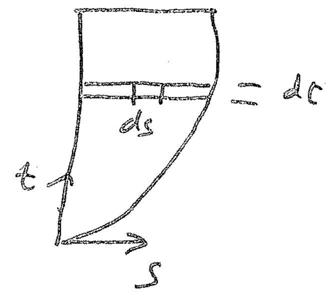
Solution:

1. (a) First find I_{st}



$$I_{st} = \int st \, dA$$

$$= \int_0^{0.4} t \left[\int_0^{\sqrt{t/10}} s \, ds \right] dt$$

$$= \int_0^{0.4} \frac{t}{2} \cdot \frac{t}{10} \, dt = \left[\frac{t^3}{60} \right]_0^{0.4} = 0.001\dot{0}\dot{6}\dot{6} \ (m^4)$$

$$I_{st} = I_{xy} + A.\bar{s}.\bar{t}$$

$$\therefore I_{xy} = I_{st} - A\bar{s}\bar{t}$$

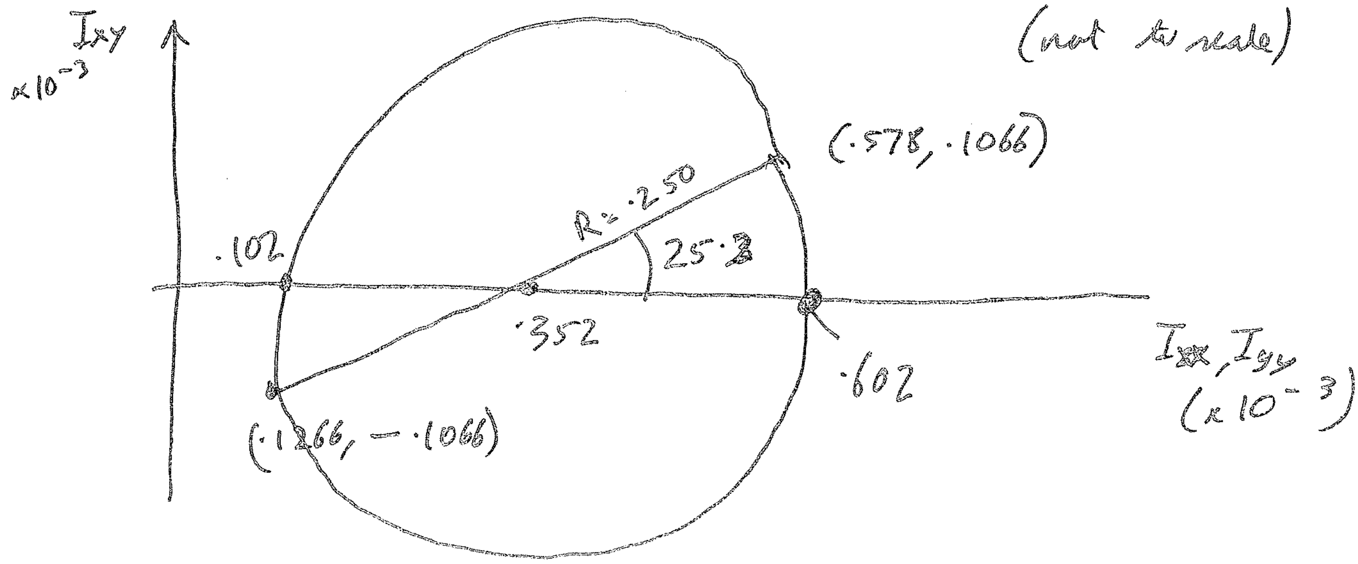
