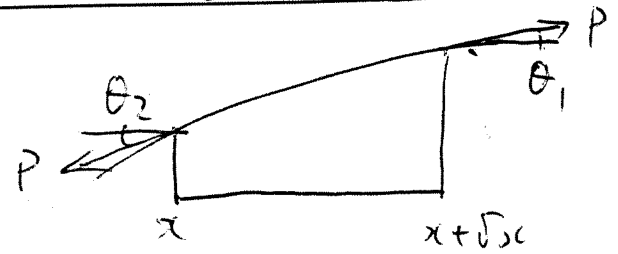
D. CEBON

PART IIA 3C6 2003 SOLUTIONS

1. (a)




$$m\,\delta x\,\frac{\partial^2 w}{\partial t^2} = -P\sin\theta_1 + P\sin\theta_2$$

$$\simeq P\left(-\frac{\partial w}{\partial x}\Big|_x + \frac{\partial w}{\partial x}\Big|_{x+\delta x}\right)$$

$$\therefore\; m\frac{\partial^2 w}{\partial t^2} = P\frac{\partial^2 w}{\partial x^2} \quad \text{in limit as } \delta x \to 0 \qquad (25\%)$$

(b) Write $w = u(x)e^{i\omega t}$

Then $P\dfrac{d^2u}{dx^2} = -m\omega^2 u$

General solution $u = A\cos kx + B\sin kx$, $k^2 = \dfrac{m\omega^2}{P}$

$u=0$ at $x=0 \rightarrow A=0$

$u=0$ at $x=L \rightarrow \sin kL = 0$

$\therefore\; kL = n\pi$, $n = 1, 2, 3 \ldots$

So nth mode is $\sin\dfrac{n\pi x}{L}$ at $\omega_n = \dfrac{n\pi}{L}\sqrt{\dfrac{P}{m}}$ (25%)

(c) Transient response: write $w(x,t) = \sum c_n \sin\dfrac{n\pi x}{L} e^{i\omega_n t}$

At $t=0$ $\dot{w} = 0$, $w = \begin{cases} a\sin 2\pi x/L & 0 \le x \le L/2 \\ 0 & L/2 \le x \le L \end{cases}$

Write $c_n = a_n + ib_n$, so $w = \sum(a_n\cos\omega_n t - b_n\sin\omega_n t)\sin\dfrac{n\pi x}{L}$

$\dot{w} = 0$ at $t=0 \rightarrow b_n = 0$

So at $t=0$, $\sum a_n \sin\dfrac{n\pi x}{L} = \begin{cases} a\sin 2\pi x/L \\ 0 \end{cases}$

(a) This is a Fourier series, so use usual method:

$$a_n \int_0^L \sin^2 \frac{n\pi x}{L}\,dx = a \int_0^{L/2} \sin\frac{2\pi x}{L} \sin\frac{n\pi x}{L}\,dx$$

$$\therefore a_n \frac{L}{2} = \frac{a}{2} \int_0^{L/2} \left[\cos\frac{(n-2)\pi x}{L} - \cos\frac{(n+2)\pi x}{L}\right] dx$$

$$\therefore a_n \frac{L}{a} = \left[\frac{L}{\pi(n-2)} \sin\frac{(n-2)\pi x}{L} - \frac{L}{\pi(n+2)} \sin\frac{(n+2)\pi x}{L}\right]_0^{L/2}$$

(provided $n \neq 2$)

$$\therefore a_n = \frac{a}{\pi}\left[-\frac{\sin n\pi/2}{n-2} + \frac{\sin n\pi/2}{n+2}\right]$$

$$= -\frac{4a \sin n\pi/2}{\pi(n^2-4)}$$

Case $n=2$: $$a_2 \frac{L}{2} = a \int_0^{L/2} \sin^2 \frac{2\pi x}{L}\,dx = \frac{a}{2}\cdot\frac{L}{2}$$

$$\therefore a_2 = a/2$$

So $$w(x,t) = \sum a_n \cos\omega_n t \sin\frac{n\pi x}{L}$$

with $$a_n = \begin{cases} a/2, & n = 2 \\ -\dfrac{4a \sin n\pi/2}{\pi(n^2-4)} & \text{otherwise.} \end{cases}$$

Since $\sin m\pi = 0$ for $m = 1, 2, 3 \ldots$, even modes are not excited except for $n = 2$.

