SECTION A *Introductory Business Economics - CRIBS*

1 (a) Discuss the concepts of the firm and the market. Illustrate your answer with appropriate examples. [5]

First of all, the students are required to address the notion of firms and markets as theoretical constructs. They can present the standard microeconomic treatment of the firm as a mechanism to transform inputs into outputs as discussed in the theory of production, and they can define the market as the mechanism through which scarce resources are allocated via the price system and trade to identify how much a good should be produced, for whom and by whom. The students should be able to emphasise the institutional foundations of firms and markets, and may discuss the legal definition of firms as well as the different degrees to which societies can decide to use the market as a way of organising economic activities. The better students will be able to provide a richer view of firms and markets by referring to the transaction and capability view of the firm. Well-chosen empirical examples will show the student's capacity to relate theory to observable firms and markets.

(b) In relation to the problem of market structure:

(i) What could be the benefits of an oligopoly compared to a perfectly competitive market and compared to a monopoly from the viewpoint of consumers? [5]

A good answer should show clear understanding of the characteristics of different market structures. It should emphasise the different equilibrium prices resulting from perfect competition, oligopoly and monopoly, and highlight the role of product differentiation and possible strategic behaviours in the three different market structures.

(ii) Explain the Bertrand model of oligopoly. [5]

The Bertrand model assumes firms that produce identical products with the same constant marginal costs and technology. They independently and simultaneously set price in order to maximise profits in the presence of some barriers to entry and assuming that consumers possess perfect information with zero transactions costs. The unique equilibrium sees firms will chose a price that is equal to the marginal cost of production (MC) resulting in a competitive outcome.

(ii) Illustrate why cartels are said to be unstable. [5]

Cartels are unstable because they tend to involve an incentive to deviate from the agreed targets in search for temporary increases in profits. The clearest way to illustrate this is to apply the Prisoner's Dilemma game in its standard form to a duopoly setting. The students should identify the equilibrium of the game and possible deviations from it. They could also discuss the nature and consequences of price wars and their implications for consumers and producers.

(c) Explain the role of government spending in the macroeconomy. What are the advantages and possible disadvantages of increasing government spending? [10]

The students should identify the motivations of government spending in the principles of efficiency and equity. A good answer will contain precise references to the problem of leaving to the market the provision of public or merit goods, and a clear articulation of the role of government spending as a tool to control aggregate demand (GDP) within a framework of active macroeconomic policy. The students should identify increased government spending as a remedy to negative effects (for example on employment) of the business cycle and as a form of support to aggregate demand against the risk of under-consumption. Among the disadvantages the students should emphasis the possible distortion of the market system (leading to efficiency losses), and the costs (e.g. increased interest rates) and risks (e.g. in the presence of unclear 'inside' and 'outside' lags) of fiscal policies based on increased in government expenditure.

P8 ~~Q1 solutions~~ Q3 cribb

(a) Stability ratio, $N = (\gamma z - \sigma_T)/s_u$

where $\gamma$ = unit weight of clay

$z$ = depth of tunnel axis below ground level

$\sigma_T$ = support pressure applied to tunnel face

$s_u$ = undrained strength of the clay

For open face tunnelling $\sigma_T = 0$

If N>5 the tunnel is unlikely to be stable

[20%]

(b) On the east side of the fault, the stability ratio for the tunnels at 15m and 25m depth (assuming the unit weight $\gamma$ of the soil is approximately 20 kN/m$^3$) would be (15 x 20)/100 = 3.0 and (25 x 20)/250 = 2.0. These stability ratios are low enough to ensure that open face tunnelling (i.e. no support to the tunnel face during excavation) would be safe.

On the west side of the fault, the stability ratio for the tunnel at 25m depth would be (25 x 20)/35 = 14.3. This is very high – if it exceeds about 5 it would be impossible to construct the tunnel under open face conditions, because the tunnel face would not be stable. Closed face tunnelling machines would be necessary, using either slurry machines or earth pressure balance machines to provide support to the tunnel face continuously during excavation.

The tunnel at a depth of 15m on the west side would be in sand (stability ratio not relevant); measures would be needed to prevent water entry. If open face tunnelling was employed, permeation grouting could be undertaken to achieve this – but it would probably be more economic to adopt closed face tunnelling machines.

Sprayed concrete linings could only be used on the east side of the fault, ie in the stiff clays but on the west side this would not be possible. Precast concrete segmental tunnel linings could be used for the tunnels on either side of the fault.

[30%]

(b) To the east of the fault, the ground comprises two layers of strong clays. The clays are of low permeability and therefore leakage of water through the walls during construction is not an issue. On the west side of the fault the top 20m is highly permeable sand which will mean potential leakage into the station could be an issue if there are defects in the walls. The walls could be constructed by the following methods:

- reinforced concrete diaphragm walls, installed in slurry-supported trenches cut in the ground (first the trench is cut – typical plan dimensions 4m x 1m - then a reinforcement cage is lowered into the trench, then concrete is poured via a 'tremie' tube inserted to the bottom of the trench, displacing the slurry)
- reinforced concrete secant pile wall in which alternate 'hard' and 'soft' bored piles are constructed – the 'soft' piles, comprising bentonite-cement

are constructed first, and then the 'hard' reinforced concrete piles cutting into the 'soft' piles

- steel sheet piles, driven or jacked into the ground

[30%]

(c) Settlement can potentially damage masonry buildings if there is <u>differential</u> settlement, which causes tensile strains in the masonry leading to cracking. Compensation grouting comprises the installation of horizontal steel tubes in the ground between the tunnel and the building foundations. Cement grout is injected from the tubes during tunnelling to compensate for the ground movements caused by tunnelling. The process prevents damaging differential building settlements movements occurring. Instrumentation is a key part of this process, because monitoring of ground and building movements is essential to inform decisions about when and where to grout.

[20%]

Question 4

$\gamma = 20\ kN/m^3$ (all soils)

$\phi' = 35°$ for sand

(a)

$K_a = \frac{1-\sin\phi'}{1+\sin\phi'} = 0.27$ (Active behind wall)

$K_p = 1/K_a = 3.68$ (Passive in front of wall)

'Active' side:

At depth $z = 6m$ (sand):

$\sigma_v = 20 \times 6 = 120\ kPa$, $u = 6 \times 10 = 60\ kPa$

$\therefore \sigma_v' = 120 - 60 = 60\ kPa$

$\sigma_h' = 0.27 \times 60 = 16\ kPa$

$\sigma_h = \sigma_h' + u = 16 + 60 = 76\ kPa$

$z = 6m$ (clay) $\sigma_h = \sigma_v - 2c_u = 120 - 2 \times 50 = 20\ kPa$

$z = 9m$ clay, $\sigma_h = \sigma_v - 2c_u = 9 \times 20 - 2 \times 50 = 80\ kPa$

'Passive' side:

At excavation level $\sigma_v = 0$, $\sigma_h = 0 + 2 \times 50 = 100\ kPa$

At 3m below excavation level, $\sigma_h = (3 \times 20) + (2 \times 50)$
$= 160\ kPa$

'Active' moments about prop

$= \left(\frac{1}{2} \times 76 \times 6 \times 4\right) + (20 \times 3 \times 7.5) + \left[\frac{1}{2} \times (80-20) \times 3 \times 8\right]$ kN-m/m

$= 912 + 450 + 720 = \underline{2082\ kN\text{-}m/m}$

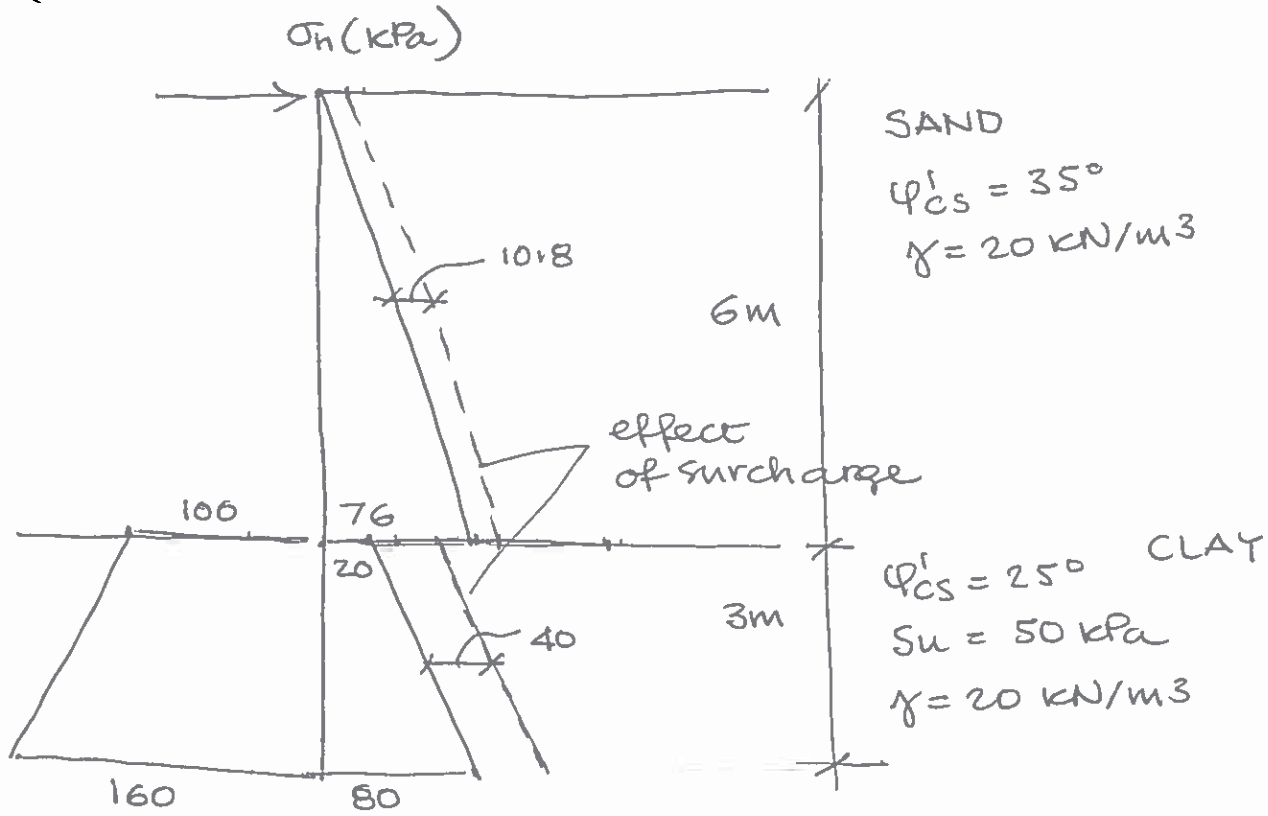
Passive moments about prop

$= (100 \times 3 \times 7.5) + \left[\frac{1}{2}(160-100) \times 3 \times 8\right] = 2250 + 720 = \underline{2970}$ kN-m/m

$\therefore$ Factor of safety $= \frac{2970}{2082} = \underline{\underline{1.43}}$

[50%]

Q.4 Solution




(b) 2 m of stockpile surcharge

$\Delta\sigma_v = 2 \times 20 = 40$ kPa

$\Delta u = 0$ in sand

$\Delta\sigma'_v = \Delta\sigma_v = 40$ kPa

$\Delta\sigma'_h = 0{,}27 \times 40 = 10.8$ kPa $= \Delta\sigma_h$

$\Delta\sigma_v = \Delta\sigma_h = 40$ kPa in clay

increase in active moment

$\Delta M_A = 10.8 \times 6 \times 3 + 40 \times 3 \times 7{,}5 = 1094.4$

safety factor $FS = \dfrac{M_P}{M_A + \Delta M_A} = \dfrac{2970}{2082 + 1094.4} = 0{,}94$

(c) On passive side, fully drained conditions occur in the 3m of clay

At $z = 0$ (measuring from excavation level)

$\sigma_v' = \sigma_h' = \sigma_h = 0$

At $z = 3m$

$\sigma_v = 3 \times 20 = 60$ kPa

$u = 3 \times 10 = 30$ kPa (hydrostatic from excavation level)

$\therefore \sigma_v' = 60 - 30 = 30$ kPa

$\phi' = 25°$, $K_p = \dfrac{1 + \sin 25°}{1 - \sin 25°} = 2.45$

$\therefore \sigma_h' = 2.45 \times 30 = 74$ kPa

$\sigma_h = \sigma_h' + u = 74 + 30 = 104$ kPa

$\therefore$ passive moment about prop $= \frac{1}{2} \times 104 \times 3 \times 8$ kN-m/m
$= \underline{1248}$ kN-m/m

$\therefore$ Factor of safety $= \dfrac{1248}{2082} = \underline{\underline{0.6}}$

Drainage of clay on passive side $\Rightarrow$ failure of wall

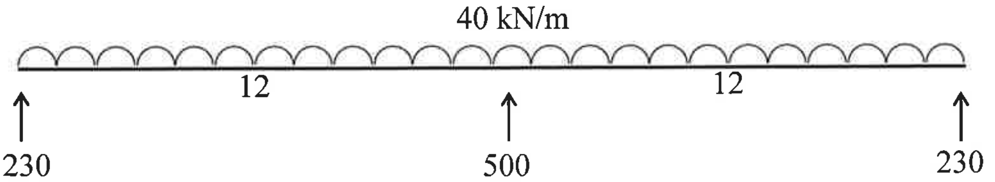
[25%]