$$= 0.001066 - .0533 . 0.075 . 0.24$$

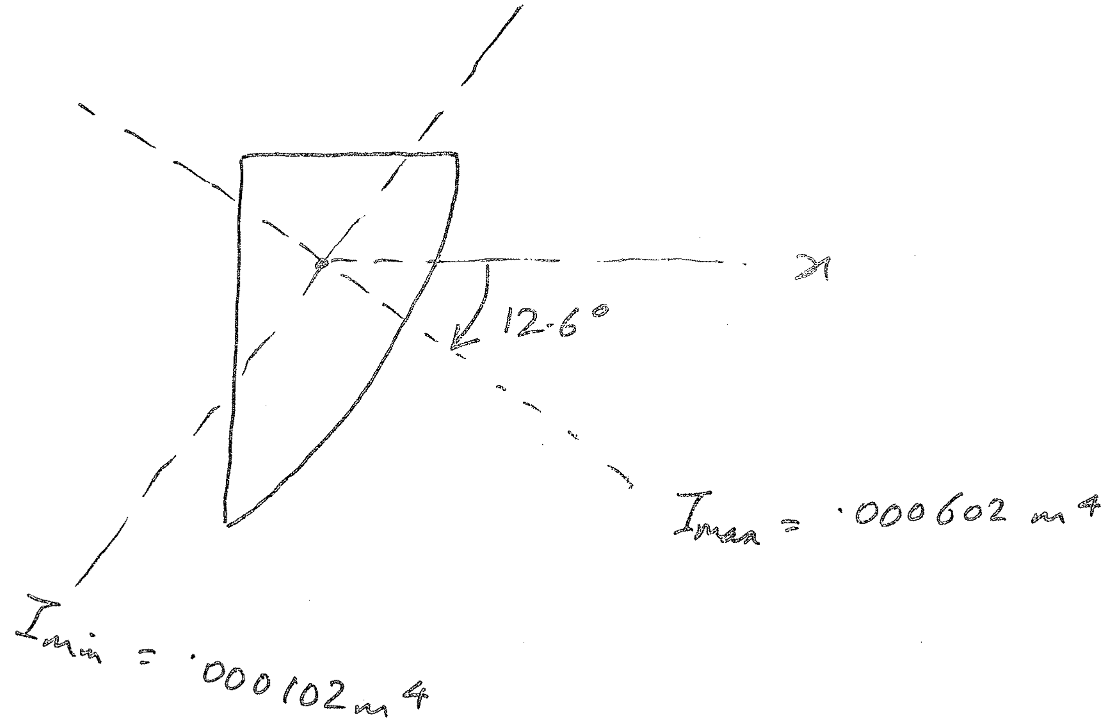
$$= +0.000\dot{1}0\dot{6}\dot{6} \ m^4.$$

(b) Need to find I_{xx} and I_{yy} about centroid

$$I_{xx} = I_{ss} - A.\bar{s}^2 = 0.000578 \ m^4$$

$$I_{yy} = I_{tt} - A\bar{t}^2 = 0.0001266 \ m^4$$

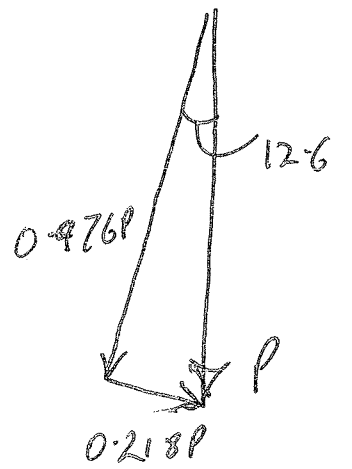




(c) Beam deflection due to tip load

$$= \frac{PL^3}{3EI} \quad \text{(data book)}$$

Resolve load parallel to principal axes.



