D. RALPH

1(a)

(i) Let g_i = amount of power generated at each node $i = 1, 3$

x_{ij} = amount of power flowing from node i to node j for $(i,j) = (1,2), (1,4), (4,1), (3,4)$, and optionally $(2,1), (4,3)$

LP:

$$\min\ c_1 g_1 + c_3 g_3$$ (optionally subtract $p_2 d_2 + p_4 d_4$) 7%

subject to

$$d_2 = x_{12} (- x_{21})$$
$$d_4 = x_{14} - x_{41} + x_{34} (- x_{43})$$ 7%

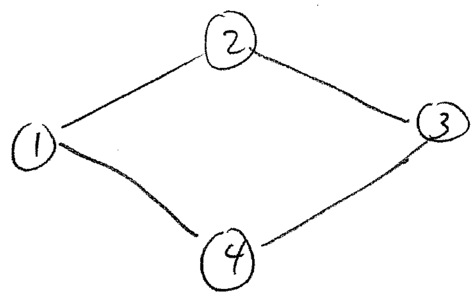
$$g_1 = x_{12} + x_{14} - x_{41} (- x_{21})$$
$$g_3 = x_{34} (- x_{43})$$ 7%

$$x_{12}, x_{14}, x_{41}, x_{34} \geq 0$$
$$(x_{21}, x_{43} \geq 0)$$
$$g_1, g_3 \geq 0$$ 4%

where the terms in brackets may be optionally omitted (will have value zero)

25%

1(a) ii)



Change constraints involving d_2 and g_3:

$$d_2 = x_{12} - x_{21} + x_{32} - x_{23}$$

$$g_3 = x_{32} - x_{23} + x_{34} - x_{43}$$

10%

where x_{23}, x_{32} are new variables.

Add new constraint on power flow in cycle:

$$\sum_{i=1}^{4} \sum_{j=1}^{i} x_{ij} - x_{ji} = 0$$

15%

↑ signed flow from node i to j (= − flow from j to i)

or $x_{12} + x_{23} + x_{34} + x_{41} = x_{14} + x_{43} + x_{32} + x_{21}$

↑ clockwise flow — anti clockwise

25%

1) b) Change to min problem: $C = (-15, -10, 10, 0, 10, 0)$

i) $B^{-T} c_B = (-5, 0, 5, 0)$

Reduced cost: $r = c_N - N^T B^{-T} c_B = (10, 5) \geq 0$ hence optimal (if feasible)] 8%

$$B = \begin{bmatrix} 5 & 3 & 0 & 0 \\ 3 & 2 & -1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \\ 4 & 0 \end{bmatrix}$$

$c_B = (-15, 10, 0, 0)$

$c_N = (-10, -10)$

Check feasibility:

$x_B = B^{-1} b = B^{-1}(18, 9, 7, 5)$
$= (3, 1, 2, 1)$
≥ 0 hence feasible] 8%

Thus basis is optimal.

Optimal vector $(3, 1, 0, 2, 0, 1)$

Optimal profit 55 (= − opt. cost)] 4%

20%

ii) Not jeopardising optimality means $r \geq 0$

If C_B changes from $(-15, -10, 0, 0)$
to $(-15+\varepsilon, 10, 0, 0)$
then r changes by $-N^T B^{-T}(\varepsilon, 0, 0, 0)$, where] 5%

$B^{-T}(1, 0, 0, 0) = (-1, 0, 3, 0)$

$-N^T B^{-T}(\text{———}) = (-2, -3)$

So $r + \varepsilon(-2, -3) = (10 - 2\varepsilon, 5 - 3\varepsilon)$
≥ 0 if $\varepsilon \leq 5$ & $\varepsilon \leq 5/3$

∴ max reduction is by amount $\varepsilon = 5/3$.] 10%

1) b)

ii) Cont.