Q5

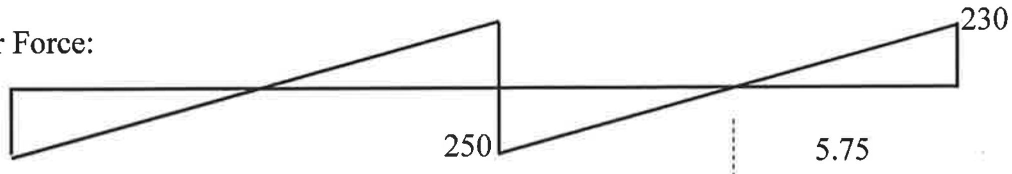
**Solution**

Total load = 960 kN

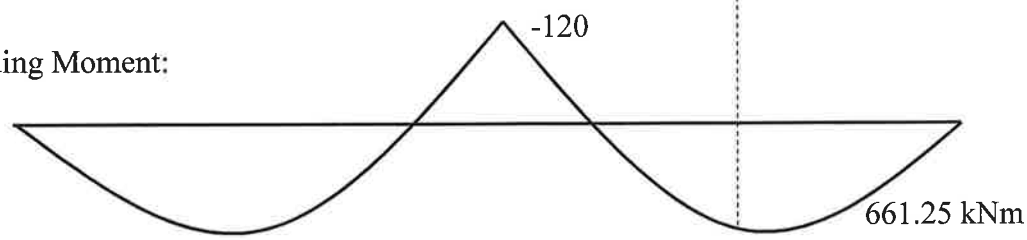
(a)




Shear Force:




Bending Moment:




(b)

Limit on singly reinforced moment capacity is:

M = $0.15 \times f_{cu} \times bd^2 = 0.15 \times 40 \times 250 \times 500^2 \times 10^{-6} = 375\ KNm$

Can be singly reinforced in hogging but not in sagging

(c)

<u>Hogging bending</u>
To find As first assume x = ½ then iterate

$$A_s = \frac{M}{0.87 f_y d\left(1 - \frac{x}{2}\right)} = \frac{120 \times 10^6}{0.87 \times 460 \times 500 \times (1 - 0.25)} = 800 mm^2$$

$$x = 2.175 \frac{f_y}{f_{cu}} \times \frac{A_s}{bd} = 0.16$$

So better estimate of x = 0.15 ⇒ As = 648 mm$^2$

(c) Present the main determinants of aggregate investment in the macroeconomy. [10]

Students should define investment as new spending on capital goods (physical assets) that will allow increased output of goods and services in the future. They should identify the role of investment as a component of the macroeconomy's aggregate demand and aggregate supply. A good answer should contain a clear identification of the objectives of investment decisions and address the role of interest rates with reference to the marginal efficiency of capital theory. The students should be able to mention the role of business confidence, the expected growth of demand, profitability, changes in the costs of purchasing capital inputs, technology shocks, and changes in the taxation on company profits or in the broader regulatory regime.

2 (a) What is the price elasticity of demand and why does it matter for consumers, producers and regulators? Illustrate your answer by providing different examples of goods or services. [5]

They students are expected to present the concept of elasticity as a way to measure the responsiveness of the quantity demand to price changes. They should include the technical definition of elasticity as the (unit free) proportionate change in quantity demanded/proportionate change in price, which is equal to the slope of the demand curve multiplied by the ratio between price and quantity. They should be able to provide examples of goods characterised by different demand elasticities and illustrate the extreme case of perfectly elastic and perfectly inelastic demand. In discussing the role of demand elasticity in the decisions made by consumers, producers and regulators, the students should consider consumer income relative to needs, the producers' profit maximising targets in relation to relatively elastic or inelastic demand curves, and the regulators objective of intervening to prevent undersupply of goods with highly inelastic demand on the ground of efficiency or equity.

(b) With reference to the problem of collusion in oligopolistic markets:

(i) Explain the rationale for collusive behaviours; [5]

In non-co-operative oligopoly firms act independently in their pursuit of profit. However, they have an opportunity to co-ordinate their choices strategically, they can collude – formally or tacitly – to maximise industry-wide profits, as shown in diagrammatic form in Lecture 4. The better students will be able to identify the conditions that make collusion possible as well as the difference between horizontal and vertical collusion.

x = 0.13 (near enough) ⇒ As = 641 mm$^2$ (Check x < 0.5 OK)

2 No. 10 mm bars + 1 No 25 mm bar = 648 mm$^2$

Fit?

2 x Cover (say 30 mm) + 2 shear bars (say 16) = 92 mm

92 + 2x10 + 1x25 = 137. Spaces: 56 mm => OK

Sagging bending

Maximum moment for sagging = 661.25 kNm

375 kNm can be carried by the concrete in compression. Comp. steel needed to carry 286.25 kNm

Assume d' = 60 mm (any reasonable value 40 – 100 mm could be accepted)

$$\therefore M = 0.15 f_{cu} b d^2 + 0.75 f_y A_s' (d - d')$$
$$661.25 \times 10^6 = 375 \times 10^6 + 0.75 \times 460 \times A_s'(500 - 60)$$

$=> A_s' = 1{,}884\ mm^2\ (4\ No\ 25 = 1{,}964\ mm^2)$

As required to balance the compressive force:

$$0.87 f_y A_s = 0.75 f_y A_s' + 0.2 f_{cu} b d => A_s = 4{,}123\ mm^2$$

5 No 32 mm bars just too small

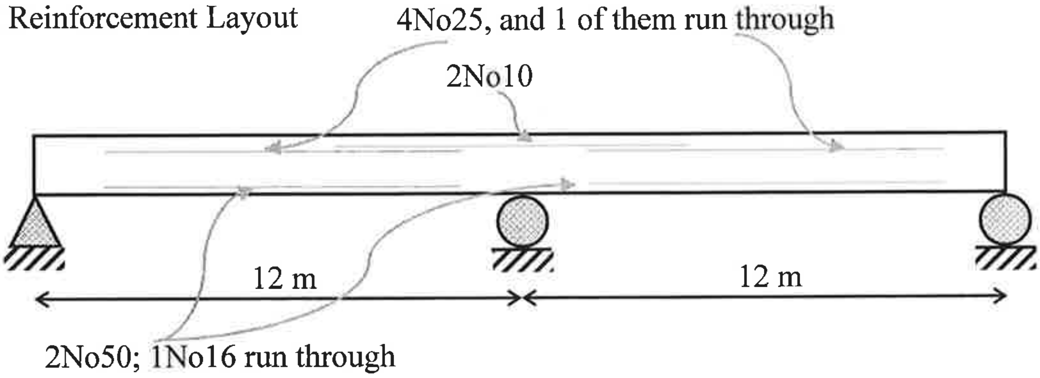
Could use 2 No 50 mm + 1 No 16 = 4,126 mm 2

Fit?

Compression area: 92 + 4x25 = 192. Spaces: 19 mm (just OK)

Tension area: 92 + 1x16 +2x50 = 208. Spaces: 21 mm (just OK)

(c) Reinforcement Layout




(d)

Maximum shear force = 250 kN

Nominal shear stress = $\frac{250\times10^3}{250\times500} = 2$ MPa

Shear links needed

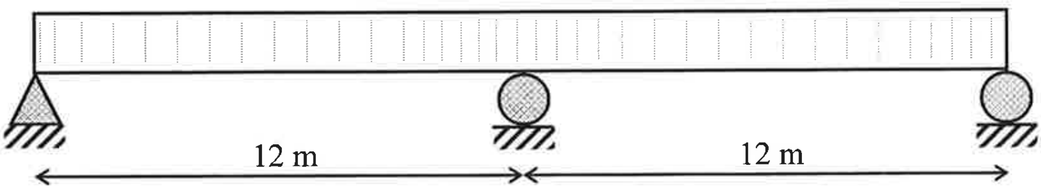
$$v_c = 0.68\sqrt[3]{\frac{100A_s}{bd}}\sqrt[4]{\frac{400}{d}} = 0.68 \times 1.4886 \times 0.946 = 0.96 \text{ N/mm}^2$$

$v_s = 2 - 0.96 = 1.04$ N/mm$^2$ (Assuming 2 No 12mm bars, $A_q$=226 mm$^2$)

$$s = \frac{0.87 f_y A_q}{v} = \frac{0.87 \times 460 \times 226}{250 \times 1.04} = 347.8\, mm\ (say\ 300) < \frac{3}{4}d, so\ OK$$

(e)

300 mm spacing at middle. Max 3/4d elsewhere.



**Engineering Tripos Part 1B**

**Paper 8, Selected Topics, Section C**

**Crib 2014/15 (M Sutcliffe)**

6 (a) There is no way to vary the synchronous speed of a cage rotor induction machine because it is not possible to make external connections to the rotor. Hence, this sort of machine will rotate at a speed very close to synchronous speed, which for a fixed-frequency grid is also fixed. An advantage is the simplicity and robustness of this sort of system, a disadvantage is that the tipspeed ratio of the turbine will vary as the wind speed varies, and consequently this type of system does not make optimal use of the power available at all wind speeds.

b) (i) p = 8 and the generator speed is 60f/p = 60×50/8= 375 rpm = 39.3 rad $s^{-1}$. This assumes that the slip is small and so the generator speed is very close to its synchronous speed.

$P = T\omega_r$, and so $T = 2\times10^6 / 39.3 = 50.9\text{kNm}$

(ii) Assume that the slip is small so that the induction machine impedance is dominated by $R_2^{'}/s$ giving the simplified torque equation:

$T = 3V^2 s/\left(\omega_s R_2^{'}\right)$ so $-50.9\times10^3 = 3\times\left(11000/\sqrt{3}\right)^2 s/\left(39.3\times2\right)$ giving $s = -0.0331$

The phase impedance is $Z = R_1 + R_2^{'}/s + \mathrm{j}\left(X_1 + X_2^{'}\right) = 2 + 2/(-0.0331) + \mathrm{j}(3+3)$ giving

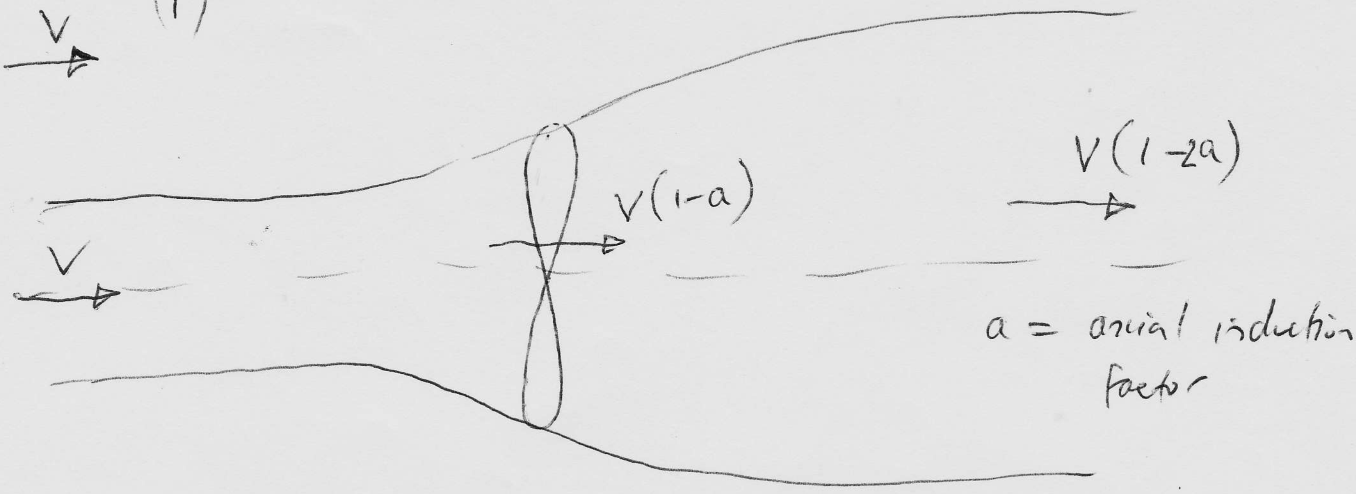
$Z = (-58.5 + j6)\Omega$

$I = V/Z = \left(11000/\sqrt{3}\right)/\left(-58.5 + j6\right) = 108\text{A}$, angle: -174°

(iii) Gearbox needs to increase turbine speed of 18 rpm to generator speed of 375 rpm giving a ratio of 375/18 = 20.8.

For part (a), many candidates knew the advantages/disadvantages of fixed-speed wind power, and that the induction machine should operate at very low slip and so the speed is essentially fixed for a fixed frequency grid. Only a few candidates pointed out that none of the techniques for adjusting the torque-speed curve are available for a cage rotor induction machine. Most candidates got the torque and speed out in part (b), but struggled with the slip and input current, caused by not using the simpler linearised version of the torque-slip equation.

