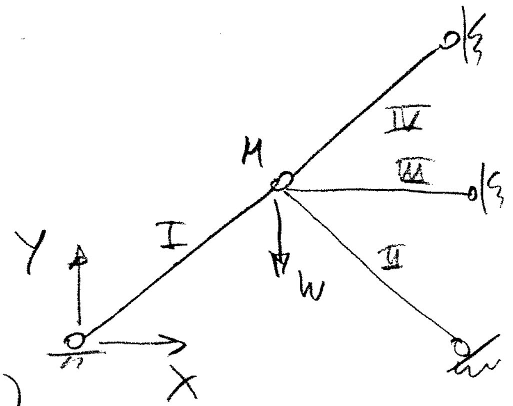
K. SOGA

1



(a)

Eqns of equilibrium for joint M:

$$\begin{cases} \frac{1}{\sqrt{2}} t_I - \frac{1}{\sqrt{2}} t_{II} - t_{III} - \frac{1}{\sqrt{2}} t_{IV} = 0 \\ \frac{1}{\sqrt{2}} t_I + \frac{1}{\sqrt{2}} t_{II} \qquad - \frac{1}{\sqrt{2}} t_{IV} = -W \end{cases}$$

in matrix form:

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & -1 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} t_I \\ t_{II} \\ t_{III} \\ t_{IV} \end{bmatrix} = \begin{bmatrix} 0 \\ -W \end{bmatrix}$$ <u>ANS</u>

(b) Solve by Gaussian elimination (row operations)

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & -1 & -1/\sqrt{2} \\ 0 & \sqrt{2} & +1 & 0 \end{bmatrix} \begin{bmatrix} t_I \\ t_{II} \\ t_{III} \\ t_{IV} \end{bmatrix} = \begin{bmatrix} 0 \\ -W \end{bmatrix}$$

2 pivots ∴ rank = 2
∴ 2 redundancies

cols without pivots = "free variables" correspond to redundant bars

□ Set $t_{III} = t_{IV} = 0$ and find rest by back-substitution: $t_{II} = -W/\sqrt{2}$, $t_I = -W/\sqrt{2}$

$$\therefore \underset{\sim}{t_0} = \left[-\frac{W}{\sqrt{2}} \; -\frac{W}{\sqrt{2}} \; 0 \; 0 \right]$$

– A set of bar forces in equilibrium with external loads.

- Set $t_{III} = 1$, $t_{IV} = 0$ and find rest for r.h.s. $= 0$. By back-substitution:

$$t_{II} = -1/\sqrt{2}, \quad t_{I} = +1/\sqrt{2}$$

$$\therefore \underset{\sim}{r}_1 = \left[\frac{1}{\sqrt{2}} \;\; -\frac{1}{\sqrt{2}} \;\; 1 \;\; 0\right]$$

— A state of selfstress.

- Set $t_{IV} = 1$, $t_{III} = 0$ and proceed as above to find

$$t_{II} = 0, \quad t_{I} = +1$$

$$\therefore r_2 = [1 \;\; 0 \;\; 0 \;\; 1]$$

— Another, independent state of selfstress.

$\therefore$ General solution of equilibrium eqtns:

$$\underset{\sim}{t} = \begin{bmatrix} -W/\sqrt{2} \\ -W/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1 \\ -1/\sqrt{2} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{[\underset{\sim}{r}_1,\ \underset{\sim}{r}_2] \ = \text{ matrix } \underset{\sim}{S}} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}}_{\underset{\sim}{\alpha}}$$

(c) Flexibility matrix:

$$\underset{\sim}{F} = \frac{L}{AE} \begin{bmatrix} \sqrt{2} & & & \\ & \sqrt{2} & & \\ & & 1 & \\ & & & \sqrt{2} \end{bmatrix}$$