If profit coeff for x_1 changes from 15 to $15 - \frac{5}{3} = 13\frac{1}{3}$,
optimal vector = same as in (i)
optimal profit = $55 - \frac{5}{3}x_1 = 55 - 5 = 50$] 5%

20%

iii) Shadow price $\lambda = B^{-T} C_B = (-5, 0, 5, 0)$ from (i).] 5%

Shadow price of changing 3rd constraint is 5, i.e.
optimal cost increases by 5Δ for a small increase Δ
in rhs of 3rd constraint from 7 to $7+\Delta$.
$\therefore$ profit decreases by 5Δ.] 5%

(This is better than decreasing by 10Δ
as suggested in the question).

10%

2 (a)

i) $x_A, x_B, \ldots, x_E$ are 0-1 variables,

$x_A = 1 \Rightarrow$ go ahead on project A

$\quad\; 0 \Rightarrow$ do not carry out project A

Constraints / goals

cash burn : hard constraint

$72x_A + \cdots + 71x_E \leq 250$] 5%

Revenue:
$128x_A + \cdots + 115x_E = 130 - u_1 + o_1$] 5%

(*) Staff:

$16x_A + \cdots + 10x_E = 35 - u_2 + o_2$] 5%

$u_2, o_2 \leq 5$] 3%

$x_A, \ldots, x_E$ are binary] 3%

u_1, u_2, o_1, o_2 nonnegative

Choose nonneg. weights w_1, w_2, v_2 with $w_1 > w_2 = v_2$,
eg. $w_1 = 2$, $w_2 = v_2 = 1$. ↑ 4%

min α

subject to $\alpha \geq \frac{u_1}{350} w_1$

$\alpha \geq \frac{u_2}{35} w_2$ 5%

$\alpha \geq \frac{o_2}{35} v_2$

and constraints (*).

[30%]

2 (a) (ii)

Update cash burn constraint

$$72x_A + \cdots + 71x_E - 14y \leq 250 \quad]\ 4\%$$

where y is new variable:

y binary] 1%

$$2y \leq x_A + x_C + x_D \quad]\ 5\%$$

The last condition forces $y = 0$ unless at least two of projects A, C, D go ahead.

10%

2(a) (iii) In order of increasing importance] 6%

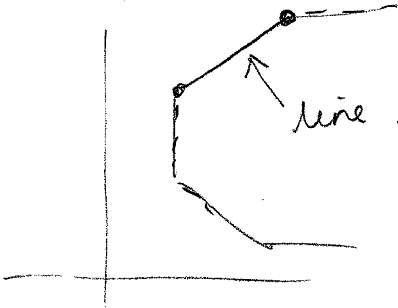
opt value - actual value meaningless] 3%

opt dec vars - useful to implement strategy but not so useful in its selection] 3%

opt deviations - tradeoffs between deviations from goals are what the decision maker thinks about when evaluating any strategy.] 3%

2 (b)

(i)

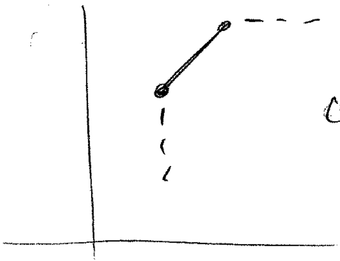


line segment is pareto-optimal

5%

(ii)

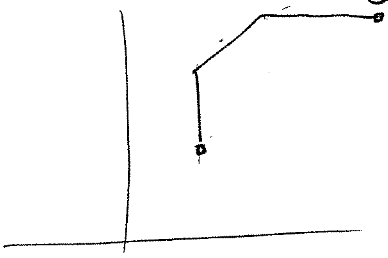
Positive weight solutions



coincide with pareto set

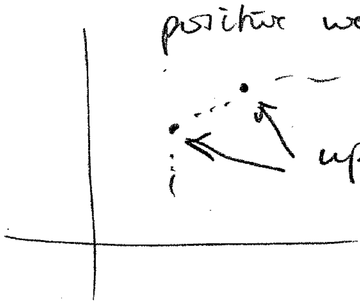
5%

nonneg weight solutions



5%

positive weights & simplex



upper left vertices only

5%

15%

2)

2 [c) Problem is already in "min" form.