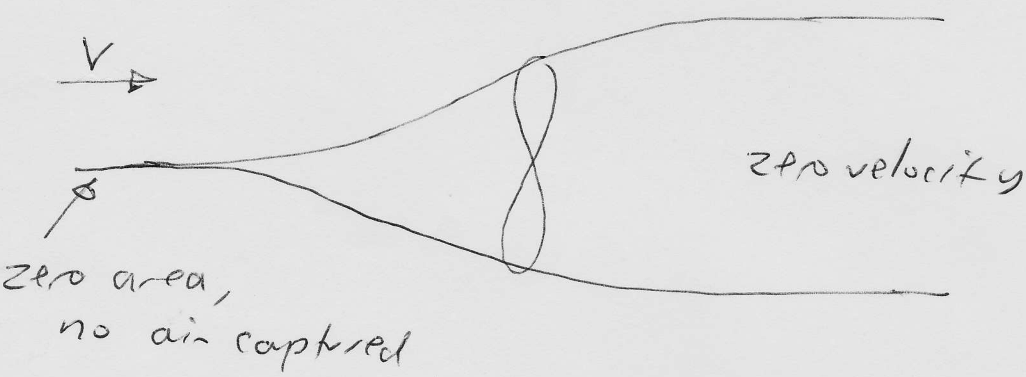
6 (c) (i)



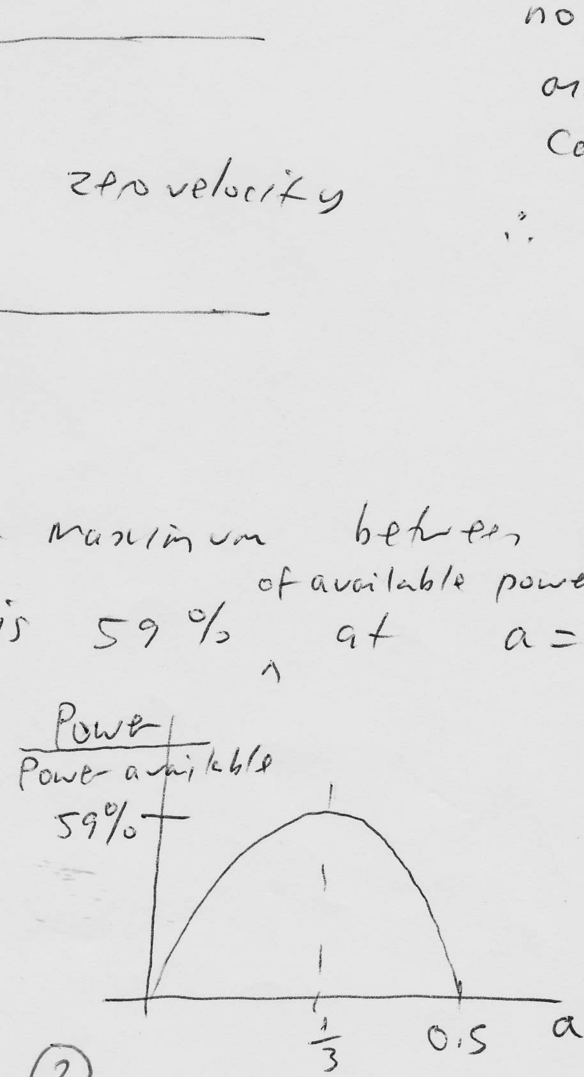

To extract energy from the wind, wind speed must drop - say to a speed $V(1-2a)$

If $a = 0$ then there is no change in speed ∴ no power

If $a = 0.5$ then the air stops but by continuity there can be no flow through the turbine so no moving air is captured ∴ no power



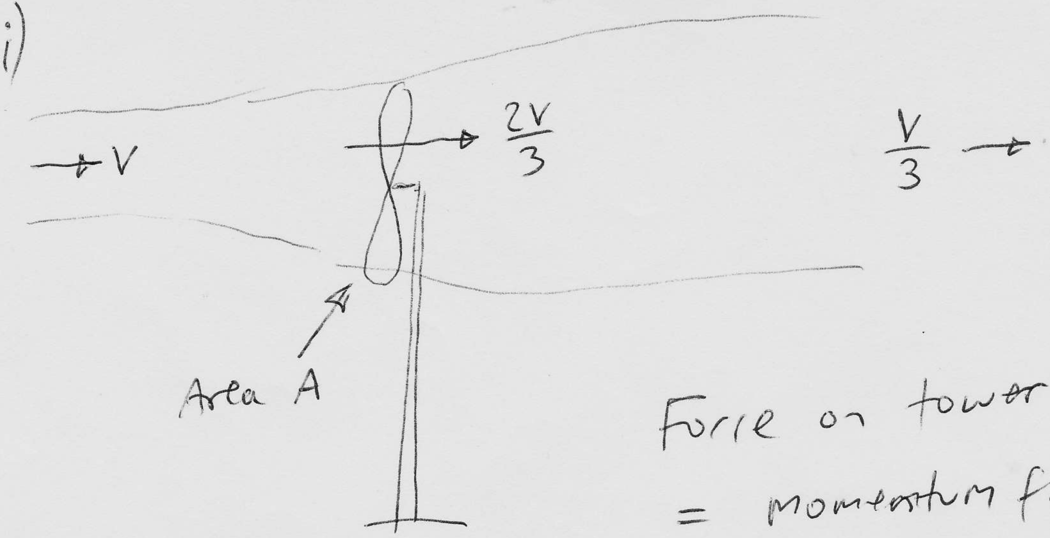

Power output has a maximum between $a = 0$ and $a = 1$ and this value is 59 % of available power at $a = \frac{1}{3}$ and is called the Betz limit




Reasonably well answered. Several sketches expected

2

6 (ii)




Force on tower

= momentum flow change

$$= \dot{m}\,\Delta v$$

$$= \rho A \frac{2V}{3}\left(V - \frac{V}{3}\right)$$

$$= \frac{4}{9}\rho A V^2$$

(iii)


The blades are subject to drag. Use

$F = C_D \frac{1}{2}\rho V^2 A$ for each blade

where $C_D$ is the drag coefficient,

and $A$ is the frontal area of the blade.

Best to feather the blades to minimize the frontal area. Also tower drag.

7 (a) Material selection will depend on a range of issues:

- mechanical performance. Weight reduction at the tower head is important to keep the other components light and cheap and to reduce self-weight loading. Also, with rotating parts, there is a danger of vibration. Both strength and stiffness are relevant, though the relative importance depends on details, specifically the scale of the blade. All of these point to materials with good specific properties such as wood, bamboo, CFRP and GFRP. CFRP is particularly good in fatigue, which will commonly be an issue.

- manufacturing issues. Key here is the need to shape the component. For the complex aerodynamic surface composites are good as they can be laid up over a mould. The size of component and number being produced is important. So making only a few may be OK to use a labour-intensive technique, such as hand lay-up of composites or hand forming of wood/bamboo parts. But for larger production runs some automation will be required, perhaps still involving some laying up of dry material then an infusion process. This may constrain the complexity of the component.

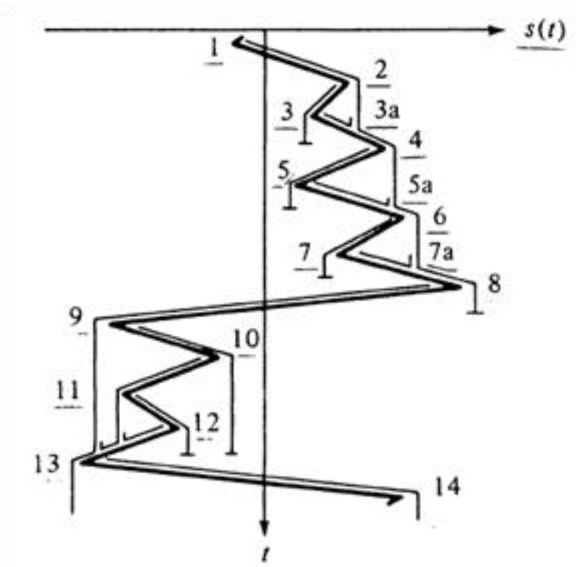
- cost is always important. Here a careful analysis needs to be done trading off the reduction in weight of the blades with the reduction in cost of other components. But in general it seems that the high mechanical demands of larger turbines is pointing these towards increasing amounts of CFRP, while smaller turbines the economics are not so favourable.

- sustainability. Need to ensure that the eco-cost of the materials is outweighed by the benefits of the power generation.

- corrosion – resistance to weathering, including salt off-shore environment

(b) A time varying signal can be split into loading cycles using the rainflow algorithm:

- Imagine rain **flowing down** the pagoda roof
- Rainflow **initiates** at each **peak** and **trough** and drips down
- When a flow-path **started** at a **trough** comes to the **tip** of the roof, the **flow stops** if the **opposite trough** is **more negative** than that at the **start** of the **path** under **consideration** [(1-8], [9-10]. A path started at a peak is stopped by a peak which is more positive than the peak at the start of the path [2-3], [4-5].
- If **rain** flowing down a roof **intercepts** a **previous path**, the **present path** is **stopped** [3-3a], [5-5a]
- A new path is not started until the path under consideration is stopped
- **Half cycles of loading** are projected distances on the stress axis [1-8], [3-3a], [5-5a]. This is the data which is needed to construct the load history.




Once the individual cycles of loading have been calculated, a spectrum of loading can be characterised by binning the cases into different mean stress and stress amplitude, or by fitting the data to mathematically distribution functions such as the exponential or Raleigh distributions.

[(a) Discussion of materials considerations tended to cover only part of the range of factors which are important, with cost and manufacturing often being omitted. The effect of scale on materials choice was also rarely mentioned. (b) Reasonably well done, with plausible sketches in the main. (c) The analysis was relatively straightforward and done fine by many people.]

7(c)(i) Miners rule

$$\sum_i \frac{N_i}{N_{fi}} = 1$$

| $S$ | $N_{fi} = \left(\frac{S}{S_0}\right)^{-m}$ | $N_i$/month |
|---|---|---|
| 100 | $6.73 \times 10^{20}$ | $10^6$ |
| 300 | $1.9 \times 10^{11}$ | $2 \times 10^5$ |
| 500 | $7.05 \times 10^6$ | $2 \times 10^4$ |

combine 0 and 50 mean stress counts for this part - this not done by some candidates

$\sum \frac{N_i}{N_{fi}}$ per month =

$$\cancel{\frac{10^6}{6.73 \times 10^{20}}} \xrightarrow{} 0 + \cancel{\frac{2 \times 10^5}{1.9 \times 10^{11}}} \xrightarrow{} 0 + \frac{2 \times 10^4}{7.05 \times 10^6}$$

$$\simeq 2.835 \times 10^{-3}$$

$$= 1/353$$

so lifetime used up in 353 months $\simeq$ 29 years

(ii) Use Goodman's rule

Assume as above that only $S = 500$ cycles contribute to life.

$$\Delta S = \Delta S_0 \left(1 - \frac{\sigma_m}{\sigma_{TS}}\right) \Rightarrow \Delta S_0 = \frac{500}{1 - \frac{50}{550}} = 550 \text{ MPa}$$

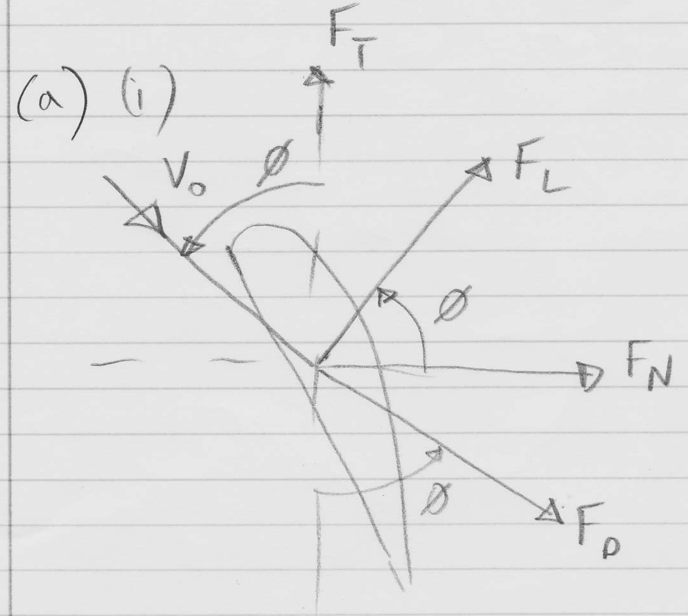
$$\sum \frac{N_i}{N_{fi}} = \underbrace{\frac{10^4}{7.05 \times 10^6}}_{\text{as before}} + \underbrace{\frac{10^4}{1.05 \times 10^6}}_{\text{dominant factor}} = \frac{1}{91.3}$$

Lifetime = 7.6 years

(d) Lifetime dominated by high stress cycles with mean stress very significant. Large value of $m$ gives this sensitivity. Predicted lifetime too short - need to redesign. Need to avoid overloads in practice and model / measure high $\sigma$ part of spectrum carefully.

