propagating waves, and specify their respective directions of travel. [2]

(b) Show that $\phi = Y(y)Z(z)e^{i(\omega t - \kappa x)}$, with $Y'' = -\alpha^2 Y$ and $Z'' = -\beta^2 Z$, is also a possible solution, and find the condition on α^2, β^2, ω and κ for it to be valid. [4]

(c) The medium is enclosed in a rectangular duct, whose axis lies in the x direction, and whose cross-section occupies the region $0 \le y \le L_y$, $0 \le z \le L_z$. The boundary conditions are:

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = 0, \quad y = L_y; \qquad \frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z = 0, \quad z = L_z.$$

Find the functions $Y(y)$ and $Z(z)$, giving also the values of α and β for which your expressions are valid. [6]

(d) Discuss how ϕ varies with x when $\alpha^2 + \beta^2 > \omega^2/c^2$. What physical behaviour would be observed for disturbances of this form? [4]

SECTION B

*Answer not more than **two** questions from this section*

4 (a) Explain what is meant by $p(A|B)$, the 'conditional probability of A given B', and give its definition. [4]

(b) Derive Bayes' rule:

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}.$$

[4]

(c) The chance of an honest citizen lying is 1 in 1000. Suppose that such a citizen is tested with a lie detector which correctly identifies both truth and falsehood 95 times out of 100.

(i) What is the probability that the lie detector indicates falsehood? [2]

(ii) In this case, what is the probability that the person is indeed lying? [4]

(d) Ten citizens, nine of whom are honest, are assembled. The tenth is a crook who lies with probability 1. A randomly chosen member of the group is tested, and the lie detector indicates a falsehood. What is the probability that this person is the crook? [6]

5 (a) For what values of t is the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & t \\ 1 & 4 & t^2 \end{bmatrix}$$

not invertible? [4]

(b) For the cases where **A** is not invertible,

(i) find bases for its four fundamental subspaces; [12]

(ii) hence, or otherwise, find the general solutions **x** to the equations $\mathbf{Ax} = [\ 0 \quad 1 \quad 3\ ]^t$. [4]

6 An over-determined set of equations is represented in matrix-vector form as $\mathbf{Ax} = \mathbf{b}$. It has the least-squares solution $\mathbf{x} = (\mathbf{A}^t\mathbf{A})^{-1}\mathbf{A}^t\mathbf{b}$.

(a) Explain how the 'QR factorisation' $\mathbf{A} = \mathbf{QR}$, with **R** upper-triangular and **Q** consisting of orthonormal columns, can be employed to simplify evaluation of **x**. [8]

(b) The matrix **A** is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$

(i) Find the QR factorisation of **A**. [8]

(ii) Hence, or otherwise, obtain the least-squares solution **x** when $\mathbf{b} = [\ 1 \quad 0 \quad 3 \quad 2\ ]^t$. [4]

END OF PAPER