objective $f(x,u) = e^{x_3^2} + u_1^2 + (u_2-u_1)^2$

2 [$Df(x,u) = (0, 0, 2x_3 e^{x_3^2}, 2u_1 + 2(u_1-u_2), 2(u_2-u_1))$

6 [

$$D^2 f(x,u) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (2+4x_3^2)e^{x_3^2} & 0 & 0 \\ 0 & 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & -2 & 2 \end{pmatrix}$$

6 [Block diagonal: * every diagonal element non neg

$0, 0, 2+(4x_3)^2 e^{x_3^2} \geq 0$

* block $\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$ also positive definite

since diag. > 0, det $= 8-4 = 4 > 0$.

Thus objective is convex

2) c) cont

2 [Linear constraints are convex.

2 [Only nonlinear constraint can be formulated as $g(u) \leq 0$ where $g(u) = u_1^2 + u_2^2 - 1$

4 [$\nabla g(u) = (2u_1, 2u_2)$

$D^2 g(u) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ pos. def.

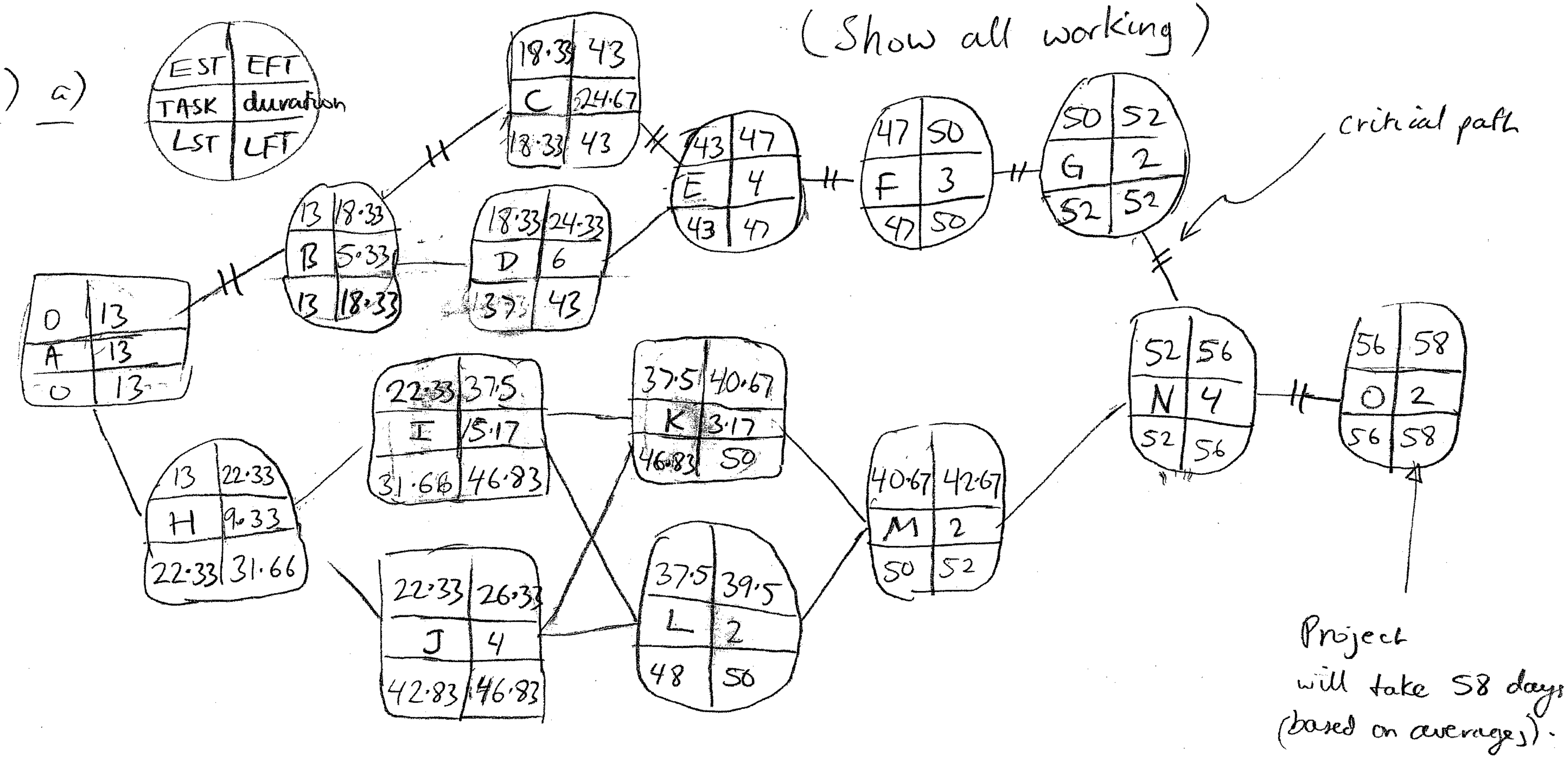
So $g(u) \leq 0$ is convex

1 [Therefore the problem is a convex NLP.

25%

3) a)

(Show all working)



Using average durations (as above), critical activities are A, B, C, E, F, G, N, O.

Variance on critical path = sum of duration variances for critical activities

$\simeq 5.56 \quad (5.555\ldots)$

Standard deviation $\simeq \sqrt{5.56} \simeq 2.358$.

[Correct durations & CPM calculations: 25%

[Standard dev: 3%]

3)