8

(a) (i)




Just a few confusing $\theta$ and $\phi$

$F_N = F_L \cos\phi + F_D \sin\phi \Rightarrow C_N = C_N \cos\phi + C_D \sin\phi$

$F_T = F_L \sin\phi - F_D \cos\phi \Rightarrow C_T = C_L \sin\phi - C_D \cos\phi$

(ii) Positive torque providing $F_T$ is +ve, i.e. $C_T$ +ve

$\therefore \quad C_L \sin\phi > C_D \cos\phi$

$\therefore \quad \dfrac{C_L}{C_D} > \cot\phi$

(b) (i) At blade mid-point $r = 4$m

Solidity $\quad \sigma = \dfrac{Bc}{2\pi r} = \dfrac{6 \times 0.5}{2\pi \times 4} = 0.119$

$\simeq 0.12$

8 (b) (ii)

$B = 6 \qquad \theta = 12° \qquad r = 3 \text{ to } 5m \qquad c = 0.5m$

$C_L = 2\pi\alpha \qquad C_D = 0.1 \qquad \text{for } 0 < \alpha < 0.3 \text{ rad}$

$V_0 = 5m/s \qquad \omega = 30 \text{ rpm} = \pi \text{ rad/s}$

$V_0/\omega r = \frac{5}{\pi \times 4} = 0.4 \qquad \sigma = \frac{Bc}{2\pi r} = \frac{6 \times 0.5}{2\pi \times 4} = 0.12$

1st guess: $a = 0 \qquad a' = 0$

$\phi = \tan^{-1}\left[\frac{(1-a)}{(1+a')}\frac{V_0}{\omega r}\right] \quad \therefore \quad \phi = 21.7°$

$\alpha = \phi - \theta \quad \therefore \quad \alpha = 9.7°$

$C_L = 0.11\alpha$ ($\alpha$ in deg) $\quad \therefore \quad C_L = 1.06 \qquad C_D = 0.1$

$C_N = C_L \cos\phi + C_D \sin\phi \quad \therefore \quad C_N = 1.025$

$C_T = C_L \sin\phi - C_D \cos\phi \quad \therefore \quad C_T = 0.300$

Recompute: $a = \left[\frac{4\sin^2\phi}{\sigma C_N} + 1\right]^{-1} \quad \therefore \quad a = 0.183$

$a' = \left[\frac{4\sin\phi\cos\phi}{\sigma C_T} - 1\right]^{-1} \quad \therefore \quad a' = 0.027$

2nd iteration

$\phi = 17.6°$
$\alpha = 5.6°$
$C_L = 0.611 \qquad C_D = 0.1$
$C_N = 0.613$
$C_T = 0.089$

thus $a = 0.167$ and $a' = 0.009$

3rd iteration

$\phi = 18.2°$
$\alpha = 6.2°$
$C_L = 0.677 \qquad C_D = 0.1$
$C_N = 0.675$
$C_T = 0.116$

thus $a = 0.171$ and $a' = 0.012$

Converged values: $\underline{\underline{0.171}} \qquad \underline{\underline{0.011}}$

(7)

8 (b) (iii) Change in angular momentum:

$$\delta T = BF_T\, r\, \delta r = \delta \dot{m}\, 2\omega r a' r$$

$$= \rho V_0 (1-a)\, 2\pi r\, \delta r \times \underbrace{2\omega r a'}_{\text{change in tangential velocity} = u_\theta} r$$

$$= \rho V_0 (1-a)\, 4\pi r^3 \omega a' \delta r$$

$\delta r = 5 - 3 = 2\,\text{m}$

$$\text{Power} = \omega\, \delta T$$

$$= \rho V_0 (1-a)\, 4\pi r^3 \omega^2 a' \delta r$$

$$= 1.2 \times 5\,(1-0.171) \times 4\pi \times 4^3 \times \pi^2 \times 0.011 \times 2$$

$$= 870\ \text{W}$$

$$\left( C_p = \frac{870}{\frac{1}{2}\rho V_0^3 \pi R^2} = \frac{2 \times 870}{1.2 \times 5^3 \times \pi \times 5^2} = 0.15 \right)$$

(b) - incorrect expressions for solidity or calculation errors crept in for (ii)

b (iii) This part not done very well

9. a)

$$\frac{p_{02}}{p_a}=\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}} \Rightarrow M=\sqrt{\left(\left(\frac{p_{02}}{p_a}\right)^{\frac{\gamma-1}{\gamma}}-1\right)\frac{2}{\gamma-1}}=\sqrt{\left(\left(\frac{45.1}{28.7}\right)^{\frac{0.4}{1.4}}-1\right)\frac{2}{0.4}}=\underline{0.830}$$

$$T_{02}=T_a\left(1+\frac{\gamma-1}{2}M^2\right)=225\times\left(1+\frac{0.4}{2}0.83^2\right)=\underline{256\ K}$$

[4]

b)

(i) $$\eta_{is,c}=0.9=\frac{\left(\frac{p_{03}}{p_{023}}\right)^{\frac{\gamma-1}{\gamma}}-1}{\frac{T_{03}}{T_{023}}-1} \Rightarrow \frac{T_{03}}{T_{023}}=294\times\left[\frac{(30)^{\frac{\gamma-1}{\gamma}}-1}{0.9}+1\right]=\underline{830.6\text{ K}}$$ [3]

(ii) Work balance for the HP shaft:

$$\dot{m}c_p\left(T_{03}-T_{023}\right)=\dot{m}c_p\left(T_{04}-T_{045}\right)$$

$$\Rightarrow T_{045}=T_{04}-\left(T_{03}-T_{023}\right)=1550-(830.6-294)=\underline{1013\text{ K}}$$ [2]

(iii) Energy from fuel = Combustor enthalpy rise

$$\dot{m}_{fuel}LCV=\dot{m}c_p\left(T_{04}-T_{03}\right)$$

$$\dot{m}_{fuel}=\dot{m}c_p\left(T_{04}-T_{03}\right)/LCV=80\times1005\times(1550-830.6)/43.6\times10^{-6}=\underline{1.35\ kg/s}$$

[2]

c) Work balance for the LP shaft:

$$\dot{m}c_p\left(T_{023}-T_{02}\right)+BPR.\dot{m}c_p\left(T_{013}-T_{02}\right)=\dot{m}c_p\left(T_{045}-T_{05}\right)$$

$$T_{05}=T_{045}-\left[\left(T_{023}-T_{02}\right)+BPR\left(T_{013}-T_{02}\right)\right]$$

$$\Rightarrow T_{05}=1013-\left[(294-256)+10.5\times(290-256)\right]=618\text{K}$$

For the LP Turbine,

$$\frac{p_{05}}{p_{045}}=\left(1-\frac{1-\frac{T_{05}}{T_{045}}}{\eta_t}\right)^{\frac{\gamma}{\gamma-1}},\ p_{045}=70\times30/6=\underline{350\text{ kPa}}\Rightarrow p_{05}=350\times\left(1-\frac{1-\frac{618}{1013}}{0.92}\right)^{\frac{1.4}{.4}}=50.8\ kPa$$

For the exhaust jet, given the nozzle is isentropic:

$$0.5V_9^2=c_p\left(T_{05}-T_9\right)=c_pT_{05}\left(1-\left(p_9/p_{05}\right)^{(\gamma-1)/\gamma}\right)$$

$$V_9=\sqrt{2c_pT_{05}\left(1-\left(p_9/p_{05}\right)^{(\gamma-1)/\gamma}\right)}=\sqrt{2\times1005\times618\times\left(1-(28.7/50.8)^{.4/1.4}\right)}=\underline{432\text{ m/s}}$$

[8]

(d) The mass flow rate of fuel represents the rate of energy input to the cycle and is therefore non-dimensionalised as a power, $A_2 p_{02}\sqrt{c_p T_{02}}$ , as follows:

$$\frac{\dot{m}_{fuel} LCV}{A_2 p_{02}\sqrt{c_p T_{02}}}$$

In contrast, the mass flow rate of air represents the momentum flow into the engine and is non-dimensionalised like a force, as follows:

$$\frac{\dot{m}\sqrt{c_p T_{02}}}{A_2 p_{02}}$$

For the test, $\dot{m}_{fuel} = \dot{m}_{fuel,cr} \times \dfrac{p_{02}}{p_{02,cr}}\sqrt{\dfrac{T_{02}}{T_{02,cr}}} = 1.35 \times \dfrac{102}{45.1} \times \sqrt{\dfrac{285}{256}} = \underline{3.22 kg/s}$

[6]

10. a)

[4]

b)

$$w_{net} = w_t - w_c = c_p\left[(T_{04} - T_{05}) - (T_{03} - T_{02})\right] = c_p T_{02}\left[\theta\left(1 - \frac{T_{05}}{T_{04}}\right) - \left(\frac{T_{03}}{T_{02}} - 1\right)\right]$$

Compressor: $\eta = \dfrac{r_T - 1}{\dfrac{T_{03}}{T_{02}} - 1}$ Turbine: $\eta = \dfrac{1 - \dfrac{T_{05}}{T_{04}}}{1 - 1/r_T}$

$$\Rightarrow \underline{\frac{w_{net}}{c_p T_{02}} = \left[\theta(1 - 1/r_T)\eta - (r_T - 1)/\eta\right]}$$

[4]

c)

$$\eta_{cyc} = \frac{w_{net}}{q_{in}} = \frac{w_{net}}{c_p(T_{04} - T_{03})} = \frac{w_{net}}{c_p T_{02}(\theta - T_{03}/T_{02})} = \frac{w_{net}}{c_p T_{02}(\theta - 1 - (r_T - 1)/\eta)}$$

$$\Rightarrow \underline{\eta_{cyc} = \frac{\theta(1 - 1/r_T)\eta - (r_T - 1)/\eta}{\theta - 1 - (r_T - 1)/\eta}}$$

[4]

(d) For maximum work,

$$\frac{\partial}{\partial r_T}\left(\frac{w_{net}}{c_p T_{02}}\right) = \theta\eta r_T^{-2} - 1/\eta = 0 \qquad \therefore \theta\eta = r_T^2/\eta, \qquad r_T = \eta\sqrt{\theta}$$

For $\eta = 0.9$, $\theta = 6$, $r_T = 0.9\sqrt{6} = 2.20$ $r = 2.20^{1.4/0.4} = \underline{15.9}$

At this pressure ratio ($r$=15.9), $\eta_{cyc} = \dfrac{6(1 - 1/2.2)0.9^2 - 1.2}{5 \times 0.9 - 1.2} = \underline{0.440}$ *or* 44%

At the higher pressure ratio ($r$=50), $r_T = 50^{.4/1.4} = 3.06$

$$\eta_{cyc} = \frac{6(1 - 1/3.06)0.9^2 - 2.06}{5 \times 0.9 - 2.06} = \underline{0.497} \quad or \quad 49.7\%$$

i.e. higher $r$ gives an efficiency increase of almost 6%

[6]

e) Civil aircraft engines need to minimise fuel burn and therefore use a high $r$ to give improved cycle efficiency for a given $\theta$. High $r$ is achieved through the use of highly loaded, multi-stage axial compressors and multiple shaft engines.

[3]

f) $\theta$ is limited by high temperature creep in the high pressure turbine rotor. This occurs when excessive metal temperatures are reached.
The technologies used to maximise $\theta$ include:
(i) single crystal casting
(ii) advanced creep resistant alloys with very high melting point
(iii) blade cooling - cooling flow within blades and film cooling over surfaces

[4]

11.a)

$$C_L = \frac{L}{0.5\rho AV^2} = \frac{L}{0.5\frac{p}{RT}AM^2\gamma RT} = \frac{L}{0.5\gamma pAM^2}$$

[2]

b) From the figure, for max $ML/D$, require $\underline{C_L = 0.5\ ,\ M = 0.84}$

At start of cruise, $\quad W_{start} = 9.81\times(285000+236000+512\times 95) = 5.588$ MN

$$p_{req} = \frac{L}{0.5\gamma C_L AM^2} = \frac{5.588\times 10^6}{0.5\times 1.4\times 0.5\times 854\times 0.84^2} = 26.5\ kPa$$

$p/p_{sl} = 0.2615 \quad \Rightarrow \underline{h = 10000\, m}$ (from data book)

$T/T_{sl} = 0.7748 \quad \Rightarrow V = M\sqrt{\gamma RT} = 0.84\times\sqrt{1.4\times 287\times 0.7748\times 288.15} = \underline{251.6\ ms^{-1}}$

[6]

c) $(ML/D)_{\max} = 17.25\ ,\quad L/D = 17.25/0.84 = 20.54$

$$\dot{m} = \frac{F_N}{V_j - V} = \frac{0.25\times L/(L/D)}{V_j - V} = \frac{0.25\times 5.588\times 10^6/(20.54)}{360-251.6} = \underline{627.4\ kg/s}$$

$$\eta_o = \eta_p\eta_{th} = \frac{F_N V}{\dot{m}_{fuel}LCV} = \frac{1}{sfc}\times\frac{V}{LCV} \quad , \ \eta_p = \frac{2V}{V_j + V} = \frac{2\times 252}{360+252} = 0.823$$

$$sfc = \frac{1}{\eta_p\eta_{th}}\times\frac{V}{LCV} = \frac{1}{0.823\times 0.47}\times\frac{251.6}{43\times 10^6} = \underline{15.1\,\text{g kN}^{-1}\text{s}^{-1}}$$

[7]

d) At the end of cruise, $W_{end} = 9.81\times(285000+0.1\times 236000+512\times 95) = 3.504$ MN

$$s_{\max} = \frac{V\ L/D}{g\ sfc}\ln\left(\frac{W_{start}}{W_{end}}\right) = \frac{251.6\times 20.54}{9.81\times 15.1\times 10^{-6}}\ln\left(\frac{5.588}{3.504}\right) = \underline{16240\text{ km}}$$

$$p_{end} = \frac{W_{end}}{0.5\gamma C_L AM^2} = \frac{3.504\times 10^6}{0.5\times 1.4\times 0.5\times 854\times 0.84^2} = 16.6\ kPa$$

$p_{end}/p_{sl} = 0.1640 \quad \Rightarrow \underline{h_{end} = 13000\, m}$ (from data book)

[6]

e) Cruising at constant altitude means that the atmospheric pressure, $p$ in the lift coefficient equation is constant. At a constant Mach number, $C_L$ will reduce during the flight leading to a reduction in $ML/D$ and therefore reduced range. To increase $C_L$ the flight Mach number $M$ should be reduced and therefore the optimal Mach number when cruising at constant altitude will decrease.