(50%)

2(a) From $\rho A \dfrac{\partial^2 w}{\partial t^2} + EI \dfrac{\partial^4 w}{\partial x^4} = 0$

For a mode, set $w = u(x)\, e^{i\omega t}$

$\therefore \dfrac{d^4 u}{dx^4} = \alpha^4 u$ with $\alpha^4 = \dfrac{\rho A}{EI}\omega^2$

General solution is $u = K_1 \cos\alpha x + K_2 \sin\alpha x + K_3 \cosh\alpha x + K_4 \sinh\alpha x$

At a free end, $\dfrac{d^2 u}{dx^2} = 0$ and $\dfrac{d^3 u}{dx^3} = 0$

$u''(0) = 0 \rightarrow -K_1 + K_3 = 0$

$u'''(0) = 0 \rightarrow -K_2 + K_4 = 0$

So $\begin{cases} u''(L) = 0 \rightarrow K_1(-\cos\alpha L + \cosh\alpha L) + K_2(-\sin\alpha L + \sinh\alpha L) = 0 \\ u'''(L) = 0 \rightarrow K_1(\sin\alpha L + \sinh\alpha L) + K_2(-\cos\alpha L + \cosh\alpha L) = 0 \end{cases}$

Determinant of these two equations must be zero,

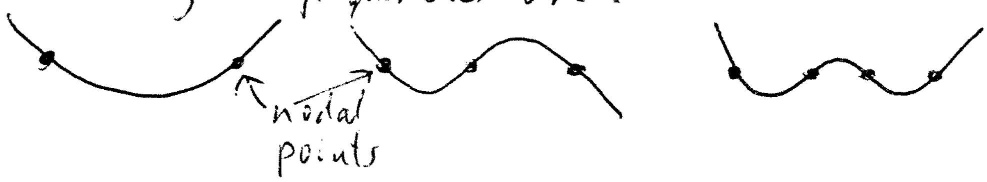
$\therefore (\cosh\alpha L - \cos\alpha L)^2 = (\sinh\alpha L + \sin\alpha L)(\sinh\alpha L - \sin\alpha L)$

$\therefore \cosh^2\alpha L - 2\cosh\alpha L \cos\alpha L + \cos^2\alpha L = \sinh^2\alpha L - \sin^2\alpha L$

$\therefore 2 = 2\cos\alpha L \cosh\alpha L$

or $\cos\alpha L \cosh\alpha L = 1$ ✓ (45%) [9]

(b) There are two rigid body modes. First 3 modes with non-zero frequencies are:




Most of sound comes from the first of these, so want to support the bar at the nodal points of this mode, approximately $L/4$ from each end.

Want supports to be <u>compliant</u> so you don't get chattering: either strings through the bar at these points, or support on rubber mounts of some kind.

(20%) [4]

2 (c) For a given mode, $\alpha L = k$ where $k$ is some suitable root of equation (*) : $k = 4.73$ for the lowest mode.

Now $\quad \omega = \sqrt{\frac{EI}{\rho A}}\,\alpha^2 = \sqrt{\frac{EI}{\rho A}}\,\frac{k^2}{L^2}$

If $\omega_1$ and $\omega_2$ are one semitone apart, $\frac{\omega_1}{\omega_2} = 2^{-1/12}$

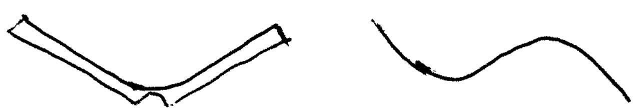
$$\therefore \frac{L_1}{L_2} = \left(\frac{\omega_2}{\omega_1}\right)^{1/2} = 2^{1/24} = 1.029$$

