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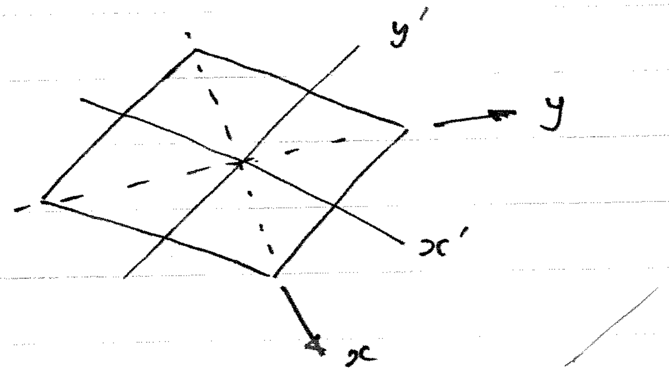
PAPER 3C5 DYNAMICS 2003 SOLUTIONS

1(a) For the plate



$I_{x'x'} = I_{y'y'} = \frac{1}{3}ma^2$

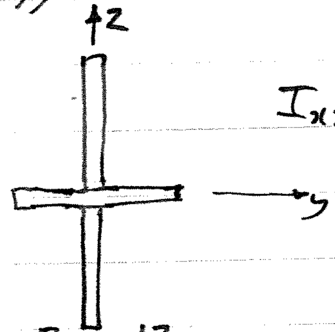
from Data book

$I_{zz} = \frac{2}{3}ma^2$ using ⊥ axes theorem.

The plate is like a disc since it has two equal moments of inertia. All inplane axes are the same.

So $I_{xx} = I_{yy} = \frac{1}{3}ma^2$ [25%]

(b) View along x

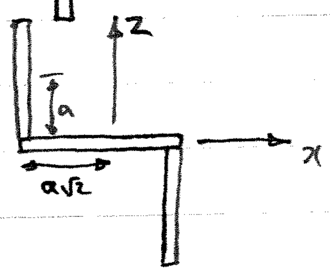


$I_{xx} = \frac{1}{3}ma^2 + \frac{1}{3}m(2a)^2$

$= \frac{5}{3}ma^2$

$I_{yz} = 0$

View along y



$I_{yy} = \frac{1}{3}ma^2 + 2\left[\frac{1}{3}\frac{m}{2}a^2 + \frac{m}{2}a^2(2+1)\right]$

$= \frac{1}{3}ma^2 + \frac{10}{3}ma^2$

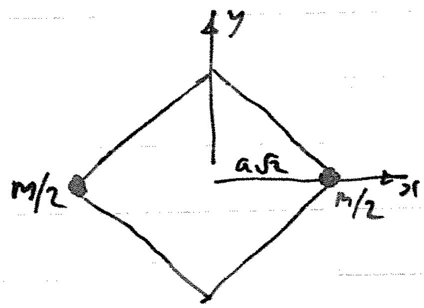
$= \frac{11}{3}ma^2$

$I_{xz} = \int xz\,dm = -2a\sqrt{2}\int_0^{2a} z\,dz\,\frac{m/2}{2a}$

$= -\frac{m\sqrt{2}}{2}\left[\frac{1}{2}z^2\right]_0^{2a}$

$= -m\sqrt{2}a^2$

View along z



$I_{zz} = \frac{2}{3}ma^2 + m(a\sqrt{2})^2$

$= \frac{8}{3}ma^2$

$I_{xy} = 0$

$$\therefore I_G = \frac{ma^2}{3}\begin{bmatrix} 5 & 0 & +3\sqrt{2} \\ 0 & 11 & 0 \\ +3\sqrt{2} & 0 & 8 \end{bmatrix}$$

[40%]

1 (c) For principal moments of inertia:

$$\begin{vmatrix} 5-\lambda & 0 & +3\sqrt{2} \\ 0 & 11-\lambda & 0 \\ +3\sqrt{2} & 0 & 8-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (11-\lambda)\left((5-\lambda)(8-\lambda) - (3\sqrt{2})^2\right) = 0$$

$$\therefore \quad (11-\lambda)\left(\lambda^2 - 13\lambda + 40 - 18\right) = 0$$

$$\therefore \quad (11-\lambda)(\lambda^2 - 13\lambda + 22) = 0$$

$$\therefore \quad (11-\lambda)(11-\lambda)(2-\lambda) = 0$$

$\therefore$ Principal moments of inertia are :

$$\frac{11ma^2}{3} \qquad \frac{11ma^2}{3} \qquad \frac{2ma^2}{3}$$

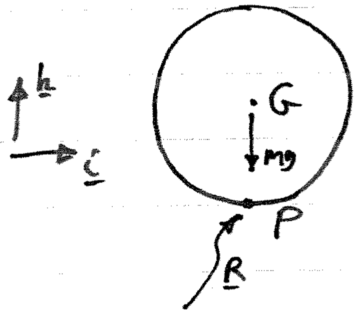
Diagonal AC is the Y axis and it was principal before transformation so it must still be principal. $I_{yy} = \frac{11ma^2}{3}$

