CRIB 3C7 2003 V. DESHPANDE

1\. (a) The fundamental assumption for continuum mechanics (including fluid mechanics and solid mechanics) is that the matter under consideration is continuously divisible and no reference is made to its discrete structure at microscopic scales well below those of the application or phenomenon of interest.

Consider a beam made of Portland cement concrete ~~and~~ which is subjected to bending. When the beam dimension is on the order of a meter and the average diameter of the aggregates is on the order of a centimeter, the concrete can be assumed to be isotropic and homogeneous. However, if the stress and strain states in a single aggregate and the surrounding cement are of particular interest, the above assumption is no longer valid. The concrete must then be treated as a heterogeneous composite material. (20%)

(b) 1. Newtonian equations of motion — equilibrium equations

2. Geometry of deformation — expressions of strains in terms of displacement gradients

3. Stress-strain relations — constitutive equations

(20%)

1

1(c)

(i) $l\sigma_{xx} + m\sigma_{xy} = X$ $\quad$ X, Y = components of surface forces per unit area

$m\sigma_{yy} + l\sigma_{xy} = Y$ $\quad$ l, m = direction cosines

For $y=0$, $|x| \leq b$: $\sigma_{yy} = -p$, $\sigma_{xy} = 0$

For $x = (b+y)$, $b \leq x \leq b+H$ $\left\{ \begin{array}{l} \sigma_{xx} - \sigma_{xy} = -\rho g y \\ \sigma_{xy} - \sigma_{yy} = \rho g y \end{array} \right.$ $\quad$ $l = \frac{\sqrt{2}}{2}$, $m = -\frac{\sqrt{2}}{2}$

For $x = -(b+y)$, $-b \geq x \geq -(b+H)$ $\left\{ \begin{array}{l} \sigma_{xx} + \sigma_{xy} = 0 \\ \sigma_{xy} + \sigma_{yy} = 0 \end{array} \right.$ $\quad$ $l = -\frac{\sqrt{2}}{2}$, $m = -\frac{\sqrt{2}}{2}$ $\quad$ (30%)

(ii) $$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 = 2C_1 x(H-y) - 2C_3 x(H-y)$$

$\Rightarrow C_3/C_1 = 1$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 = C_3(H-y)^2 - 3C_2(H-y)^2$$

$\Rightarrow C_2/C_1 = 1/3$ $\quad$ (20%)

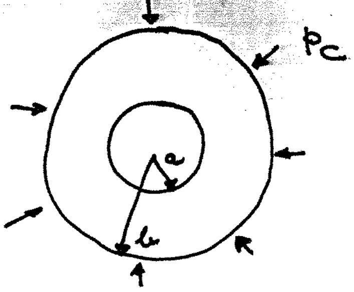
(iii) $\sigma_{yy} = -p$, $\sigma_{xy} = 0$ when $y = 0$, $|x| \leq b$

$\Rightarrow$ $C_2 H^3 = -p$, $\quad C_3 x H^2 = 0$

$\Rightarrow$ $C_2 = -p/H^3$, $\quad C_3 = 0$

Contradicting with (ii) $\Rightarrow$ Not possible $\quad$ (10%)

2(a)



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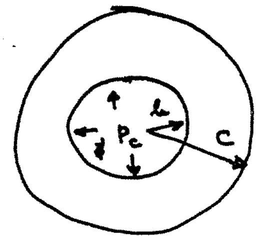


Ⓑ

Cylinder A : $\sigma_{rr} = A - \frac{B}{r^2}$, $\sigma_{\theta\theta} = A + \frac{B}{r}$ $(a \le r \le b)$

B.C. $\sigma_{rr} = 0$ at $r = a$; $\sigma_{rr} = -p_c$ at $r = b$

$$\Rightarrow \sigma_{rr} = \frac{-p_c b^2}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right), \quad \sigma_{\theta\theta} = \frac{-p_c b^2}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right)$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu \sigma_{rr}\right) = \frac{-p_c b^2}{E(b^2 - a^2)}\left(1 + \frac{a^2}{r^2} - \nu + \nu \frac{a^2}{r^2}\right)$$