(15%)

So the bar for the higher note needs to be about 3% shorter

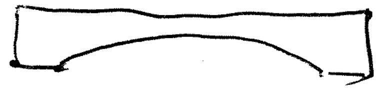
(d) $\omega \propto (\alpha L)^2$, so frequency ratios are $1 : 2.76 : 5.40$
So main requirement is probably to change the 2.76 ratio to 3, by lowering the first mode relative to the second. This could be attempted by adding masses to the bar in carefully-chosen places, or more realistically by carving material from the bar so the cross-section is no longer uniform. As a first guess, a thin patch near the centre might help:

Modes



bigger effect on this mode.

So aim is to find a profile $A(x)$ which shifts mode 1 relative to mode 2. For a <u>small</u> change in profile, can use Rayleigh's principle to find the frequency shift by using the <u>original</u> modes as approximations in the new Rayleigh quotient. Procedure: find a profile which gives the desired response in this approximation, then solve equation again with new profile and iterate.
Result in practice is usually



(20%)

3. <u>Kinetic Energy:</u>

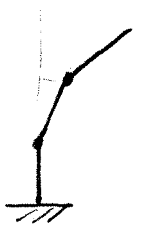
(a) $$T = \tfrac{1}{2}m\left(\frac{\dot y_2+\dot y_3}{2}\right)^2 + \tfrac{1}{2}\left(\frac{ml^2}{12}\right)\left(\frac{\dot y_3-\dot y_2}{l}\right)^2$$
$$+\tfrac{1}{2}(2m)\left(\frac{\dot y_1+\dot y_2}{2}\right)^2 + \tfrac{1}{2}\left(\frac{2ml^2}{12}\right)\left(\frac{\dot y_2-\dot y_1}{l}\right)^2$$
$$+\tfrac{1}{2}(3m)\left(\frac{\dot y_1}{2}\right)^2 + \tfrac{1}{2}\left(\frac{3ml^2}{12}\right)\left(\frac{\dot y_1}{l}\right)^2$$
$$= \frac{m}{2}\left\{\frac{5}{3}\dot y_1^2 + \dot y_2^2 + \frac{\dot y_3^2}{3} + \frac{2}{3}\dot y_1\dot y_2 + \frac{1}{3}\dot y_2\dot y_3\right\}$$

$$V = \tfrac{1}{2}k\left[\frac{y_3-y_2}{l} - \frac{y_2-y_1}{l}\right]^2 + \tfrac{1}{2}k\left[\frac{y_2-y_1}{l} - \frac{y_1}{l}\right]^2 + \tfrac{1}{2}S\left(\frac{y_1}{l}\right)^2$$

$$= \frac{k}{2l^2}\left[(y_3+y_1-2y_2)^2 + (y_2-2y_1)^2\right] + \frac{S\,y_1^2}{2l^2}$$

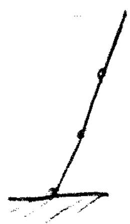
(30%)

(b)

(i) 

Rigid base
2 top links moving

$S \gg k$

(ii) 

2 top joints rigid
all movement at base

$S \ll k$

(20%)

(c) for $S \gg k$, approx mode shape is $y_1 = 0,\ y_2 = 1,\ y_3 =$.