[30%]

(d) The assembly is an "AAC" body just like a cylinder. Any forces applied to the assembly and to a cylinder of the same mass and with the same AAC values will cause identical responses.

[5%]

2. (a)



Consider the point P directly below G at all times. $\underset{\sim}{v}_P = \underset{\sim}{v}_G$

The weight force $m\underline{g}$ and any contact reaction at P have no moment about P so consider using the formulation

$$\underline{Q}_P = \dot{\underline{h}}_P + \dot{\underline{r}}_P \times \underline{p}$$

with $\underline{Q}_P = 0$ and since $\underline{p} = m\underline{v}_G$ we have $\dot{\underline{r}}_P \times \underline{p} = 0$

So $\boxed{\dot{\underline{h}}_P = 0}$

$$\underline{h}_P = A\omega_1 \underline{i} + A\omega_2 \underline{j} + C\omega_3 \underline{k} \quad \text{with} \quad A = \tfrac{7}{5}ma^2, \quad C = \tfrac{2}{5}ma^2$$

$\underline{h}_P = \text{const}$ so $\omega_1 = \text{const}$

$\omega_2 = \text{const}$

$\omega_3 = \text{const}$

No slip at P $\quad \underline{v}_G + \underset{\sim}{\omega} \times (-a\underline{k}) = 0$

$\therefore \quad \underline{v}_G + a\omega_1 \underline{j} - a\omega_2 \underline{i} = 0$

Let $\underline{v}_G = v_1 \underline{i} + v_2 \underline{j} + v_3 \underline{k}$

so $v_1 = a\omega_2 = \text{constant}$

$v_2 = -a\omega_1 = \text{zero (also constant)}$

$v_3 = 0$

Note there is no constraint on ω_3 so ANY spin is allowable. [20%

2 (b) Just as for (a) but now $\underline{Q}_P = -Fa\underline{i}$

So $-Fa\underline{i} = A\dot{\omega}_1 \underline{i} + A\dot{\omega}_2 \underline{j} + A\dot{\omega}_3 \underline{k}$