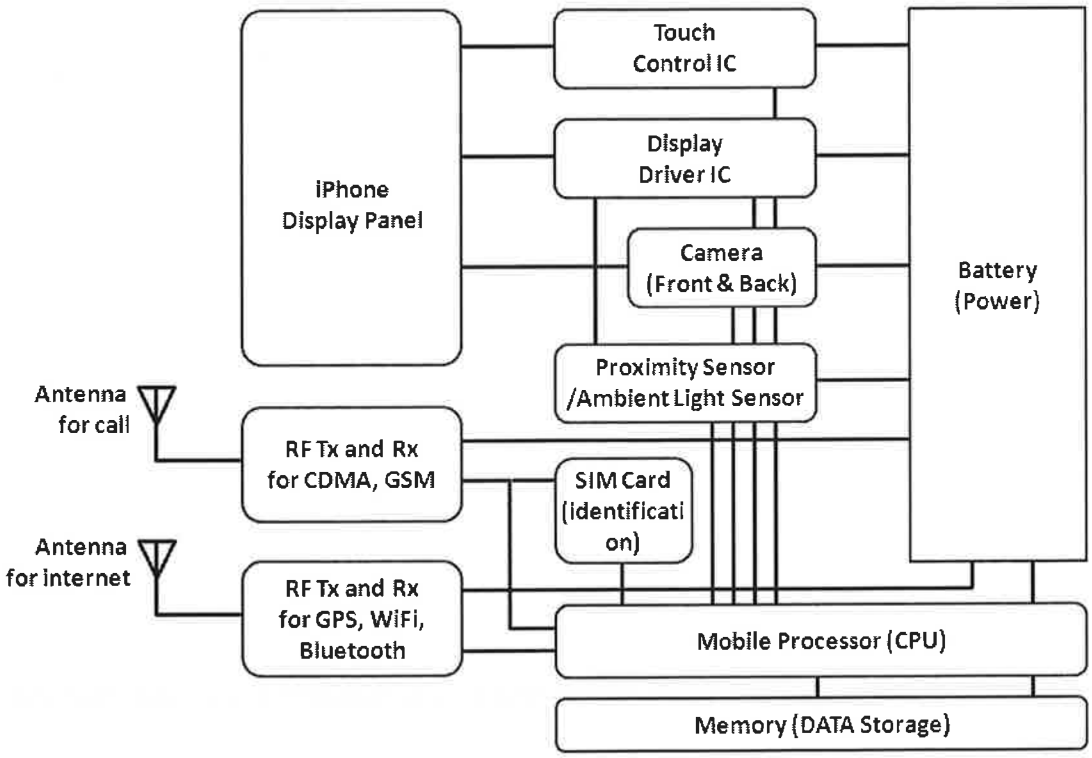
[4]

**Dr C. A. Hall**

## Question 12

(a) With the aid of a block diagram, illustrate with brief descriptions the various principal parts (i.e. functional blocks) and their interconnection in an *iPhone 5*? [5 marks]

**➔Answer:**




- **Mobile Processor (CPU) which controls / manages all blocks (0.5 marks)**
- **Display IC which controls display arrays (0.5 marks)**
- **Touch IC which controls touch input (0.5 marks)**
- **Memory elements which stores data (0.5 marks)**
- **Antenna and Tx/Rx block for CDMA / GSM (0.5 marks)**
- **Antenna and Tx/Rx block for GPS/WiFI/Bluetooth (0.5 marks)**
- **Battery to supply power (0.5 marks)**
- **SIM card for user identification (0.5 marks)**
- **Front and Back Camera modules (0.5 marks)**
- **Proximity and Ambient Sensors (0.5 marks)**

(b) The 1.94 inch × 3.48 inch display screen in an *iPhone 5*, referred to as Retina display, has 632 x 1134 pixels. Assuming all pixels are the same size, what is the display resolution?

**➔Answer:**

**Since we assumed that all pixels are the same size and square-sized, we can approximate the resolution as follows:**

**632 / 1.94 = 326 pixels per inch or 326 ppi or 128 ppcm**

**or alternatively, 1134 / 3.48 = 326 ppi (5 marks)**

(c) Name three basic thin film transistor (TFT) technologies that are of current interest in display screens? You can tabulate their relative performance in terms of the key performance attributes: mobility, temporal stability, spatial uniformity, and cost. [5 marks]

**➔Answer:**

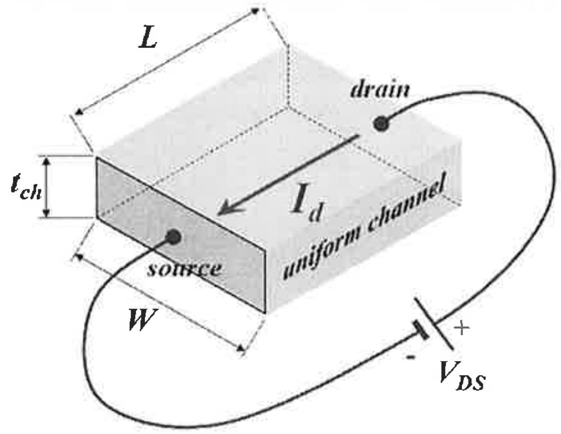
| **Attribute** | **a-Si:H (hydrogenated amorphous silicon)** | **LTPS (low temperature poly-silicon)** | **AOS (amorphous oxide semiconductor)** |
|---|---|---|---|
| Mobility ($cm^2V^{-1}s^{-1}$) | **< 1** | **30-100** | **<50** |
| Temporal stability | **poor** | **high** | **high** |
| Spatial uniformity | **high** | **low** | **medium** |
| Cost | **low** | **high** | **low** |

***Note that each one in red is 0.3 marks, and fully filled table is full marks (5 marks).**

(d) A thin film transistor is being made from the layered semiconductor $WSe_2$ (Tungsten Diselenide), whose electron mobility ($\mu$) is 500 $cm^2V^{-1}s^{-1}$. The transistor will have channel width $W = 200$ μm, gate length $L = 20$ μm, and channel thickness ($t_{ch}$) of 3 monolayers. Each monolayer is 0.68 nm thick. Assuming that the gate induces a uniform carrier density ($N$) in the channel, what is $N$ if the drift current ($I_d$) is 0.1 μA for a source-drain voltage of 5 V

[10 marks]

**➔Answer:**




**Current equation based on drift process (7 marks)**

$$I_d = W \cdot t_{ch} \cdot e \cdot N \cdot \mu \cdot \frac{V_{DS}}{L} \rightarrow N = \frac{L \cdot I_d}{W \cdot t_{ch} \cdot e \cdot \mu \cdot V_{DS}}$$

**Calculation of N (3 marks)**

$$N = \frac{(20\times10^{-4}\,cm)\times(0.1\times10^{-6}\,A)}{(200\times10^{-4}\,cm)\times(3\times0.68\times10^{-7}\,cm)\times(1.6\times10^{-19})\times(500cm^2V^{-1}s^{-1})\times(5V)}$$

$$= 1.22549\times10^{14}\,cm^{-3} \;\; or \; = 1.22549\times10^{20}\,m^{-3}$$

*Parameters values given in the question:

$W = 200\mu m = 200 \times 10^{-4} cm,$

$L = 20\mu m = 20 \times 10^{-4} cm,$

$channel\ thickness\ (t_{ch}) = 3 \times 0.68nm = 3 \times 0.68 \times 10^{-7} cm,$

$\mu = 500cm^2V^{-1}s^{-1},$

$I_d = 0.1 \times 10^{-6} A,$

$e = 1.6 \times 10^{-19},$

$V_{DS} = 5V,$

# Question 13

(a) Explain how conductivity is controlled in a semiconductor, using the format of bullet points. [5 marks]

**➔Answer:**

- **Conductivity (σ) can be increased with increasing mobility (μ), following *σ=enμ*, where e is the elementary charge (1 mark).**
- **It can also be increased as increasing carrier density (n), following *σ=enμ*, where e is the elementary charge (1 mark).**
- **Depending on the impurity concentration, and assuming full ionization, conductivity decreases with increasing temperature at medium temperatures, since mobility is reduced due to lattice scattering (1 mark).**
- **At very high temperatures, there is thermal activation of electrons, which serves to increase the conductivity despite decreasing mobility (1 mark).**
- **Conductivity decreases at very low temperatures where there is impurity free-out, and despite mobility increasing with decreasing temperature, the decreasing carrier density dominates (1 mark).**

(b) Using the format of bullet points, compare Si and GaAs for use in high performance field effect transistors. [5 marks]

**➔Answer:**

- **Speed of n-channel Si field effect transistor (FET) is higher than the GaAs FET at higher field strengths because of the higher velocity saturation in Si, despite the (six times) lower electron mobility of Si compared to GaAs FET (1 mark).**
- **Speed of p-channel Si FET can be similar to that of p-channel GaAs FET since hole mobility of Si is similar to that of GaAs (1 mark).**
- **Si FET can be an inversion type transistor while inducing electrons on p-type substrate since the interface trap density is small enough due to a good matching with $SiO_2$ gate insulator (1 mark).**

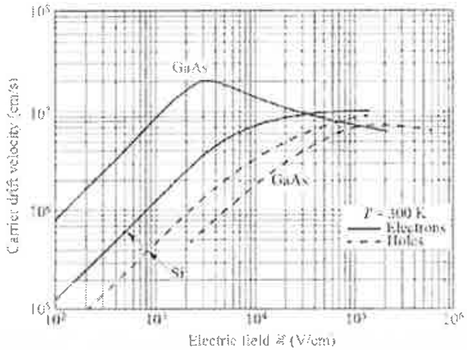
- **GaAs has much more interface traps due to mismatch with gate dielectric, suggesting that it is difficult to make channel inversion, thus it is mostly used for depletion mode transistor (1 mark).**
- **Si FET is normally off due to inversion operation, thus lower power consumption compared to GaAs FET which is normally on due to depletion mode operation (1 mark).**

(c) Sketch the velocity vs. electric field characteristic of an electron in Si and GaAs, and explain the various features of those characteristics. What causes the limiting velocity?

[10 marks]

**➔Answer:**

**Sketch of the velocity vs. E-field (5 marks):**




**Origin of the velocity saturation (5 marks):**

At sufficiently high electric fields, the carriers start interacting with optical phonons and the drift velocity becomes less and less dependent on the applied field thus approaching saturation.

(d) If the mobility in a semiconductor $WS_2$ is 0.04 $m^2V^{-1}s^{-1}$, and the limiting velocity ($v_s$) is $2.5\times10^5$ m/s, using Newton's Laws or otherwise, relate the mobility ($\mu$) to the effective electron mass ($m^*$) and mean free time ($\tau$) between collisions. Hence, from the limiting velocity $v_s$, calculate the energy lost ($E_{lost}$) per collision in eV. Assuming $m^*=m_o$, calculate the mean free path length ($l$) between collisions, where, $m_o$ is the free mass ($9.1\times10^{-31}$ kg) and e the elementary charge ($1.6\times10^{-19}$ coulombs). [5 marks]

**➔Answer:**

**Mobility and effective mass / mean free time, and calculation (2 marks):**

$$\mu = \frac{e\tau}{m^*} \Rightarrow \tau = \frac{\mu m^*}{e}$$

$$\therefore \tau = \frac{\left(0.04m^2V^{-1}s^{-1}\right)\times\left(1\times9.1\times10^{-31}kg\right)}{1.6\times10^{-19}\,coulombs} = 2.275\times10^{-13}\,s$$

*Parameters values given in the question:

$\mu = 0.04 m^2 V^{-1} s^{-1}$,

$m^* = 1 \times m_o = 1 \times 9.1 \times 10^{-31} kg$,

$e = 1.6 \times 10^{-19} coulombs$

**Energy loss Calculation (2 marks):**

$$E_{lost} = \frac{1}{2} m^* v_s^2$$

$$E_{lost} = 0.5 \times (1 \times 9.1 \times 10^{-31} kg) \times (2.5 \times 10^5 m/s)^2 = 2.84375 \times 10^{-20} J$$

$$\Rightarrow \frac{2.84375 \times 10^{-20} J}{1.6 \times 10^{-19} coulombs} = 0.17773 eV \approx 0.18 eV$$

*Parameters values given in the question:

$m^* = 1 \times m_o = 1 \times 9.1 \times 10^{-31} kg$,

$v_s = 2.5 \times 10^5 m/s$

**Mean free path calculation (1 mark):**

$$v_s = \frac{l}{\tau} \Rightarrow l = v_s \tau$$

$$l = (2.5 \times 10^5 m/s) \times (2.275 \times 10^{-13} s) = 5.6875 \times 10^{-8} m = 56.875 nm$$

*Parameters values given in the question:

$v_s = 2.5 \times 10^5 m/s$

$\tau = 2.275 \times 10^{-13} s$

# Question 14

(a) What is the evidence that electrons can behave like waves? [5 marks]

**➔Answer:**

Moving electrons seem to be guided to an interference pattern just like waves of light or just like photons of light **(5 marks):**

(b) Derive an expression for a wavelength $\lambda$ of an electron in terms of its kinetic energy $E_K$ and its effective mass $m^*$. [5 marks]

**➔Answer:**

**Wavelength and the Kinetic energy can be stated as (4 marks):**

$$\lambda = \frac{2\pi}{h}$$

$$E_K = \frac{\hbar^2 k^2}{2m^*} \text{ or } k = \frac{\sqrt{2m^* E_K}}{\hbar}$$

**The relationship between wavelength and other parameters (1 mark):**

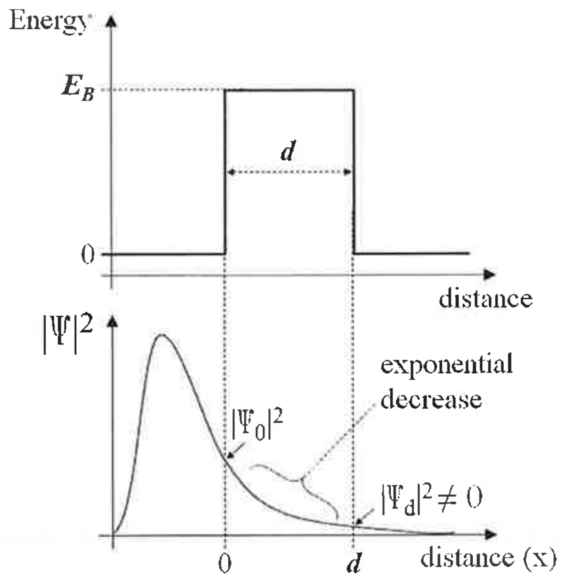
$$\therefore \lambda = \frac{2\pi\hbar}{\sqrt{2m^* E_K}} \text{ or } \frac{h}{\sqrt{2m^* E_K}}$$

(c) Explain what is meant by quantum mechanical tunnelling, for example by sketching the wavefunction of an electron travelling from left to right in the above diagram. Here, $E_B$ is the barrier height, and $E_K$ is the electron's kinetic energy. [5 marks]

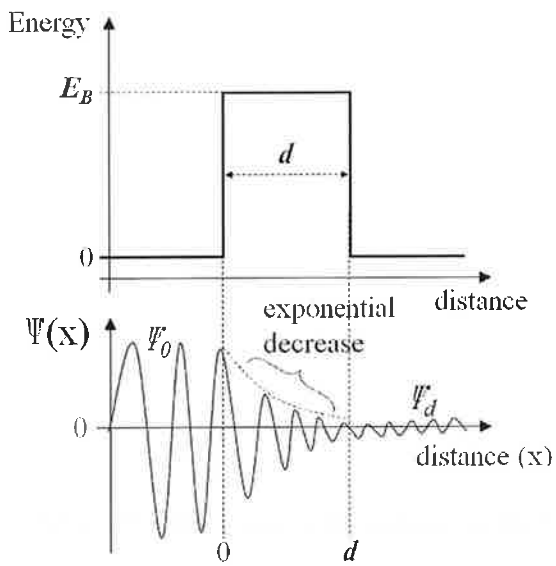
➔**Answer:**

- **Quantum mechanical tunnelling of electrons happens through a barrier of finite height and thickness (1 mark).**

**Sketch of the wavefunction of an electron travelling (3 marks):**




or




- **Since the boundary condition at x=d is not zero, there is probability of the presence of electrons, thus tunnelling from left to right (1 mark).**

(d) Derive an approximate expression for the tunnelling probability ($T$) in terms of the parameters $E_K$, $m^*$, $E_B$, and the barrier thickness is $d$. [5 marks]

➔**Answer:**

**In the barrier we have (2 mark):**

$$k = \frac{\sqrt{2m^*(E_B - E_K)}}{\hbar} \quad \textit{in barrier}$$

**The tunnelling probability in the presence of the barrier is (2 mark):**

$$T \equiv \left(\frac{\Psi_d}{\Psi_0}\right)^2 = \exp(-2kd)$$

**Thus the tunnelling probability can be stated as (1 mark):**

$$\therefore T = \exp\left(-2\frac{\sqrt{2m^*(E_B - E_K)}}{\hbar}d\right)$$

(e) Calculate $T$ for the case of 1.5 nm thick $SiO_2$ layer, assuming that the effective electron mass ($m^*$) is 0.5 of the free mass ($m_o$), $E_K$ = 0.5 eV, and $E_B$ = 3.5 eV, taking $m_o = 9.1\times10^{-31}$ kg, the elementary charge = $1.6\times10^{-19}$ coulombs, and Planck's constant $h = 6.626 \times 10^{-34}$ kg·m$^2$·s$^{-1}$. [5 marks]

**➔Answer:**

**The probability is (5 marks):**

$$T = \exp\left(-2\frac{\sqrt{2\times(0.5\times9.1\times10^{-31}\,kg)\times(3.5-0.5)\times1.6\times10^{-19}\,coulombs}}{(6.626\times10^{-34}\,kg\cdot m^2\cdot s^{-1})/2\pi}\times(1.5\times10^{-9}\,m)\right)$$

$$\therefore T = \exp(-18.8014) = 6.83346\times10^{-9} \approx 6.83\times10^{-9}$$

*Parameters values used:

$m^* = 0.5m_o = 0.5\times9.1\times10^{-31}\,kg,$

$E_B = 3.5eV = 3.5\times1.6\times10^{-19}\;\;V,$

$E_K = 0.5eV = 0.5\times1.6\times10^{-19}\;\;V,$

$d = 1.5nm = 1.5\times10^{-9}\,m,$

$h = 6.626\times10^{-34}\,kg\cdot m^2\cdot s^{-1},$

28/1/15

# Solutions (section F)

RC var 2.

Q15 a

(a) (i) Spatial lowpass filtering will tend to blur the image as it is approximately equivalent to convolving the image with a circular 'out-of-focus' blurring kernel. Unit gain at low frequencies will ensure that the intensities of pixels in larger smooth regions are unaffected by the filter. Lowpass filtering tends to reduce the visibility of noise as it produces an averaging effect on the pixels within the point-spread-function (psf) of the filter.

(b) (ii) Spatial highpass filtering will tend to sharpen edges in the image, and is used approximately to correct for the effects of blur (ie to perform deblurring). Unit gain at low frequencies again ensures that larger smooth regions are unaffected. It does however tend to increase the visibility of any noise in the image.

(c) (iii) Scaling of pixels by 2 will double the brightness of the red, green and blue components of each pixel. This will produce a much brighter image with stronger colours everywhere, except in brighter regions where one or more of the colour components are likely to saturate at a maximum value (typically 255) and make the brightest pixels tend towards 'white' (255,255,255).

(d) (iv) Adding 64 to pixel values will just make all pixels lighter by a constant amount. This is equivalent to flooding the whole image with a 'grey' light (RGB = 64,64,64) in addition to all the reflected light from objects, and so the image will appear 'misty'.

2. (b) YUV is a linear conversion from RGB colour space that can be achieved by multiplying each 3-element vector R,G,B by a 3x3 matrix C to give an output vector Y,U,V. Y represents the apparent intensity or brightness of the pixel and typically $Y = 0.3R + 0.6G + 0.1B$, while U and V represent colour-difference components that are proportional to (B - Y) and (R - Y) respectively.

   HSV is a non-linear conversion from RGB to hue, saturation and value (H,S,V). V is the max value from R, G and B; S is 1.0 minus the ratio of the min value of R, G and B to V; and H is a number between 0 and 1 which indicates approximately where the RGB colour lies on the visible spectrum. Usually V is also normalised to lie between 0.0 and 1.0, so that all three components lie in this range. HSV is an attempt to represent colour in a perceptually meaningful way, with V representing the intensity (brightness), S the saturation (colour strength), and H the hue of the colour.

3. (c) To increase the strength of colours is easiest with HSV, as one just scales up S by a constant factor (limiting it to 1.0 to keep within the allowed colour space). With YUV pixels, we must scale up both U and V by the same amount to increase the colour strength without altering the hue. With RGB pixels, we can scale all three components up by the same

Q1b

Q1.(a) Raw pixels — sensitive to changes in nature and distribution of lighting and changes in optics of camera (ie. changes to contrast + brightness)

<u>Normalize</u> — edge dection or use gradients
— normalized intensities by subtracting mean and normalising by variance.

to make less sensitive.

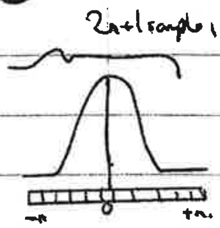
(b)(i) Low-pass filter to reduce noise before differentiation — gaussian kernel
Use sampl 1D Gaussian. $g_\sigma(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$

$= g_\sigma(x,y) * I(x,y)$

$$S(x,y) = \sum_{-n}^{n}\sum_{-n}^{n} I(x-u, y-v)\, g_\sigma(u,v)$$

$$g_\sigma(u,v) = g_\sigma(u) \cdot g_\sigma(v)$$

$$= \sum_{-n}^{n}\sum_{-n}^{n} I(x-u, y-v)\, g_\sigma(u)\, g_\sigma(v)$$




where $2n+1$ samples of $g_\sigma(x)$ for smoothing filter.

(ii) Need to compute samples of $S(x, y, \sigma_i)$ for $\sigma_i = 2^{i/s}\sigma_0$
(log scale spacing)

— 2D convolution by 2×1D convolutions. — 2 1D convolution, incremental blur
— Incremental blur $\left\{ \begin{array}{l} g_{\sigma_{j+i}} = g_{\sigma_j} * g_{\sigma_k} \\ \sigma_{i+1} = 2^{\frac{1}{s}}\sigma_i \end{array} \right.$
— when $\sigma_i$ doubles then <u>subsample</u> image to $\frac{1}{4}$ size
$\sigma \to 2\sigma$ then new octave.

1(c).

(i) Blob-like shapes:

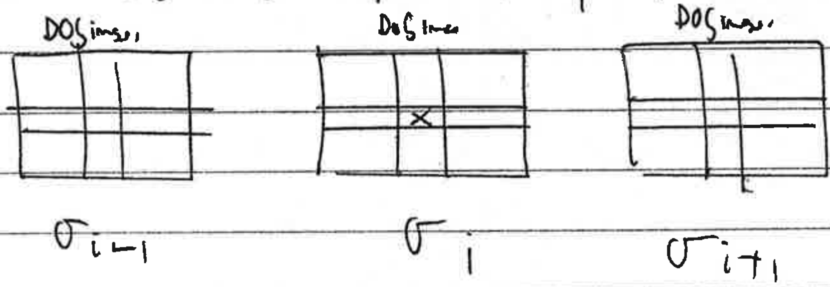
— convolve image with $\nabla^2 G_{\sigma_i}(\ast)$

— search for max or min in $\nabla^2 S(x,y,\sigma_i^2)$ over scale $\sigma_i$ and image position

— more efficient to use DOG

$$\nabla^2 S(x,y,\sigma_i^2) \simeq S(x,y,\sigma_{i+1}^2) - S(x,y,\sigma$$

— evaluate 26 neighbours for local max/min




— max min is blob centre with scale $\sigma_i$

— orientation → Look at gradients histogram and find dominant orientation.

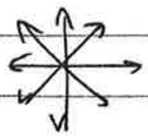
ii) Take 16x16 pixels at $S(x,y,\sigma_i)$ centred a blob centre and with orientation given by dominant orientation

1(c)(iii).

— Sample 16×16 gradients from $S(x, y, \sigma_i)$

(viii) — Smooth gradients with $g \cdot (1.5\sigma)$

— Produce orientation histograms in 4×4 blocks (cell)
Each bin records gradient magnitude, interpolated over & dirⁿ



— concatenate 16 histograms (HOGs) to 128D vector

— normalize to a unit vector and truncate ~~max~~ entries to 0.2
so no effect of very strong highlights

— Invariance to lighting — derivatives + normalisation
— Invariance to ~~small~~ translations and scale
— Invariance to small mis-alignment
— Invariance to small viewpoint changes

— Cannot cope with features on occluding boundaries or large viewpoint changes.

Q17.

(a) Using $1/\sqrt{2\pi} \approx 0.399$

$$p(x_1 = 0|M_1) = \frac{1}{\sqrt{2\pi}} \exp\{0\} \approx 0.399$$

$$p(x_2 = 1.4|M_1) = \frac{1}{\sqrt{2\pi}} \exp\{-1.4^2/2\} \approx 0.150$$

$$p(x_1 = 0|M_2) = \frac{1}{4} = 0.25$$

$$p(x_2 = 1.4|M_2) = \frac{1}{4} = 0.25$$

Using this we see that $p(x_1|M_1) > p(x_1|M_2)$ but $p(x_2|M_1) < p(x_2|M_2)$. So $x_1$ is more probable under $M_1$ and $x_2$ is more probable under $M_2$.