3 (c) Cont

$$\text{Rayleigh: } \omega^2 = \frac{V_{max}}{T^*} = \frac{\frac{k}{2\ell^2}\left[1^2+1\right]}{M/2\left[1+3+1\right]} = \frac{k}{m\ell^2}\left(\frac{2}{5}\right)$$

$$\therefore \omega \approx \frac{1}{\ell}\sqrt{\frac{2k}{5m}} = \frac{0.632}{\ell}\sqrt{k/m}$$ 15%

(d) To find a more accurate estimate of the frequency, require a stationary value of Rayleigh's quotient

Assume the mode shape $[0 \;\; 1 \;\; \alpha]^T$

Substitute into Rayleigh: $\omega = \dfrac{V_{max}}{T^*}(\alpha)$

Stationary value $\dfrac{d\omega}{d\alpha} = 0 \Rightarrow \hat{\alpha}$

Minimum freq is $\underline{\underline{\omega^2(\hat{\alpha})}}$

This would be an exact value if the mode was really $[0 \; 1 \; \alpha]^T$ But in practice it will be slightly different ie $[\varepsilon, 1, \alpha]^T$ with $\varepsilon \ll 1$ The error will be small.

(20%)

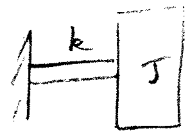
(e) Including gravity: Sideways movement makes c.g. of each element move down a little → reduces PE $\therefore$ Effective stiffness is reduced & mode frequencies all lower.

(15%)

4 (a)

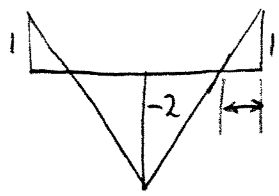
<u>Mode 1</u>: Rigid body rotation : $\omega_1 = 0$, $u^{(1)} = [1 \ \ 1 \ \ 1]^T$

<u>Mode 2</u>: Anti-symmetric, disk 2 stationary (node)




$\omega_2 = \sqrt{k/J}$, $u^{(2)} = [1 \ \ 0 \ \ -1]^T$

<u>Mode 3</u>: Symmetric, 2 nodes, for zero net angular momentum, mode shape is




$u^{(3)} = [1 \ \ -2 \ \ 1]^T$

⅓ shaft ⇒ stiffness = 3k $\quad \omega_3 = \sqrt{3k/J}$

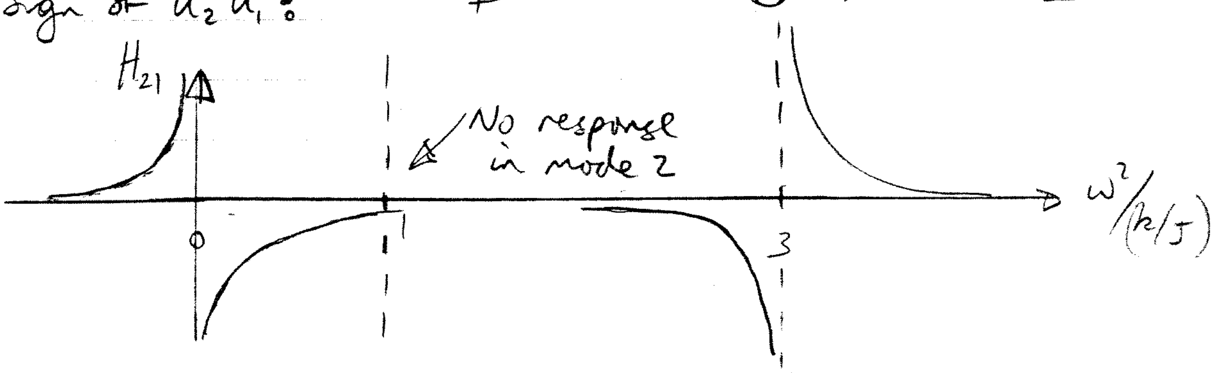
(30%)