So $A\dot{\omega}_1 = -Fa$ ①

$\omega_2 = \text{Const}$ $\therefore$ $v_1 = a\omega_2 = \text{constat}$

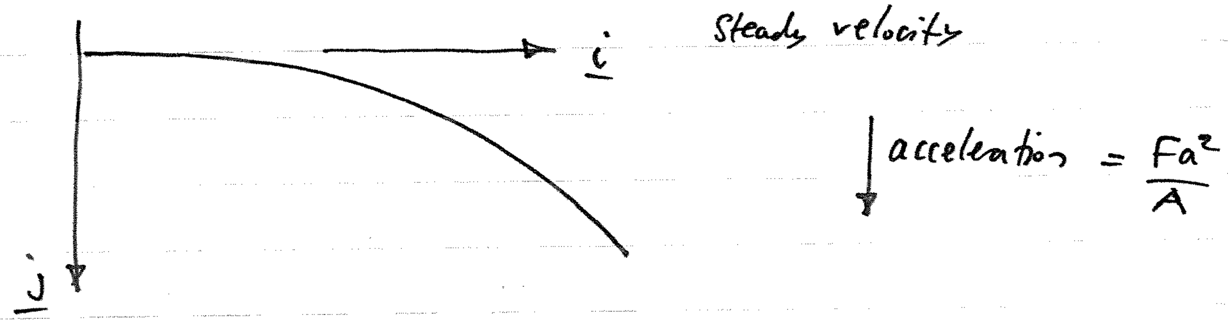
$\omega_3 = \text{Const}$

① → $\dot{\omega}_1 = -\dfrac{Fa}{A}$ but $v_2 = -a\omega_1$

$\therefore$ $\dot{v}_2 = -a\dot{\omega}_1 = \dfrac{Fa^2}{A}$

$\dot{v}_2$ is the constant acceleration in the $\underline{j}$ direction

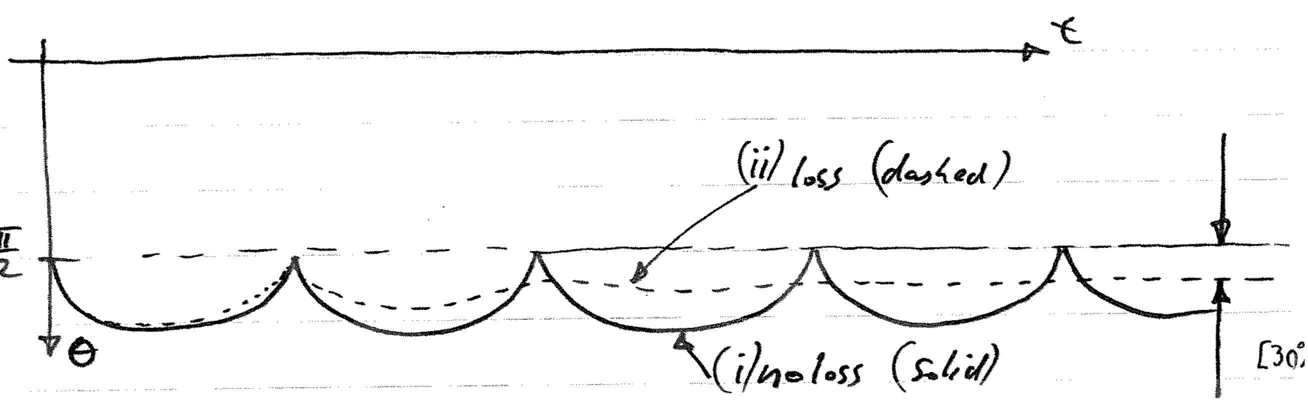
and since v_1 is constant this is just like projectile motion



[60%

(c) Again note that ω_3 is unconstrained, so spin does not affect the motion [20%

3(a)



[30%]

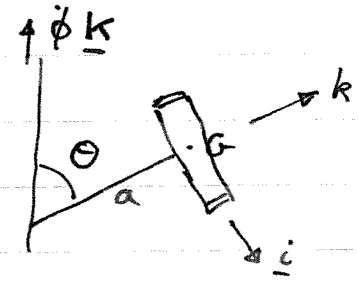
this is the permanent steady state value of the θ offset.

(b) To begin with, $KE = \frac{1}{2}C\omega^2$.

and $\underline{h} \cdot \underline{K} = 0$

ie there is no moment of momentum about a vertical axis.

Subsequent motion: PE + KE & $\underline{h} \cdot \underline{K}$ are conserved.



Velocity of G: $\underline{U} = a\dot{\theta}\underline{i} + a\sin\theta\dot{\phi}\underline{j}$

Angular velocity of rotor: $\underline{\omega} = \omega_1\underline{i} + \omega_2\underline{j} + \omega_3\underline{k}$

$\omega_1 = -\dot{\phi}\sin\theta$

$\omega_2 = \dot{\theta}$

$\omega_3 = \dot{\phi}\cos\theta + \omega \quad = \omega$ for fast spin

$$\therefore \quad KE = \tfrac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) + \tfrac{1}{2}A\left(\dot{\phi}^2\sin^2\theta\right) + \tfrac{1}{2}A\dot{\theta}^2 + \tfrac{1}{2}C\omega^2$$

At extreme position $\dot{\theta} = 0$

$$\therefore \quad KE = \tfrac{1}{2}(ma^2 + A)\sin^2\theta\,\dot{\phi}^2 + \tfrac{1}{2}C\omega^2$$

Put $\theta = \frac{\pi}{2} + \alpha \quad \therefore \quad \left.\begin{aligned}\sin\theta &= \cos\alpha \simeq 1\\ \cos\theta &= -\sin\alpha \simeq -\alpha\end{aligned}\right\}$ small α

Energy Conservation: $KE + PE = \frac{1}{2}(ma^2 + A)\dot{\phi}^2 + \frac{1}{2}C\omega^2 - mga\alpha = \frac{1}{2}C\omega^2$

$$\therefore \quad \boxed{(ma^2 + A)\dot{\phi}^2 = 2mga\alpha} \qquad (1)$$

Moment of Momentum: $\underline{h} = (A + ma^2)\omega_1\underline{i} + (A + ma^2)\omega_2\underline{j} + C\omega\underline{k}$

$$\underline{h} \cdot \underline{K} = (A + ma^2)(-\dot{\phi}\sin\theta)(-\sin\theta) + C\omega\cos\theta = 0$$

$$\therefore \quad (A + ma^2)\dot{\phi}\cos^2\alpha - C\omega\sin\alpha = 0$$

3(b) cont

$$\therefore \quad \boxed{(A+ma^2)\,\dot{\phi} = C\omega\,\alpha} \qquad (2)$$

eliminate $\dot{\phi}$ from (1) & (2)

$$\frac{(2)^2}{(1)} \quad \therefore \quad (A+ma^2) = \frac{C^2\omega^2\alpha}{2mga}$$

$$\therefore \quad \boxed{\alpha = \frac{2mga\,(A+ma^2)}{C^2\omega^2}}$$

[40%]