(b) The maximum likelihood Gaussian assuming iid data $x_1$ and $x_2$ has a mean of $0.7 = (0 + 1.4)/2$ and a variance of $0.7^2 = 0.49$. This should be derived by writing down the log likelihood and maximising as a function of $\mu$ and $\sigma$.

(c) Using the results from (a), Bayes rule, and this equal prior we can show the following:

$$\begin{aligned}
p(M_2|x_1, x_2) &= \frac{p(x_1, x_2|M_2)P(M_2)}{p(x_1, x_2|M_1)P(M_1) + p(x_1, x_2|M_2)P(M_2)} \\
&= \frac{p(x_1|M_2)p(x_2|M_2)}{p(x_1|M_1)p(x_2|M_1) + p(x_1|M_2)p(x_2|M_2)} \\
&= \frac{1/16}{1/16 + 0.399 \times 0.150} \\
&= 0.511
\end{aligned}$$

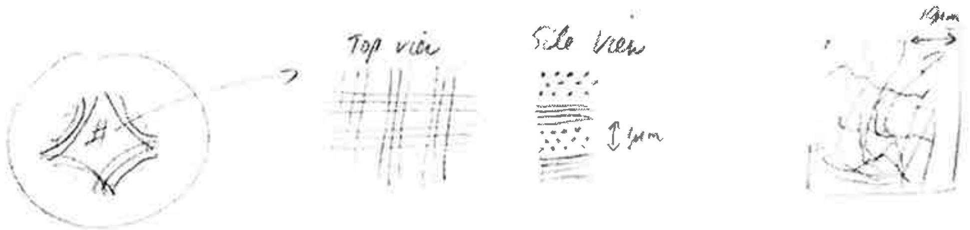
So the posterior for $M_2$ is only slightly higher than the prior given this data.

## SECTION G

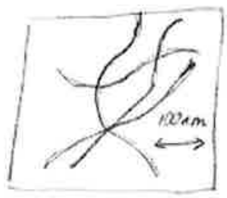
Bioengineering

18 (a) Describe, with sketches, the collagen microstructural organisation in the cornea, sclera and vitreous humour of the eye. How does the microstructure relate to the differing functions of these components? [7]

**Answer**:




**Cornea** (See above) predominant structural element is stroma, where collagen fibres are crystalline with uniform diameter and even spacing. In the centre of the eye this provides good strength, toughness and transparency. Collagen fibres on the edge of the eye are directed towards muscle attachment points.



**Sclera** More random arrangement of fibres as transparency not needed (see above), but strength and toughness is. Low level of inter-weaving between layers to improve toughness.

**Vitreous humour** low density random gel network with collagen fibres (see above) gives modest structural properties and transparency

(b) What are the roles of aqueous humour in the eye? [4]

**Answer**: Provides pressurisation of eye. Regulation needed to maintain IOP; high IOP leads to glaucoma.

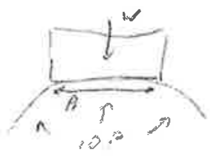
•Nourishes cornea and lens.

•Clears debris from eye (e.g. red cells from haemorrhage).

(c) Describe, with appropriate sketches, methods for measuring:

(i) intraocular pressure of a patient's eye; [3]

**Answer**:

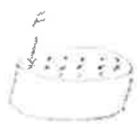


Tonometry measures IOP indirectly by deformation of the cornea. Goldmann tonometry:

- •topical anaesthetic applied
- •place head on cornea with force W
- •flattened area A: IOP = W/A
- •correction needed for tear film and bending stresses
- •empirically noted that these cancel out when A = 7.35 $mm^2$
- •further correction for abnormal corneal thickness.

(ii) lens stiffness in post-mortem tissue. [3]

**Answer**:



Indentation testing of lens across the surface e.g. Test 8 points across lens, several rows when possible. Calculate shear modulus, G (Pa), from load and penetration depth using appropriate formula.

(d) (i) Why are sequential recruitment models useful for characterising the elastic response of soft tissue? [1]

**Answer**: Typically the collagen network structure gives a non-linear strain-

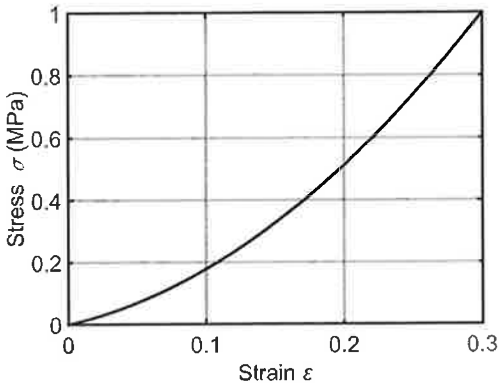
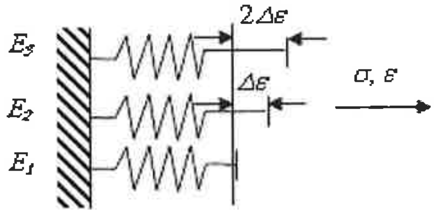


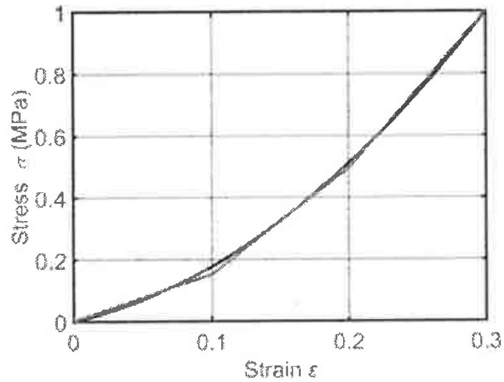

Fig. 1

hardening response which can be modelled by sequential recruitment of springs as strain increases.

(ii) The stress-strain data given in Fig. 1 is to be modelled using a three-spring sequential recruitment model. Sketch an appropriate model configuration and estimate appropriate parameter values for the model. [7]

**Answer**:

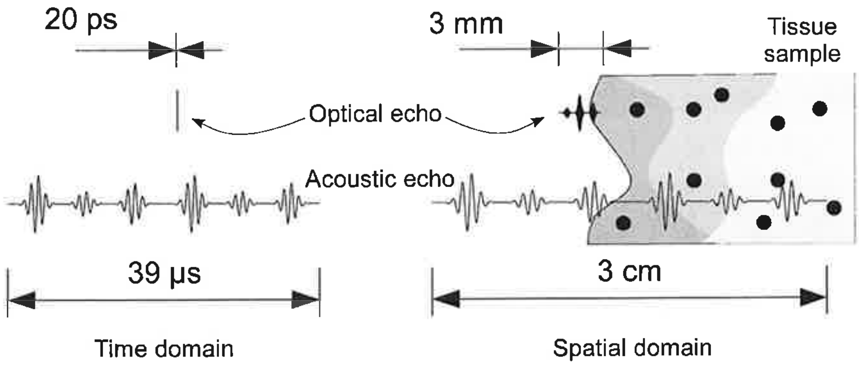



The figure above illustrates a suitable model, with $\Delta\varepsilon = 0.1$. With this model, matching the stress-strain response with three straight-line segments as illustrated, gives the total stiffness over the first, second and third segments of the line as: 0.16/0.1 = 1.6 MPa, (0.50-0.16)/0.1 = 3.4 MPa and (1 - .50)/0.1 = 5.0 MPa, respectively and the corresponding stiffnesses of the first, second and third springs as: E1 = 1.6 MPa, E2 = 3.4 - 1.6=1.8 MPa and E3 = 5.0 - 3.4 = 1.6 MPa, respectively.

19 (a) Ultrasound (US) and optical imaging systems can both use *pulse-echo* techniques to form cross-sectional images of the fundus. Explain what is meant by a pulse-echo technique, quantify the differences in the speed of travel of the pulse between US and optical systems, and describe the consequences of these speeds for cross-sectional imaging of the fundus. [6]

**Answer**: A *pulse-echo* technique is one which sends a pulse of sound (US) or light (optical) into the tissue, which is then reflected and the echo is used to image the tissue. The pulse is broadband with low-coherence, i.e. it has a fairly high bandwidth relative to the centre frequency, which means there are only a fairly small number of oscillations in each pulse.




The figure above shows how the speed differs between the two techniques. There is roughly a factor of $10^5$ between the speed of sound and light, although a much smaller factor between the spatial duration of each of the signals, since the wavelengths are also different by about $10^6$. The major consequence of this difference in speed is in how the echo is recorded. The US echo is slow enough to be recorded as it arrives, i.e. we can sample the time signal fast enough to separate out the responses from various different depths. The optical echo is much too fast for this, and there is only time to register a single intensity for each echo: in fact we are likely to need multiple echos to register the intensity with sufficient signal-to-noise ratio.

This means in optical imaging we need other ways to be able to associate an echo with a given depth, for instance confocal imaging in Scanning Laser Ophthalmology, or the interferometer in Optical Coherence Tomography, both of which ensure that only echos from a given depth are received at the photo-diode.

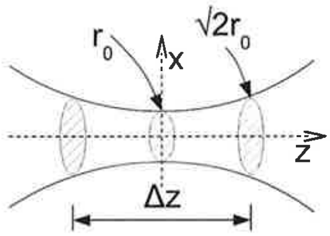
(b) In spectral Optical Coherence Tomography (OCT), several lenses are used to focus light into the eye. The beam cross-section from each lens is a disc of radius $r$ given by:

$$r^2(z) = r_0{}^2 \left(1 + \left(\frac{\lambda z}{\pi r_o{}^2}\right)^2\right)$$

where $z$ is the axial distance from the focal point, $r_0$ is the radius at the focal point, and $\lambda$ is the wavelength of the light. Assume that all lenses and imaged material are in air and have a refractive index $n = 1$.

(i) Using the equation above, derive an expression for the axial (depth) resolution of a lens, in terms of its Numerical Aperture (NA) and the light wavelength $\lambda$. [6]

**Answer**:




With symbols as defined above, depth resolution $\Delta z$, we require that the radius $r = \sqrt{2}r_0$ at $z = \frac{\Delta z}{2}$, in order that the beam cross-sectional area at this point is twice the value than at the focal spot. The equation for the radius then gives:

$$2r_0{}^2 = r_0{}^2 \left(1 + \left(\frac{\lambda \Delta z}{2\pi r_0{}^2}\right)^2\right)$$

Hence:

$$2 = \left(1 + \left(\frac{\lambda \Delta z}{2\pi r_0{}^2}\right)^2\right)$$

Taking the square-root and simplifying gives:

$$\Delta z = \frac{2\pi r_0{}^2}{\lambda}$$

Noting that NA $\approx \frac{r}{z}$ at the lens, which is at relatively large $z$, in which case, from the original equation:

$$r(z) \approx \frac{\lambda z}{\pi r_0}$$

hence

$$\mathrm{NA} \approx \frac{r}{z} \approx \frac{\lambda}{\pi r_0} \quad \text{i.e.} \quad r_0 = \frac{\lambda}{\pi \mathrm{NA}}$$

 (cont.

Substituting this in to the earlier equation gives:

$$\Delta z = \frac{2\lambda}{\pi(\mathrm{NA})^2}$$

(ii) The spectrum of the reflected light in spectral OCT is sampled by an array of photo-diodes with a sampling frequency equal to the bandwidth, $\Delta\lambda$, of the laser pulse. Derive an approximate expression for the resulting spacing of depth samples, in terms of $\lambda$ and $\Delta\lambda$. [5]

**Answer**: Depth samples are spaced at $tc$, where $t$ is the sampling period and $c$ is the speed of light. So if $\Delta d$ is the depth sample spacing, then:

$$\begin{aligned}
\Delta d &= tc \\
&= \frac{c}{\Delta F} \\
&= \frac{c}{\frac{c}{\lambda_{\min}} - \frac{c}{\lambda_{\max}}} \\
&= \frac{1}{\frac{1}{\lambda - \frac{1}{2}\Delta\lambda} - \frac{1}{\lambda + \frac{1}{2}\Delta\lambda}} \\
&= \frac{(\lambda + \frac{1}{2}\Delta\lambda)(\lambda - \frac{1}{2}\Delta\lambda)}{(\lambda + \frac{1}{2}\Delta\lambda) - (\lambda - \frac{1}{2}\Delta\lambda)} \\
&= \frac{\lambda^2 - \frac{\Delta\lambda^2}{4}}{\Delta\lambda} \\
&\approx \frac{\lambda^2}{\Delta\lambda}
\end{aligned}$$

(iii) Suggest typical values for each of the terms in the formulae derived in (i) and (ii), and hence estimate typical lens depth resolution and OCT depth sample spacing, assuming all imaging is in air. [4]

**Answer**: Typical values would be $\mathrm{NA} = 0.4$, $\lambda = 800$ nm and $\Delta\lambda = 50$ nm. In this case, we have:
$\Delta z \approx 3\ \mu$m
$\Delta d \approx 13\ \mu$m

(iv) Comment on the significance of lens depth resolution and OCT depth sample spacing for determining depth resolution of spectral OCT. [4]

**Answer**: Neither of these measures really determines depth resolution. It is the interferometer which controls the depth setting, and hence the pulse width which is most important. $\Delta z$ is only the depth resolution of the lens, but this generally has to have low NA in order to image over a reasonable depth, and in any case light is still collected over other depths, just from a wider field of view. $\Delta d$ is the sample spacing, so represents the maximum depth resolution the system can have, but is not otherwise directly connected to the actual system resolution.