Cylinder B

with B.C $\sigma_{rr} = -p_c$ at $r = b$ & $\sigma_{rr} = 0$ at $r = c$

$$\sigma_{rr} = \frac{p_c b^2}{c^2 - b^2}\left(1 - \frac{c^2}{r^2}\right) ; \quad \sigma_{\theta\theta} = \frac{p_c b^2}{c^2 - b^2}\left(1 + \frac{c^2}{r^2}\right)$$

$$\Rightarrow \varepsilon_{\theta\theta} = \frac{p_c b^2}{E(c^2 - b^2)}\left(1 + \frac{c^2}{r^2} - \nu + \nu\frac{c^2}{r^2}\right)$$ (25%)

2(b) At $r = b$ of disk A

$$u^A = u(r = b) = (r\,\varepsilon_{\theta\theta})_{r=b}$$

$$= \frac{p_c b}{E}\left(-\frac{b^2/a^2 + 1}{b^2/a^2 - 1} + \nu\right)$$

at $r = b$ of disk B

$$u^B = \frac{p_c b}{E}\left(\frac{c^2/b^2 + 1}{c^2/b^2 - 1} + \nu\right)$$

compatibility requires $\delta = u^B - u^A$

$$\Rightarrow \frac{p_c}{E} = \frac{\delta}{b}\left[\frac{b^2/a^2+1}{b^2/a^2-1} + \frac{c^2/b^2+1}{c^2/b^2-1}\right]^{-1} \qquad (35\%)$$

(c) (i) for compound disk due to p alone

$$\sigma_{rr} = A - \frac{B}{r^2}, \quad \sigma_{\theta\theta} = A + \frac{B}{r^2} \qquad (a \le r \le c)$$

B.C. $\sigma_{rr} = -p$ at $r = a$, $\sigma_{rr} = 0$ at $r = c$

$$\Rightarrow \sigma_{rr} = \frac{pa^2}{c^2-a^2}\left(1 - \frac{c^2}{r^2}\right), \quad \sigma_{\theta\theta} = \frac{pa^2}{c^2-a^2}\left(1 + \frac{c^2}{r^2}\right)$$

complete field

$$\left.\begin{aligned} \sigma_{rr} &= \frac{-p_c b^2}{b^2-a^2}\left(1 - \frac{a^2}{r^2}\right) + \frac{pa^2}{c^2-a^2}\left(1 - \frac{c^2}{r^2}\right) \\ \sigma_{\theta\theta} &= \frac{-p_c b^2}{b^2-a^2}\left(1 + \frac{a^2}{r^2}\right) + \frac{pa^2}{c^2-a^2}\left(1 + \frac{c^2}{r^2}\right) \end{aligned}\right\} \quad a \le r \le b$$

(25%)

$$\left.\begin{aligned} \sigma_{rr} &= \frac{p_c b^2}{c^2-b^2}\left(1 - \frac{c^2}{r^2}\right) + \frac{pa^2}{c^2-a^2}\left(1 - \frac{c^2}{r^2}\right) \\ \sigma_{\theta\theta} &= \frac{p_c b^2}{c^2-b^2}\left(1 + \frac{c^2}{r^2}\right) + \frac{pa^2}{c^2-a^2}\left(1 + \frac{c^2}{r^2}\right) \end{aligned}\right\} \quad b \le r \le c$$

(ii) when $p = 0$ $\quad |\sigma_{rr} - \sigma_{\theta\theta}|_{r=a} = \dfrac{2p_c}{1 - a^2/b^2}$ disk A

$$|\sigma_{rr} - \sigma_{\theta\theta}|_{r=b} = \frac{2p_c}{1 - b^2/c^2} \quad \text{disk B} \qquad (15\%)$$

$|\sigma_{rr} - \sigma_{\theta\theta}|_{r=a} = |\sigma_{rr} - \sigma_{\theta\theta}|_{r=b}$ for simultaneous yielding

$$\Rightarrow a/b = b/c$$

3

(a) $\phi = r\left[A\cos\theta + B\sin\theta + C\theta\cos\theta + D\theta\sin\theta\right]$

$$\frac{\partial\phi}{\partial r} = A\cos\theta + B\sin\theta + C\theta\cos\theta + D\theta\sin\theta$$