A set of 2 compatibility eqtns obtained from:

$$\underset{\sim}{S}^T \underset{\sim}{F} \underset{\sim}{S} \underset{\sim}{\alpha} = -\left(\underset{\sim}{S}^T \underbrace{\underset{\sim}{e}_0}_{\text{initial extensions } = 0} + \underset{\sim}{S}^T \underset{\sim}{F} \underset{\sim}{t}_0\right)$$

$$\underset{\sim}{S}^T \underset{\sim}{F} = \frac{L}{AE}\begin{bmatrix} -1 & 1 & -1 & 0 \\ -\sqrt{2} & 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$\underset{\sim}{S}^T \underset{\sim}{F}\, \underset{\sim}{S} = \frac{L}{AE}\begin{bmatrix} 1+\sqrt{2} & 1 \\ 1 & 2\sqrt{2} \end{bmatrix}$$

$$\underset{\sim}{S}^T \underset{\sim}{F}\, \underset{\sim}{r_0} = \frac{L}{AE}\begin{bmatrix} 0 \\ -W \end{bmatrix}$$

Hence, system of compatibility eqtns is:

$$\begin{bmatrix} 1+\sqrt{2} & 1 \\ 1 & 2\sqrt{2} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ +W \end{bmatrix} \quad \therefore \begin{bmatrix} 1+\sqrt{2} & 1 \\ 0 & \frac{3+2\sqrt{2}}{1+\sqrt{2}} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ +W \end{bmatrix}$$

solution: $\alpha_1 = -0.1716\,W$, $\alpha_2 = +0.4142\,W$

$$\therefore \underset{\sim}{r} = \left(\begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/\sqrt{2} & 1 \\ -1/\sqrt{2} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -0.1716 \\ +0.4142 \end{bmatrix}\right) W = \begin{bmatrix} -0.4142 \\ -0.5858 \\ -0.1716 \\ 0.4142 \end{bmatrix} W$$

Ans

Hence no need to consider external forces.

(d) The changes in bar forces due to the given lack of fit could be obtained by considering:

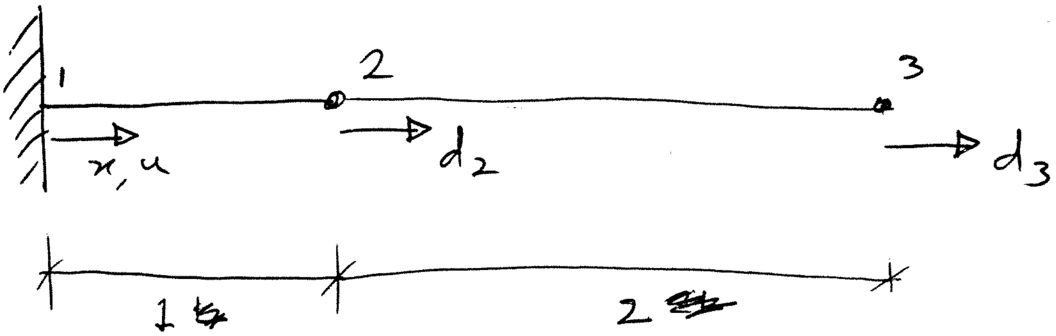
$$\underset{\sim}{e_0} = \begin{bmatrix} +L/100 \\ +L/100 \\ 0 \\ 0 \end{bmatrix}$$

and by solving the compatibility eqtns

$$\underset{\sim}{S}^T \underset{\sim}{F}\, \underset{\sim}{S}\, \underset{\sim}{\alpha}^* = -\underset{\sim}{S}^T \underset{\sim}{e_0}$$

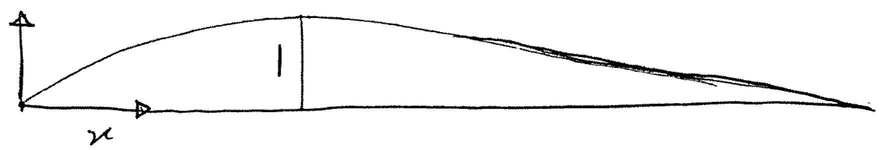
and then finding $\Delta \underset{\sim}{r} = \underset{\sim}{S}\, \underset{\sim}{\alpha}^*$.