a) cont.

If $z \sim N(0,1)$ (standard normal random variable)

then normal tables yield

$$P(z \leq Z) = 0.95$$

for $Z = 1.645$

So if t = random length of project,

then $t \sim N(58, 2.358)$,

hence $$\frac{T-58}{2.358} = Z = 1.645$$

or $T \simeq 61.879$ or 61.9 weeks

7%

35%

b) If the actual critical path realised in the project is as predicted by PERT, the above analysis suggests there is 5% chance of overrunning the 61.9 week "limit". [The assumptions behind the normal distribution used for the random completion time, t above, are debatable ...]. 5

However, if a non critical activity is significantly delayed, an unforeseen (by PERT) critical path may result. Thus the actual risk of overrunning T could be expected to exceed 5%. 15% 10

3)

(b) (i)

$$\min \; -x_1 - x_2 - 2x_1x_2 + \frac{1}{2}(x_1^2 + 3x_2^2) = f(x_1, x_2)$$

subj to $\quad 1 \le x_1 \le 2$

$\qquad\qquad 1 \le x_1 \le 2$

$x^0 = (3/2, 3/2)$, $\quad \nabla f(x_1, x_2) = \begin{pmatrix} -1 + x_1 - 2x_2 \\ -1 - 2x_1 + 3x_2 \end{pmatrix}$ 5%

Steepest descent direction:

$$d^0 = -\nabla f(\tfrac{3}{2}, \tfrac{3}{2})$$

$$= -\left[\left(-1 + \tfrac{3}{2} - 3, \; -1 - 3 + \tfrac{9}{2}\right)\right]$$

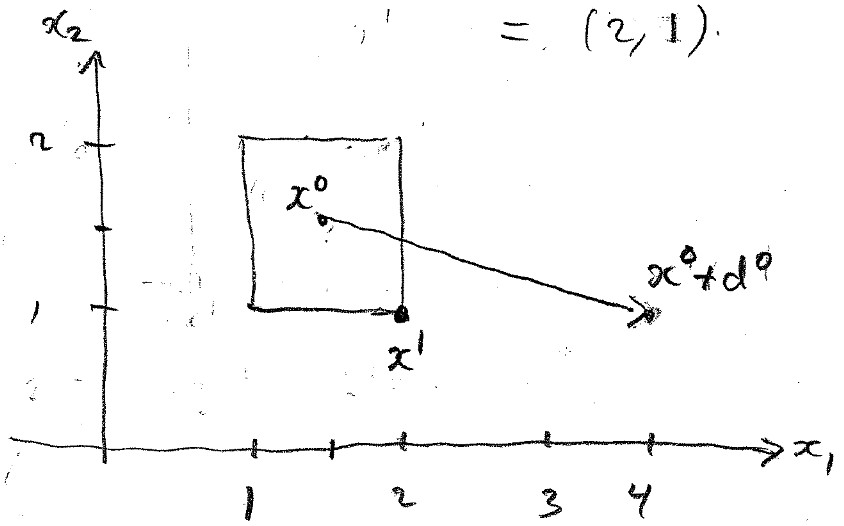
$$= \left(\tfrac{5}{2}, -\tfrac{1}{2}\right)$$