20 (a) Describe the physical phenomenon to which Snell's law applies, and provide the associated formula, defining each of the quantities involved. [5]

**Answer**: When light passes through the boundary between two different media, it's direction (angle of incidence) changes. The formula for this given by Snell's law is:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{c_1}{c_2} = \frac{r_2}{r_1}$$

where $\theta_1$ and $\theta_2$ are the angle of incidence in the two media (the angle between the light ray and the line perpendicular to the tangent of the boundary where light passes through it, i.e. the 'normal') , $c_1$ and $c_2$ are the speed of light in the two media, and $r_1$ and $r_2$ are the refractive indices of the two media. Roughly speaking, when light enters 'denser' media, it bends away from the boundary (and towards the normal).

(b) For each of the optical elements listed below, name one animal whose optical apparatus makes use of it:

•pinhole camera;

•parabolic lens;

•scanning objective;

•flat cornea;

•negative lens;

•reflective mirror;

•a structure whose main job is to correct for the spherical aberration of other structures in the eye.

[7]

**Answer**:

•Pinhole camera: Nautilus

•Parabolic lens: Pontella

•Scanning objective: Copella

•Flat cornea: Penguin, Seal

•Negative lens: Eagle

•Reflective mirror: Scallop

•A structure whose main job is to correct for the spherical aberration of other structures in the eye: Pontella, Human, Scallop

(c) The angular resolution of the human eye at the fovea in bright light is $\sim 0.01°$. What is the minimal radius of a compound eye required to achieve this resolution for green light (with wavelength 500 nm)? [8]

**Answer**: The resolution of the compound eye is limited by diffraction. The point spread function, $\alpha$ (in radians), due to diffraction is

$$\alpha = \frac{2\lambda}{D}$$

where $\lambda$ is the wavelength of the light, and $D$ is the diameter of the lens. To achieve angular resolution $\rho$, the minimal angle between the axes of adjacent ommatidia is $\alpha/2$. So

$$\rho = \frac{\alpha}{2} = \frac{\lambda}{D}$$

and from this (substituting $\lambda = 500\,\text{nm}$ and $\rho = 0.01° \simeq 1.75 \cdot 10^{-4}$)

$$D = \frac{\lambda}{\rho} \simeq 2.86 \cdot 10^{6}\,\text{nm} = 2.86\,\text{mm}$$

From simple trigonometry, assuming that the longitudinal section of an ommatidium is an isosceles triangle with base $D$, sides $r$ (the radius of the eye), and a vertex angle $\rho$ opposite the base, we also know that

$$\sin\frac{\rho}{2} = \frac{D/2}{r}$$

which can be expressed, using the formula derived for $D$ above, as

$$= \frac{\frac{\lambda}{2\rho}}{r}$$

from which we get the expression for $r$:

$$r = \frac{\frac{\lambda}{2\rho}}{\sin\frac{\rho}{2}}$$

and substituting $\lambda = 500\,\text{nm}$ and $\rho = 0.01° \simeq 1.75 \cdot 10^{-4}$ we obtain

$$r \simeq 16.33 \cdot 10^{9}\,\text{nm} = 16.33\,\text{m}$$

This is clearly unrealistic and explains why the resolution of compound eyes is inferior to that of the human eye.

 (cont.

(d) Describe how retinal receptive fields depend on the ambient light level. [2]

**Answer:** At high ambient light level, retinal receptive fields have a centre-surround organisation, where they are excited by a bright spot surrounded by a darker annular region, or a dark spot surrounded by a bright annular region. At low ambient light levels, retinal receptive fields become simply sensitive to a (roughly circular) area of brighter or darker region, but without an opponent surround.

(e) Explain, with reasons, why efficient coding predicts different retinal receptive fields at different ambient light levels. [3]

**Answer:** This is because the relative magnitude of noise and signal in the RGC inputs varies with ambient light level: the lower the light level, the lower the signal-to-noise ratio is. At lower signal-to-noise ratios, denoising dominates, and RGCs act as simple low pass filters (centre with no surround). At high signal-to-noise ratios, whitening becomes important, RGC filters have non-monotonic power spectra resulting in a centre-surround structure.

**END OF PAPER**

**Crib:**

21 (a) Explain what is meant by the following three types of funding:

(i) grant;

(ii) debt;

(iii) equity. [9]

*Grant = a 'gift' from an organisation such as Innovate UK or NESTA (National Endowment for Science, Technology and the Arts). The type of projects that are eligible for funding through this route may be very restricted.*

*Debt = borrowing money from a bank or specialist finance organisation. You can only borrow money if you can convince the bank that you can repay the money, plus interest, exactly when they want it. This is usually very hard for a new company to do.*

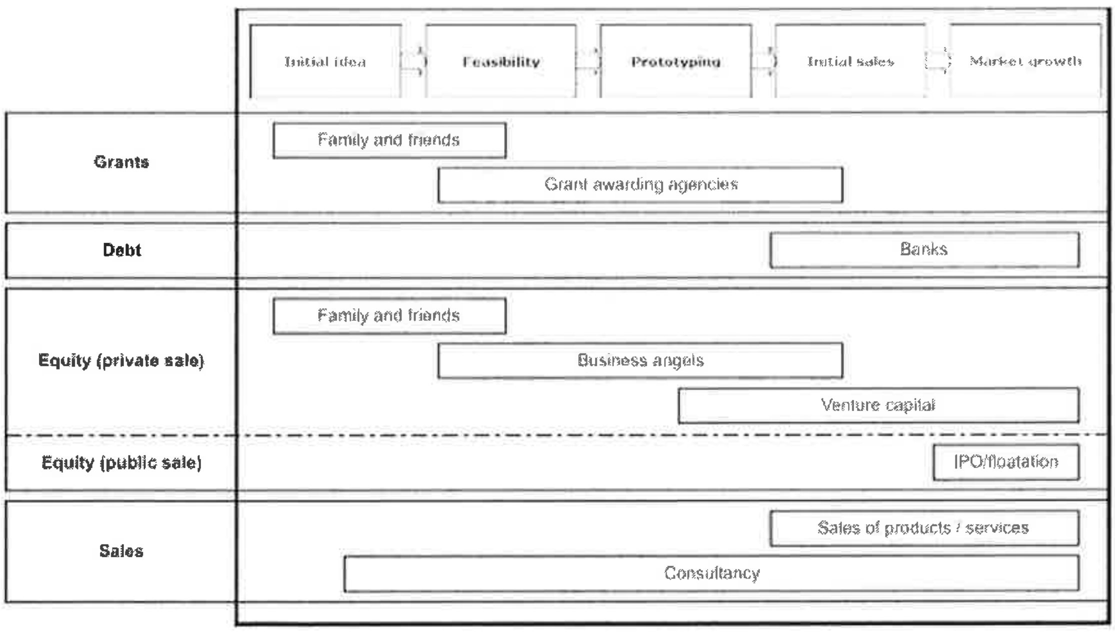
*Equity = selling part of the ownership (shares) of your business in return for cash. The assumption is that whoever buys part of your business will wish to sell this part of the business to someone else in the future at a higher price. The only reason for doing this is if the investor can be convinced that:*

*(a) the business is really going to grow (and quite fast) and that*

*(b) there will be someone else willing and able to buy their share in the company at a later date*

(b) Sketch a diagram that illustrates how different sources of the three types of funding described in (a) are likely to be appropriate at different stages of the commercialisation of a new idea.

[6]



(c) (i) Describe the main elements of a business plan suitable for attracting investment from a Venture Capitalist (VC). [4]

(ii) Discuss the specific elements of the business plan that the VC would focus upon for a product-based business idea.

[6]

*(i)*

*The market - Who has the problem that you attempting to fix?*

*Product or service - What solutions are going to be used to address the problem?*

*Management team - Who is going to do it? What is their track record?*

*Business operations - How are you going to do it?*

*Financial projections - How and when will money be made?*

*Marketing strategy - How will get people to buy your product / service*

*Resources required - What do you need to start your business?*

*Exit opportunities - How will your investors get their returns?*

*(ii)*

*The basic issue is that, in general, VCs dislike product based business plans as they typically offer higher risks and more rigidities than service or software plans. For a product-based business, the VC would be concerned about issues of product development (costs, timing, milestones), scale up (prototyping, initial run, full run), design of the operations (In-house? Outsourced?), risk mitigation, use of subcontractors, etc. Basic answers should highlight the core issue with some supporting evidence, where stronger answers would drill deeper into these issues and provide some examples.*

22 (a) Describe what is meant by the term *purchase stakeholders.*

[5]

*Initiator – begins the buying process and gathers information*

*Influencer – persuades or guides, has some role in influencing the decision, but is not primary decision maker*

*Decider – holds the power/purse strings*

*Buyer – conducts the transaction*

*User – the actual end user, who interacts with the product.*

*In some cases, these may all be one person, in others, each role may be multiple people/groups.*

(b) Explain how *user observations* and *personas* could be used to understand the needs of potential buyers of a new type of bicycle helmet.

*User observation*

*Basic answers should draw upon the material below to describe where and how they would observe potential users.*

There are many areas, where people either cannot answer direct questions, are too embarrassed to answer, or simply do not know the answer.

For example "what would you like me to cover in a lecture on understanding user needs?"

Observation enables insights on issues that the consumer might not have considered for themselves.

For example, "what makes you cross when cooking?" might elicit some interesting answers, but seeing someone struggling to open a tin of tomatoes might provide some real inspiration.

But, be careful, as people modify their behaviour when they know they are being studied, and are often a lot more careful about what they are doing.

Observation can also be time consuming to do and analyse, and therefore costly.

Observation will not answer 'why' any particular action has been done. For that reason, it is often good to combine observation with interviews. For example "I noticed that you frowned when you opened the case, why was that?" or "the pen obviously makes you cross, why did you buy that particular one?"

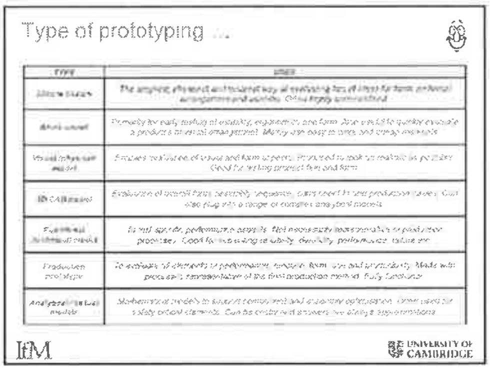
*Persosas are a representation of a key stakeholder, a hypothetical 'archetype'. They remove the subjectivity of design choices. The basic answer should develop some examples based on the structure given below.*

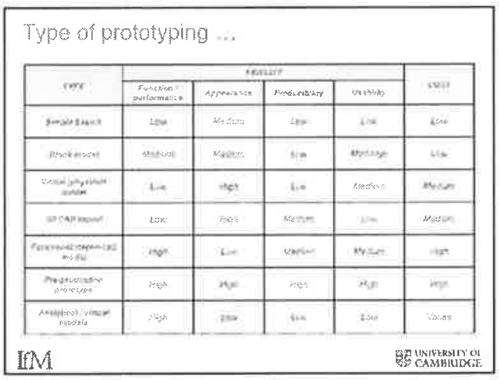

*Advanced answers could then link both user observation and personas to the role of the Kano model.*

(c) Discuss how different forms of *prototyping* could be used in the development of a new type of bicycle helmet.

[10]

*Basic answer should draw upon the material below, and consider the context-specific issues (i.e. how to test for safety, etc)*



23 Printers and ink are an example of a product plus consumables business model.

(a) Describe three other examples of product plus consumables business models.

[6]

*Could be any from razors and blades, guns and ammunition, Nerf guns and tags, electronic devices and batteries, Nespresso and capsules, et al.*

(b) Discuss the advantages and disadvantages of the following business models:

(i) product plus consumables;

(ii) service enabled by a product;

(ii) service only. [9]

*(i) Sell a product plus consumables: Pro = Long-term revenues on the back of the sale of each product; Con = Need infrastructure to deliver consumables.*

*(ii) Sell a service enabled by a product: Pro = Long-term revenues on the back of the sale of each product, Con = Need infrastructure to provide services.*

*(iii) Sell a service: Pro = No manufacturing costs; Con = Can be hard to scale up the business (in many cases can only grow through recruiting and training lots of people).*

(c) Discuss the reasons why a small firm might seek to partner with a large firm in order to implement a product plus consumables business model.

[10]

*The product plus consumables business model is effectively two business models in one. As such, a smaller firm may struggle to do these two things at once. If they can find a partner that sees mutual benefit in doing this, then that could work well. The basic answer should discuss issues such as these. Stronger answers, as discussed in Lecture 10, should address the fact that there are many challenges for a small firm seeking to collaborate with a large firm – alignment of objectives, complexity of management structures, communication problems, et al.*