Q.2



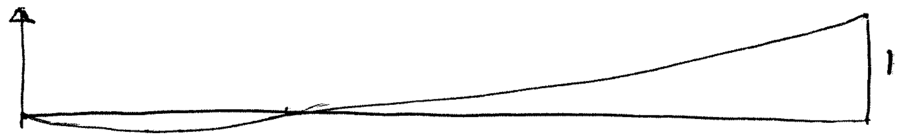
(a) Shape functions

n_2



$n_2(1) = 1$

n_3



$n_3(3) = 1$

To find n_2 $\quad n_2(0) = n_2(3) = 0$ Hence

$$n_2 = x(3-x)/2 = \frac{3}{2}x - \frac{1}{2}x^2$$

To find n_3 $\quad n_3(0) = n_3(1) = 0$ Hence

$$n_3 = x(1-x)/-6 = \frac{-x + x^2}{6}$$

$\therefore$

$$u(x) = \begin{bmatrix} \frac{1}{2}(3x - x^2) & \frac{1}{6}(-x + x^2) \end{bmatrix} \begin{bmatrix} d_2 \\ d_3 \end{bmatrix}$$

Shape function matrix $\underset{\sim}{N}$

(b) To find the stiffness matrix, we need the strain shape function matrix $\underset{\sim}{B}$

$$\underset{\sim}{B} = \frac{d}{dx}\underset{\sim}{N} = \left[\tfrac{1}{2}(3-2x) \quad \tfrac{1}{6}(-1+2x)\right]$$

Stiffness matrix is given, in general by:

$$\underset{\sim}{K} = \int_V \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, dV = \int_0^3 \underset{\sim}{B}^T E\, \underset{\sim}{B}\, A\,dx$$

$$= EA \int_0^3 \begin{bmatrix} \frac{3}{2} - x \\ -\frac{1}{6} + \frac{x}{3} \end{bmatrix} \begin{bmatrix} \frac{3}{2} - x & -\frac{1}{6} + \frac{x}{3} \end{bmatrix} dx$$

$$= EA \int_0^3 \begin{bmatrix} \left(\frac{3}{2} - x\right)^2 & \left(\frac{3}{2}-x\right)\left(-\frac{1}{6} + \frac{x}{3}\right) \\ \text{Symmetric} & \left(-\frac{1}{6} + \frac{x}{3}\right)^2 \end{bmatrix} dx$$

$$= EA \begin{bmatrix} \frac{9}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{7}{12} \end{bmatrix}$$ see below

$$\int_0^3 \left(\frac{3}{2} - x\right)^2 dx = \int_0^3 \frac{9}{4} - 3x + x^2 \, dx$$

$$= \left[\frac{9}{4}x - \frac{3x^2}{2} + \frac{x^3}{3}\right]_0^3$$

$$= \left\{\frac{9}{4}3 - \frac{3}{2}.9 + \frac{1}{3}.27 - 0\right\}$$

$$\cancel{= \frac{27 - 18 + 4}{4 \times 3}} = \frac{27}{4} - \frac{27}{2} + 9$$

$$\cancel{= \frac{13}{12}} = \frac{9}{4}$$

$$\int_0^3 \left(\frac{3}{2} - x\right)\left(\frac{-1}{6} + \frac{x}{3}\right) dx = \int_0^3 -\frac{1}{4} + \frac{x}{2} + \frac{x}{6} - \frac{x^2}{3}\, dx$$

$$= \int_0^3 -\frac{1}{4} + \frac{2}{3}x - \frac{x^2}{3}\, dx$$

$$= \left[-\frac{1}{4}x + \frac{x^2}{3} - \frac{x^3}{9} \right]_0^3$$

$$= \left\{ -\frac{3}{4} + 3 - 3 - 0 \right\}$$

$$= -\frac{3}{4}$$

$$\int_0^3 \left(\frac{-1}{6} + \frac{x}{3}\right)^2 dx = \int_0^3 \frac{1}{36} - \frac{2x}{18} + \frac{x^2}{9}\, dx$$

$$= \frac{1}{9}\left[\frac{x}{4} - \frac{x^2}{2} + \frac{x^3}{3} \right]_0^3$$

$$= \frac{1}{9}\left\{ \frac{3}{4} - \frac{9}{2} + 9 - 0 \right\}$$

$$= \frac{1}{9}\left(\frac{3}{4} + \frac{18}{4} \right)$$

$$= \frac{21}{36}$$

$$= \frac{7}{12}$$

3 (a)