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \left[-A\sin\theta + B\cos\theta + C(\cos\theta - \theta\sin\theta) + D(\sin\theta + \theta\cos\theta)\right]$$

$$\frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} = \left[-A\cos\theta - B\sin\theta - C(2\sin\theta + \theta\cos\theta) + D(2\cos\theta - \theta\sin\theta)\right]$$

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = \frac{1}{r}\left[-2C\sin\theta + 2D\cos\theta\right]$$

$$\nabla^4\phi = \frac{1}{r^3}\left[-2C\sin\theta + 2D\cos\theta\right]\left[2-1-1\right] = 0$$

$\Rightarrow$ valid Airy stress function (20%)

(b) (i) $\sigma_{\theta\theta} = \sigma_{r\theta} = 0$ on AB & AC (10%)

(ii) $\sigma_{\theta\theta} = \dfrac{\partial^2\phi}{\partial r^2} = 0$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0$$

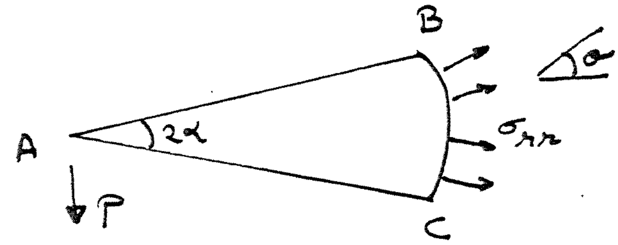
$\Rightarrow$ stresses derived from ϕ satisfy boundary conditions on AB & AC. (20%)

(c)

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{1}{r}\left[-2C\sin\theta + 2D\cos\theta\right]$$

σ_{rr} should be an odd function of θ for bending problem considered $\Rightarrow$ $D = 0$

$$\sigma_{rr} = -\frac{2C}{r}\sin\theta$$



$\sigma_{r\theta} = 0$

$$\therefore\ P = \int_{-\alpha}^{\alpha} \sigma_{rr}\sin\theta\, r\, d\theta = -2C\int_{-\alpha}^{\alpha}\sin^2\theta\, d\theta$$

$$= -C\int_{-\alpha}^{\alpha}(1-\cos 2\theta)\,d\theta$$

$$= C(\sin 2\alpha - 2\alpha)$$

$$C = \frac{P}{(\sin 2\alpha - 2\alpha)}$$

$$\sigma_{rr} = -\frac{2P\sin\theta}{(\sin 2\alpha - 2\alpha)r}$$ (30%)

(d) σ_{rr} given in (c) is an exact solution for the end loaded elastic tapered cantilever that is supported by only radial forces. otherwise σ_{rr} given above is accurate only far away from the supported end. (20%)

4(a)

(i) Upper bound theorem: The rate of work done by applied loads equated to the rate of plastic work dissipated in a kinematically admissible mode of deformation provides an upper bound on the true load for plastic deformation

(ii) Lower bound theorem: A stress distribution in equilibrium with the applied loads that does not violate the yield criterion provides a lower bound on the load for plastic deformation

By ignoring equilibrium in (i) or compatibility in (ii) we simplify the problem.

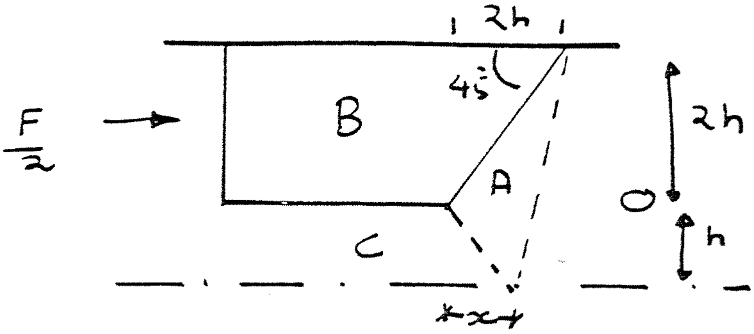
(i) is useful to design say metal forming m/c. Here we need to ensure that the m/c is capable of providing the required force to bring about the deformation

(ii) is useful in "safe design" of mechanical or structural components.