(c) For steady motion (precession) small α

$$Q_2 = C\omega\dot{\phi}$$

$$\therefore \quad mga = C\omega\dot{\phi} \qquad (3)$$

But (2) (conservation of moment of momentum about $\underline{K}$) must always be satisfied

$$\therefore \quad \frac{(2)}{(3)} \text{ gives } \quad \frac{A+ma^2}{C\omega} = \frac{C\omega\,\alpha_s}{mga}$$

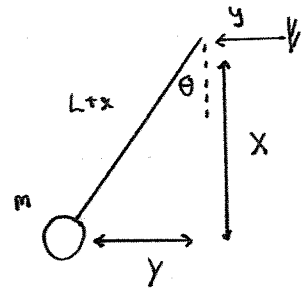
$$\therefore \quad \alpha_s = \frac{mga\,(A+ma^2)}{C^2\omega^2}$$

Note that $\alpha_s = 2\alpha$

ie steady α is half the peak α reached.

[30

4. a)



Mass position $X = (L+x)\cos\theta$

$Y = y + (L+x)\sin\theta$

Kinetic energy $T = \frac{1}{2}m\,(\dot{X}^2 + \dot{Y}^2)$

$T = \frac{1}{2}m\left\{\left[\dot{x}\cos\theta - (L+x)\dot{\theta}\sin\theta\right]^2 + \left[\dot{y} + \dot{x}\sin\theta + (L+x)\dot{\theta}\cos\theta\right]^2\right\}$

$\underline{T = \frac{1}{2}m\left\{\dot{x}^2 + \dot{y}^2 + (L+x)^2\dot{\theta}^2 + 2\dot{y}\dot{\theta}(L+x)\cos\theta + 2\dot{x}\dot{y}\sin\theta\right\}}$

$V_g = -MgX = -Mg(L+x)\cos\theta$ gravity

$V_s = \frac{1}{2}kx^2$ spring

$\underline{V = -Mg(L+x)\cos\theta + \frac{1}{2}kx^2}$ [10]

b) For x: $\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}}\right] - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0$

$\frac{d}{dt}\left[m\dot{x}\right] - m(L+x)\dot{\theta}^2 - m\dot{y}\dot{\theta}\cos\theta + \frac{d}{dt}\left[m\dot{y}\sin\theta\right] - Mg\cos\theta + kx = 0$

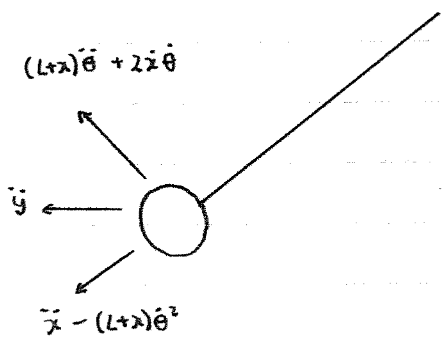
$\underline{M\ddot{x} - m(L+x)\dot{\theta}^2 + M\ddot{y}\sin\theta - Mg\cos\theta + kx = 0}$ — (1)

For θ: $\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}}\right] - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0$

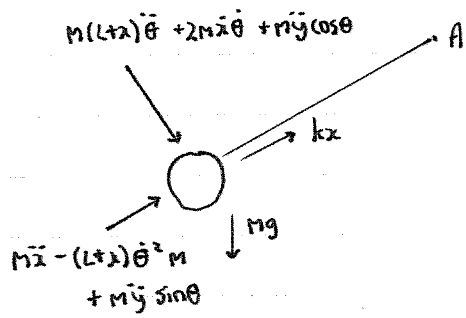
$\frac{d}{dt}\left[m(L+x)^2\dot{\theta} + m\dot{y}(L+x)\cos\theta\right] + M\dot{y}\dot{\theta}(L+x)\sin\theta - M\dot{x}\dot{y}\cos\theta + Mg(L+x)\sin\theta = 0$