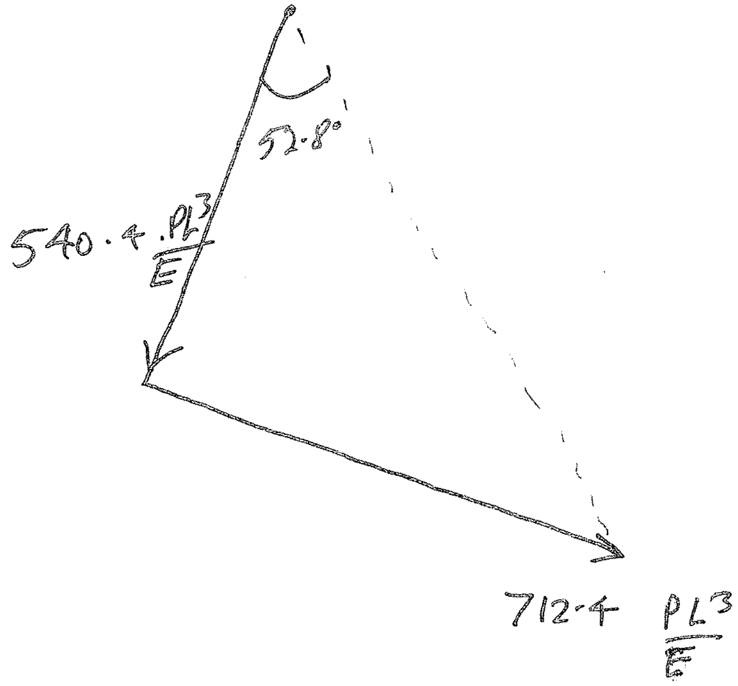
∴ Deflection about major axis

$$= \frac{PL^3}{E} \cdot \frac{0.976}{3 \cdot .000602}$$

$$= \cancel{1621}\ 540.4\ \frac{PL^3}{E}$$

Deflection about minor axis

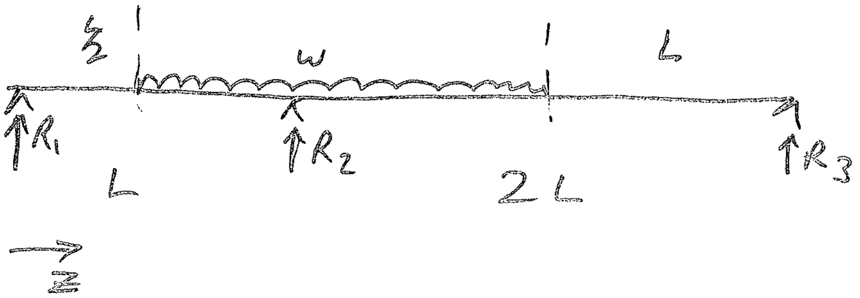
$$= \frac{PL^3}{E} \quad \frac{0.218}{3 \cdot .000102} = 712.4\ \frac{PL^3}{E}$$



$\therefore$ Magnitude of deflection $= 894.2 \frac{PL^3}{E}$

At an angle of 52.8° to the principal axes
$= 40.2^\circ$ to the vertical.

2



$$-EI\frac{d^2v}{dz^2} = -R_1 z - R_2\{z-L\} + \frac{w}{2}\{z-L/2\}^2 - \frac{w}{2}\{z - \tfrac{3L}{2}\}^2$$

Integrate twice.

$$-EI\,v = -R_1\frac{z^3}{6} - \frac{R_2}{6}\{z-L\}^3 + \frac{w}{24}\{z-L/2\}^4 - \frac{w}{24}\{z-3L/2\}^4 + Az + B$$

where A & B are constants of integration.

B.C at $z=0$ $\quad v=0 \Rightarrow B=0$ (1)

(The students may choose to leave this as one of their equations since they have been taught to think of this as a general method).

at $z=L$, $\quad v=0 \quad \Rightarrow 0 = -\frac{R_1L^3}{6} + \frac{w}{24}\left(\frac{L}{2}\right)^4 + A.L + B$ (2)

$z=3L \quad v=0 \quad \Rightarrow \quad 0 = -R_1\frac{27L^3}{6} + \frac{w}{24}\cdot\frac{625}{16}L^4 - R_2\frac{8L^3}{6} - \frac{w}{24}L^4 + 3AL + B$ (3)

Overall equilibrium $R_1 + R_2 + R_3 = \frac{3wL}{2}$ (4)

$$\frac{3wL}{2} \cdot \frac{5L}{4} = R_2 L + 3R_3 L. \quad (5)$$

Equations (1) to (5) form a set of simultaneous equations in the unknowns R_1, R_2, R_3, A & B.

The students are not expected to solve these equations.

They could simplify them by noting that equation (1) eliminates B trivially, and if they took moments about the R.H end they would only have one equation that involves R_3.
They could thus write a set of three equations (2), (3) & a new version of (5) in terms of A, R_1 & R_2 leaving R_3 to be found from equation 4.

However, they are being taught this method not as a hand method, so they should not be looking for tricks like this.

(b) Equation (2) would now become.

$$-EI.\delta = -\frac{R_1 L^3}{6} + \frac{w}{24}\left(\frac{L}{24}\right)^4 + AL + B.$$

don't forget this bit!

Nothing else would change.

(c) To find the influence line for moment at the centre of the shorter span, remove all loads and apply a unit rotation at that point

∴ Moment equation is simply

$$-EI\frac{d^2v}{dZ^2} = -R_1 Z - R_2\{Z-L\}$$

Integrate

$$-EI\frac{dv}{dZ} = -R_1\frac{Z^2}{2} - R_2\frac{\{Z-L\}^2}{2} + \{Z-L/2\}^0 + A$$

This term allows for hinge discontinuity

Integrate again

$$-EIv = -R_1\frac{Z^3}{6} - R_2\frac{\{Z-L\}^3}{6} + \{Z-L/2\}^1 + AZ + B$$

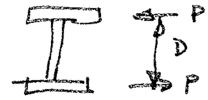
Apply boundary conditions and overall equilibrium as before.