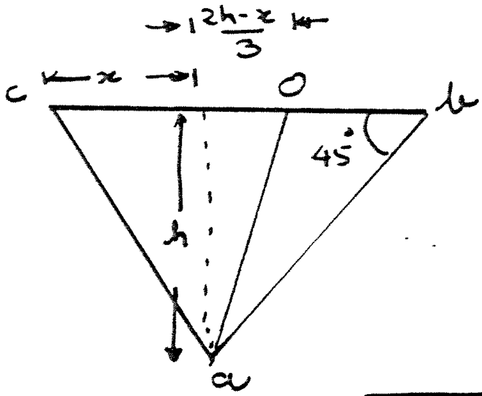
(30%)

b

(i)



Choose scale of velocity diagram such that the vertical velocity of A is proportional to h



$$\frac{F}{2} v_B = k\left(l_{AC} v_{AC} + l_{OA} v_{OA}\right)$$

$$\frac{F}{2}\left[h-\frac{1}{3}(2h-x)\right] = k\left[\sqrt{x^2+h^2}\,\sqrt{x^2+h^2} + \sqrt{(2h-x)^2+9h^2}\,\frac{\sqrt{(2h-x)^2+9h^2}}{3}\right]$$

$$\frac{F}{2k} = \frac{k\left[(x^2+h^2) + \frac{(2h-x)^2+9h^2}{3}\right]}{h-\frac{2h-x}{3}}$$

$$\frac{F}{2k} = 4\,\frac{(x^2-hx+4h^2)}{h+x}$$

(4%)

(ii) Min value when

$$\frac{d}{dx}\left(\frac{F}{2k}\right) = 0$$

$$\frac{d}{dx}\left(\frac{F}{2k}\right) = 4\,\frac{\left[(h+x)(2x-h) - (x^2 - hx + 4h^2)\right]}{(h+x)^2} = 0$$

$$x^2 + 2xh - 5h^2 = 0$$

inadmissible

$$\frac{x}{h} = (\sqrt{6} - 1) \text{ or } (-\sqrt{6} - 1)$$

$$(F)_{min} = 8kh\,\frac{\left[6 + 1 - 2\sqrt{6} - \sqrt{6} + 1 + 4\right]}{\sqrt{6}}$$

$$= 8kh\left(\frac{12 - 3\sqrt{6}}{\sqrt{6}}\right) \simeq 15.2\,kh. \quad (10\%)$$

(iii) Assume that friction occurs between A & B with a friction factor f. Then there is an additional term on the r.h.s of the work eqn & the modified eqn is

$$\frac{F}{2} = \frac{4k}{(h+x)}\left(x^2 + 4h^2 - hx + 3h^2 f\right)$$

$\because$ additional term $fk\sqrt{2}h\,2\sqrt{2}h$

This can be minimised w.r.t x to give the min. upper load F_{min}. (20%)

Engineering Tripos Part IIA THIRD YEAR

Paper 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\,dr$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \dfrac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2 \varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial \gamma_{r\theta}}{\partial \theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial \varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial \varepsilon_{rr}}{\partial r} + \dfrac{\partial^2 \varepsilon_{rr}}{\partial \theta^2}$
or (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial \sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right\} \times \left\{\dfrac{\partial^2 \phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2 \phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2 \phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2 \phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2 \phi}{\partial x \partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial \phi}{d\theta}\right\}$

3. **Torsion of prismatic bars**

Prandtl stress function: $\sigma_{zx}\ (=\tau_x) = \frac{dF}{dy}$, $\sigma_{zy}\ (=\tau_y) = -\frac{dF}{dx}$

Equilibrium: $T = 2\int_A F dA$

Governing equation for elastic torsion: $\nabla^2 F = -2G\beta$ where β is the angle of twist per unit length.

4. **Total potential energy of a body**

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D]\, \underset{\sim}{\varepsilon}\, dV$, $W = \underset{\sim}{P}^T \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. **Principal stresses and stress invariants**

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1 \sigma_P^2 + I_2 \sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} .$$

6. **Equivalent stress and strain**

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}} \left\{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}} \left\{ d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2 \right\}^{1/2}$

7. **Yield criteria and flow rules**

Tresca

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3 - \sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}} .$$