$\underline{m(L+x)^2\ddot{\theta} + 2m\dot{x}\dot{\theta}(L+x) + m\ddot{y}(L+x)\cos\theta + Mg(L+x)\sin\theta = 0}$ — (2) [15]

c)



Accelerations



Forces

Resolve along string ⇒ eqn (1) ✓

Moments about A ⇒ eqn (2) ✓ [25]

d) For $\lambda = 0$ and θ small, equation (2) becomes:-

$$mL^2\ddot{\theta} + mgL\theta = -mL\ddot{y} \Rightarrow \ddot{\theta} + \omega_n^2\theta = -\left(\tfrac{1}{L}\right)\ddot{y} = \left(\tfrac{A}{L}\right)\sin\omega t \quad ; \quad \omega_n^2 = g/L$$

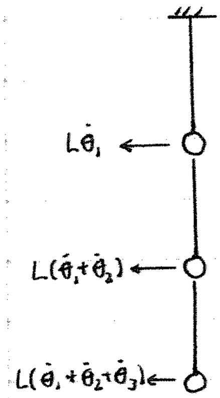
The solution is $$\underbrace{\theta = \frac{A\sin\omega t}{L(\omega_n^2-\omega^2)}}_{P.I} + \underbrace{\alpha\cos\omega_n t + \beta\sin\omega_n t}_{C.F.}$$

Validity : θ must remain small $\Rightarrow$ small, non-resonant motion $y(t)$ [20]

5 a) If the generalised coordinates are $q_1, q_2 \ldots q_N$, then:-

$$\underline{M_{ij} = \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}} \quad ; \quad \underline{k_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}}$$ [5]

b) Assuming that the angles are small, the velocities are:-



Thus $T = \frac{1}{2} m L^2 \{ \dot{\theta}_1^2 + (\dot{\theta}_1+\dot{\theta}_2)^2 + (\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)^2 \}$

$$M_{ij} = \frac{\partial^2 T}{\partial \dot{\theta}_i \partial \dot{\theta}_j} \Rightarrow M = \frac{1}{2} m L^2 \begin{pmatrix} 2+2+2 & 2+2 & 2 \\ \text{symm} & 2+2 & 2 \\ & & 2 \end{pmatrix}$$

$$V = -mgL\cos\theta_1 - mgL(\cos\theta_1 + \cos\theta_2) - mgL(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)$$

$$k_{ij} = \frac{\partial^2 V}{\partial \theta_i \partial \theta_j} = mgL \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus the equations of motion can be written in the form (with $\omega_0^2 = g/L$):-

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \omega_0^2 \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} FL/mL^2$$ [6

note generalised force vector

$\delta W = L(\delta\theta_1 + \delta\theta_2 + \delta\theta_3) F$

c) For $\ddot{\theta}_1 = 0$:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{pmatrix} + \omega_0^2 \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} FL/mL^2$$

Consider free harmonic vibration : must have $(-\omega^2 M + k)\underset{\sim}{\theta} = \underset{\sim}{0}$, so that :-

$$\begin{pmatrix} 2(\omega_0^2-\omega^2) & -\omega^2 \\ -\omega^2 & \omega_0^2-\omega^2 \end{pmatrix}\begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Determinant $= 0 \quad \Rightarrow \quad 2(\omega_0^2-\omega^2)^2 - \omega^4 = 0$

$$\Rightarrow \quad 2\omega_0^2 - 4\omega_0^2\omega^2 + \omega^4 = 0 \quad \Rightarrow \quad \omega^2 = \frac{4\omega_0^2 \pm \sqrt{16\omega_0^4 - 8\omega_0^4}}{2}$$

$$\Rightarrow \quad \underline{\omega^2 = \omega_0^2 (2 \pm \sqrt{2})}$$

For $\omega^2 = \omega_0^2(2-\sqrt{2})$ $\quad \begin{pmatrix} 2(-1+\sqrt{2})\omega_0^2 & -\omega_0^2(2-\sqrt{2}) \\ & \end{pmatrix}\begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \theta_3 = \frac{2(\sqrt{2}-1)}{2-\sqrt{2}}\,\theta_2 = \underline{\sqrt{2}\,\theta_2}$

For $\omega^2 = \omega_0^2(2+\sqrt{2})$ $\quad \begin{pmatrix} 2(-1-\sqrt{2})\omega_0^2 & -\omega_0^2(2+\sqrt{2}) \\ & \end{pmatrix}\begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \theta_3 = \frac{-2(1+\sqrt{2})}{2+\sqrt{2}}\,\theta_2 = \underline{-\sqrt{2}\,\theta_2}$

[50]