$$x^0 + d^0 = (4, 1)$$

Proj. gradient for $t = 1$ 15%

$$x^1 = y^0(1) = \text{Proj } x^0 + d^0 \text{ onto feas. set}$$

$$= (2, 1).$$



b) (i) cont

$x^1 = (2,1)$, $d^1 = -Df(2,1)$

$= -(-1+2-2, -1-4+3)$

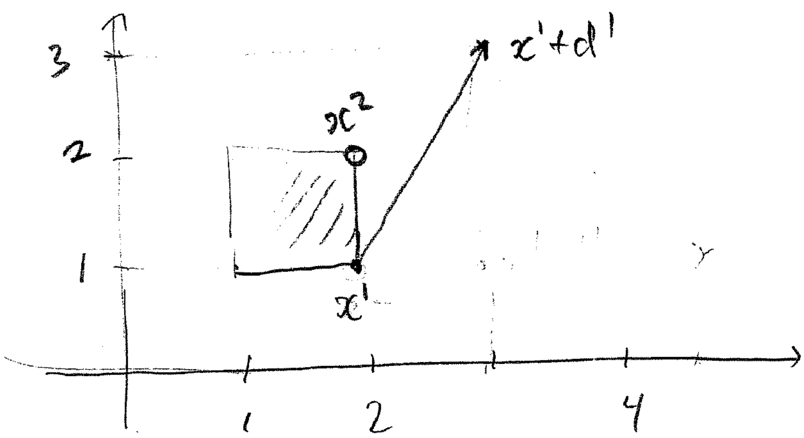
$= (1,2)$

$x^1+d^1 = (3,3)$

$x^2 = y^1(1) =$ Proj of x^1+d^1 onto box

$= (2,2)$

15%



Note: x^1 cannot be stationary because $x^2 \neq x^1$, i.e. x^2 is not the projection of $x^1 - Df(x^1)$ onto the feasible set

5%

40%

3 b) ii)

Is this problem is a convex program? – It's a min problem.

- its constraints are linear, and
- its objective function is quadratic with Hessian

$$\begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$$ NOT pos. def. because $\det = -1 < 0$

] 5%

hence nonconvex objective

Thus cannot tell if a stationary point is either a local min or a global min.] 5%

[10%]

4(a)

Let x = amount of sawdust mulch bought in a particular month

y = amount of mulch not exceeding 1000 kg

z = amount in excess of 1000 kg

I.e.

(1)
$$\begin{cases} x, y, z \geq 0 \\ y \leq 1000 \\ z \geq x - 1000 \\ y + z = x \end{cases}$$
10%

Also need binary variable b s.t.

$b = 1$ if $x \leq 1000$ (and $y = x$, $z = 0$)

$b = 0$ otherwise. (and $y = 1000$, $z = x - 1000$)

So

(2)
$$\begin{cases} b \text{ binary} \\ 1000 - y \leq 1000b \\ z \leq M(1-b) \end{cases}$$
15%

where $M \geq \max x - 1000$ subj to original constraints 5%

4(a) cont.