Global level

Specify body forces + Applied boundary forces

↓

Gauss Point level

Evaluate constitutive matrix D and body forces

↓

Element level

Derive element stiffness matrix $K_{element}$ and nodal forces

↓

Global level

Assemble element stiffness to form global stiffness matrix K and boundary vectors using the nodal forces and boundary forces

↓

Global level

Compute nodal displacements **a**

↓

Gauss point level

Compute strains ε using the nodal displacements **a** and shape functions

↓

Gauss point level

Compute stress using $\sigma = D\varepsilon$

(b). Dirichlet boundary

$\boldsymbol{u} = 0$ for $x_2 = 0$

Neumann boundary

$-\boldsymbol{\sigma} e_1 = \rho_w g(h-x_2)$ for $x_1 = 0$

$\boldsymbol{\sigma} e_2 = 0$ for $x_2 = h$

$\sigma \boldsymbol{\beta} = 0$ for $(x_2-h) = -h(x_1-w_1)/(w_2-w_1)$

Body force

$\boldsymbol{b} = -\rho_c g e_2$

(c) Define a test function v

$$v\left[-\frac{d}{dx}(AV)+Q\right]=0$$

$$\int_0^L v\left[\frac{d}{dx}(Ak\frac{dH}{dx})+Q\right]dx=0$$

Integrate by parts to derive the weak form.

$$\left[vAk\frac{dH}{dx}\right]_0^L-\int_0^L\frac{dv}{dx}Ak\frac{dH}{dx}dx+\int_0^L vQdx=0$$

$$\int_0^L\frac{dv}{dx}Ak\frac{dH}{dx}dx=-(vAV)_{x=L}+(vAV)_{x=0}+\int_0^L vQdx$$

$$\int_0^L\frac{dv}{dx}Ak\frac{dH}{dx}dx=v(0)A\bar{V}+\int_0^L vQdx$$

($\because$ V=0 at x=L (no flux))

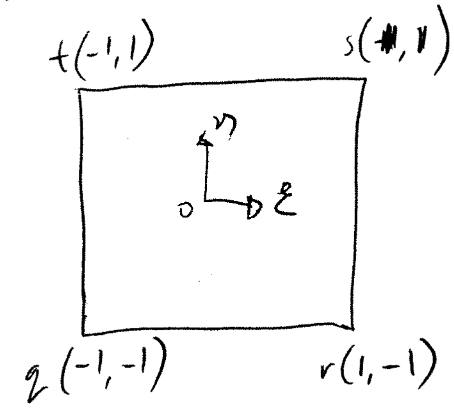
Hence, the problem is restated as to find H such that the following equations hold for all $\{v|v(L)=0\}$

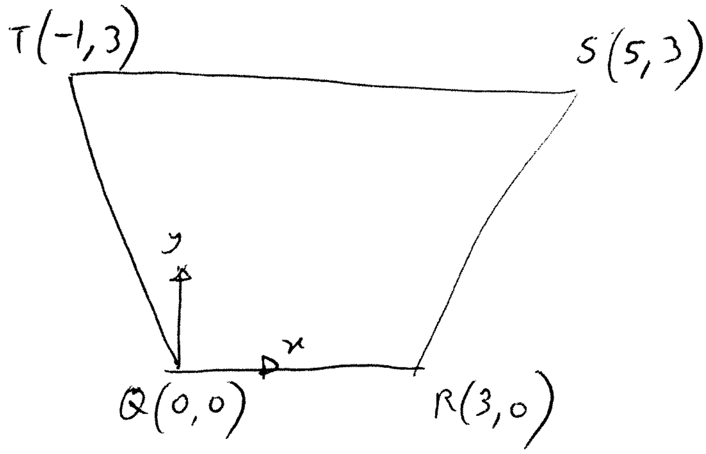
$$\int_0^L\frac{dv}{dx}Ak\frac{dH}{dx}dx=v(0)A\bar{V}+\int_0^L vQdx$$