3D4

Q3. Short spans $\quad M_{cr} \approx \frac{\pi}{L}\sqrt{\cancel{GJ}EI_y}\;\frac{\pi}{L}\frac{\sqrt{E\Gamma}}{\cancel{\sqrt{GJ}}} = \frac{\pi^2}{L^2}E\sqrt{I_y\Gamma}$

$$= \frac{\pi^2}{L^2}EI_y\,D/2$$

Force in flange $\quad P = \frac{M}{D}$



Critical compressive force P_{cr} for Euler buckling of flange $= \text{“}\frac{\pi^2 EI}{L^2}\text{”} = \frac{\pi^2 E}{L^2}\left(\frac{I_y}{2}\right) \leftarrow$ one flange only

$\therefore$ Corresponding moment $M = P.D = \frac{\pi^2}{L^2}EI_y\frac{D}{2}$ which is same as expression above.

ii) $457 \times 152 \times 74$ kg/m UB . $L = 8$m

$D = 462$ mm $\qquad G = 81 \times 10^3$ N/mm²

$I_y = 1047 \times 10^4$ mm⁴ $\qquad E = 210 \times 10^3$ N/mm²

$J = 65.9 \times 10^4$ mm⁴

$$\Gamma \approx \frac{I_y D^2}{4} = 1047 \times 10^4\,\frac{(462)^2}{4} = 0.5587 \times 10^{12}\ \text{mm}^6$$

$$\frac{\pi^2 E\Gamma}{L^2 GJ} = \frac{\pi^2}{(8000)^2}\,\frac{210}{81}\,\frac{0.5587 \times 10^{12}}{65.9 \times 10^4} = 0.3390$$

$$(1 + 0.339)^{1/2} = \underline{1.157}$$

$$\frac{\pi}{L}\sqrt{GJEI_y} = \frac{\pi}{8\,\text{m}}\sqrt{210 \times 81} \times 10^3\,\frac{\text{N}}{\text{mm}^2}\,\sqrt{65.9 \times 1047} \times 10^4\ \text{mm}^4$$

$$= 13\,453 \times 10^7\ \frac{\text{N mm}^2}{\text{m}} = 134.5\ \text{kNm}$$

$$\therefore M_{cr} = 134.5\ \text{kNm} \times 1.157 = \underline{155.7\ \text{kNm}}$$

iii) Warping is deviation from "plane-sections-remain-plane" hypothesis, exhibited strongly by open-section beams [I, I, etc].

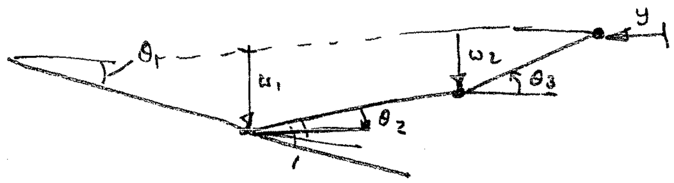
Analysis that ignores warping actually ignores warping <u>restraint</u>, the propensity of the structural element warping deformations. Ignoring warping means we assume structural element is completely free to warp, with no associated gain in strain energy. Including this extra strain energy gives increased stiffness, thus higher buckling load.

b) Shanley's insight was to recognise that structures could deform <u>during</u> a loading increment, and thus curvature could build up without any strain reversal, thereby avoiding inelastic unloading curve.

3D4 Structural Analysis and Stability. Cribs.

Q4. a) Around any static equilibrium configuration, the total potential energy function can be expanded as a Taylor Series. Since the constant term is arbitrary, and the linear term is zero at an equilibrium, the significant term is the quadratic term. For small deformations about the equilibrium, the potential energy function is thus approximately a quadratic form. The principal directions of this quadratic form are the eigenvectors and the associated eigenvalues are the stiffnesses against deformation into ~~any~~ the principal directions (or shapes). Stability of the equilibrium requires all eigenvalues to be positive. Buckling occurs the first time an eigenvalue stops being positive.

b)



Total Potential energy = Internal strain energy − External W.D

$$= \left[\tfrac{1}{2}G(\theta_1+\theta_2)^2 + \tfrac{1}{2}k w_2^2\right] - Py \underbrace{- Qw_1 - Qw_2}_{= -w^TQ}$$