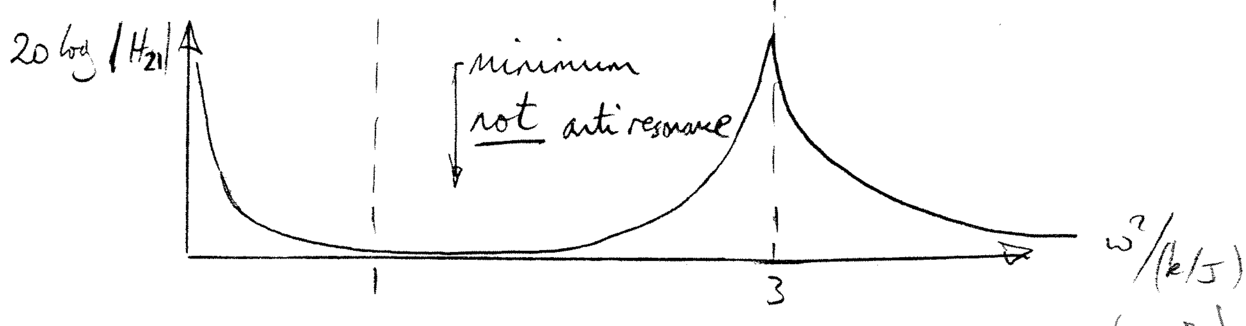
(b) Harmonic response formula

$$H_{rs} = \frac{\theta_r}{T_s} = \sum_{n=1}^{3} \frac{u_r^{(n)} u_s^{(n)}}{\omega_n^2 - \omega^2} \quad \text{with } r = 1 \ \& \ s = 2$$

$$H_{21} = \frac{\theta_2}{T_1} = \frac{u_2^{(1)} u_1^{(1)}}{\omega_1^2 - \omega^2} + \frac{u_2^{(2)} u_1^{(2)}}{\omega_2^2 - \omega^2} + \frac{u_2^{(3)} u_1^{(3)}}{\omega_3^2 - \omega_2^2}$$

Sign of $u_2 u_1$: $\quad + \quad\quad 0 \quad\quad -$

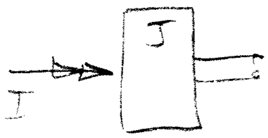



(30%)

4 (c)

Consider rotor 1 during the impact :




$$J\ddot{\theta}_1 + k(\theta_1 - \theta_2) = I\,\delta(t)$$

$$\int_{0^-}^{0^+} J\ddot{\theta}_1\,dt + \cancel{\int_{0^-}^{0^+} k(\theta_1-\theta_2)} = \int_{0^-}^{0^+} I\,\delta(t)\,dt$$

$$= 0$$

ie $J\dot{\theta}_1(0^+) = I \Rightarrow \dot{\theta}_1(0^+) = I/J$ — ①

So the initial conditions are $\theta_1(0) = \theta_2(0) = \theta_3(0) = 0$

$\dot{\theta}_1(0) = I/J\ ,\ \dot{\theta}_2(0) = \dot{\theta}_3(0) = 0$

Subsequent motion will be a combination of the 3 modal responses. Motion after impulse will be :

$$\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} = \underbrace{\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & -2\\ 1 & -1 & 1\end{bmatrix}}_{\text{e-vectors}} \begin{Bmatrix} a t\\ b\sin\omega_2 t\\ c\sin\omega_3 t\end{Bmatrix} \qquad \text{— ②}$$

Initial vels by differentiating

$$\begin{Bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{Bmatrix} = \begin{Bmatrix} I/J\\ 0\\ 0\end{Bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 1 & 0 & -2\\ 1 & -1 & 1\end{bmatrix} \begin{Bmatrix} a\\ b\omega_2\\ c\omega_3\end{Bmatrix} \Rightarrow \begin{Bmatrix} a\\ b\\ c\end{Bmatrix} = \begin{Bmatrix} I/3J\\ I/2J\omega_2\\ I/6J\omega_3\end{Bmatrix} \quad ③$$

Substitute ③ into ②

$$\theta_2(t) = \frac{I}{3J}t - \frac{2I}{6J\omega_3}\sin\omega_3 t$$

Put $\omega_3 = \sqrt{3k/J}$ & $t = \sqrt{J/k}$

gives $\theta_2\left(\sqrt{J/k}\right) = \frac{I}{3J}\sqrt{J/k} - \frac{2I}{6J\sqrt{3k/J}}\sin\sqrt{3} = \underline{\underline{0.143\,I/\sqrt{Jk}}}$

(40%)