$H(x=L)=\bar{H}$

Parent element

Q.4





(a)

shape functions from (data sheet?)

$$n_q = (1-\xi)(1-\eta)/4 \qquad n_s = (1+\xi)(1+\eta)/4$$

$$n_r = (1+\xi)(1-\eta)/4 \qquad n_t = (1-\xi)(1+\eta)/4$$

(b)

Geometric mapping:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} n_q & 0 & n_r & 0 & n_s & 0 & n_t & 0 \\ 0 & n_q & 0 & n_r & 0 & n_s & 0 & n_t \end{bmatrix}}_{\underset{\sim}{N}} \begin{bmatrix} x_Q \\ y_Q \\ x_R \\ y_R \\ \vdots \end{bmatrix} \quad \text{here} \quad \begin{bmatrix} x_Q \\ y_Q \\ x_R \\ y_R \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 5 \\ 3 \\ -1 \\ 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3n_r + 5n_s - n_t \\ 3n_s + 3n_t \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 3(1+\xi)(1-\eta) + 5(1+\xi)(1+\eta) - (1-\xi)(1+\eta) \\ 3(1+\xi)(1+\eta) + 3(1-\xi)(1+\eta) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} (1+\xi)(8+2\eta) - (1-\xi)(1+\eta) \\ 6(1+\eta) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8+2\eta+8\xi+2\xi\eta-1-\eta+\xi+\xi\eta \\ 6(1+\eta) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7+9\xi+\eta+3\xi\eta \\ 6(1+\eta) \end{bmatrix}$$

(c) Point $p(0.75, 0.75)$ corresponds to:

$$\begin{bmatrix} x \\ y \end{bmatrix}_p = \frac{1}{4}\begin{bmatrix} 7 + 9\times\frac{3}{4} + \frac{3}{4} + 3\times\frac{3}{4}\times\frac{3}{4} \\ 6\left(1+\frac{3}{4}\right) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} \frac{1}{16}(112 + 108 + 12 + 27) \\ 6\left(\frac{7}{4}\right) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} \frac{1}{16}\times 259 \\ \frac{21}{2} \end{bmatrix} = \begin{bmatrix} 4.047 \\ 2.625 \end{bmatrix}$$

$$\underset{\sim}{u} = N\underset{\sim}{d}$$

(d) The line is not straight due to non-linear terms $\xi\eta$

A straight line would pass through $\left[\frac{0.75}{2}\times 5, \frac{0.75\times 3}{2}\right]$

$$= \begin{bmatrix} 4.375 \\ 2.625 \end{bmatrix}$$

(e) Exactly same matrix $\underset{\sim}{N}$ is used to transform displacements

So $\begin{bmatrix} u \\ v \end{bmatrix} = \underset{\sim}{N}\begin{bmatrix} 0.1 \\ 0.2 \\ 0 \\ 0 \\ 0.15 \\ 0 \\ -0.05 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1n_q + 0.15n_s - 0.05n_t \\ 0.2n_q + 0 \end{bmatrix}$

for $\xi = 0.75$
$\eta = 0.75$

$n_q = \frac{1}{16}$

$n_s = \frac{49}{64}$

$n_t = \frac{7}{64}$

$$= \begin{bmatrix} \frac{0.1}{16} + 0.15\times\frac{49}{64} - 0.05\times\frac{7}{64} \\ 0.2\times\frac{1}{16} \end{bmatrix} = \begin{bmatrix} 0.116 \\ 0.0125 \end{bmatrix}$$