Now $y = 3a - a\cos\theta_1 - a\cos\theta_2 - a\cos\theta_3$

$$= 3a - a\left(1-\frac{\theta_1^2}{2}+..\right) - a\left(1-\frac{\theta_2^2}{2}+\right) - a\left(1-\frac{\theta_3^2}{2}+..\right)$$

$$= \frac{a}{2}\left(\theta_1^2+\theta_2^2+\theta_3^2\right)$$

$$\theta_1 \simeq \tan\theta_1 = \frac{w_1}{a} \qquad \theta_2 \simeq \frac{(w_1-w_2)}{a} \qquad \theta_3 \simeq \frac{w_2}{a}$$

$$\therefore y = \frac{a}{2}\left(\theta_1^2+\theta_2^2+\theta_3^2\right) = \frac{a}{2}\left(\frac{w_1^2}{a^2} + \frac{w_1^2-2w_1w_2+w_2^2}{a^2} + \frac{w_2^2}{a^2}\right)$$

$$= \frac{1}{2a}\left[2w_1^2 - 2w_1w_2 + 2w_2^2\right] = \frac{1}{2a}\begin{bmatrix} w_1 & w_2\end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2\end{bmatrix}\begin{bmatrix} w_1 \\ w_2\end{bmatrix}$$

Also $$(\theta_1+\theta_2)^2 = \left(\frac{2w_1-w_2}{a}\right)^2 = \frac{4w_1^2 - 4w_1w_2 + w_2^2}{a^2}$$

$$= \frac{1}{a^2}\begin{bmatrix} w_1 & w_2\end{bmatrix}\begin{bmatrix} 4 & -2 \\ -2 & 1\end{bmatrix}\begin{bmatrix} w_1 \\ w_2\end{bmatrix}$$

$$\therefore \Pi(\underline{w}) = -w^TQ + \frac{1}{2}\begin{bmatrix} w_1 & w_2\end{bmatrix}\left\{\begin{bmatrix} 4G/a^2 & -2G/a^2 \\ -2G/a^2 & G/a^2\end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & k\end{bmatrix} - \begin{bmatrix} 2P/a & -P/a \\ -P/a & 2P/a\end{bmatrix}\right\}\begin{bmatrix} w_1 \\ w_2\end{bmatrix}$$

$$= -w^TQ + \frac{1}{2}w^TK_{TOT}\,w$$

where $K_{TOT} = \begin{bmatrix} c & d \\ d & e\end{bmatrix}$ with $c = (4G/a^2 - 2P/a)$ $\quad d = -2G/a^2 + P/a$

$e = G/a^2 + k - 2P/a$.

ii) ~~If~~ $G = \frac{1}{3}ka^2$

$$c = \frac{4}{3}\frac{ka^2}{a^2} - \frac{2P}{a} = \frac{1}{3}\left(\cancel{\frac{4ka^2 - 6P}{a}}\right) = (4ka - 6P)/3a$$

$$d = -\frac{2}{3}\frac{ka^2}{a^2} + \frac{P}{a} = \left(-\frac{2}{3}ka + 3P\right)/3a$$

$$e = \frac{4}{3}k - \frac{2P}{a} = c = (4ka - 6P)/3a.$$

$$K = \begin{bmatrix} c & d \\ d & c \end{bmatrix} \qquad \begin{vmatrix} c-\lambda & d \\ d & c-\lambda \end{vmatrix} = 0 \Rightarrow (c-\lambda)^2 - d^2 = 0$$

$$c - \lambda = \pm d$$

$$\therefore \lambda = c \pm d$$

$$\therefore \lambda_a = \frac{1}{3a}(4ka - 6P - 2ka + 3P) = (2ka - 3P)\frac{1}{3a}$$

$$\lambda_b = \frac{1}{3a}(4ka - 6P + 2ka - 3P) = (6ka - 9P)\frac{1}{3a}$$

$\therefore$ Buckling occurs when $2ka - 3P = 0 \rightarrow P = \frac{2}{3}ka$

or $6ka - 9P = 0 \rightarrow P = \frac{2}{3}ka$ again!

$\therefore$ Any pair of linearly independent shapes are eigenvectors

e.g $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore$ Buckling mode shape not determined. It could buckle into any shape (– imperfections may play a rôle in determining buckled shape).