Then objective would also change in the $-0.2x$ term (subtract linear cost of mulch in pounds) 4%

to $-0.2y - 0.15z$

34%

4)

b)

$[0,1] \cup [2,3]$

$b_1 \qquad b_2$

ⓐ $\begin{cases} b_1, b_2 \text{ - binary} \\ b_1 + b_2 = 1 \end{cases}$] 6%

where $b_1 = 1 \iff x \in [0,1]$

$b_2 = 1 \iff x \in [2,3]$

ⓑ

$0 \le x \le 3$] 5%

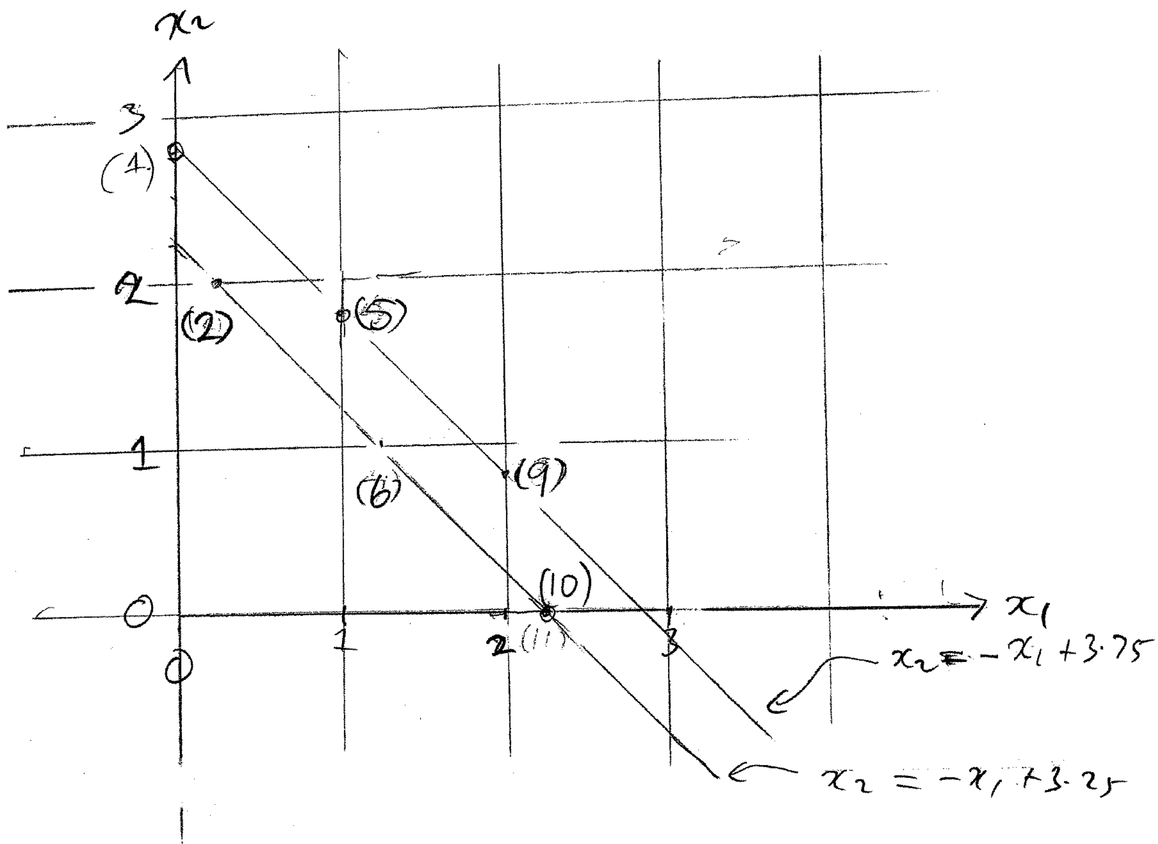
① — $2 - x \le 2b_1$ (so $x < 2 \Rightarrow b_1 = 1 \Rightarrow b_2 = 0 \Rightarrow x \le 1$)] 11%

② — $x - 1 \le 3b_2$ (so $x > 1 \Rightarrow b_2 = 1 \Rightarrow b_1 = 0 \Rightarrow x \ge 2$ from ①)] 11%

Complete formulation uses ⓐ and ⓑ.

33%

4) c)



Solution to "LPk" is denoted "(k)" in the above diagram,

For example:

LP1: max x_2 subj to
$$\left.\begin{array}{l} x_1 + x_2 \le 2.75 \\ x + x_2 \ge 2.25 \\ x_1, x_2 \ge 0 \end{array}\right\} \quad (1)$$

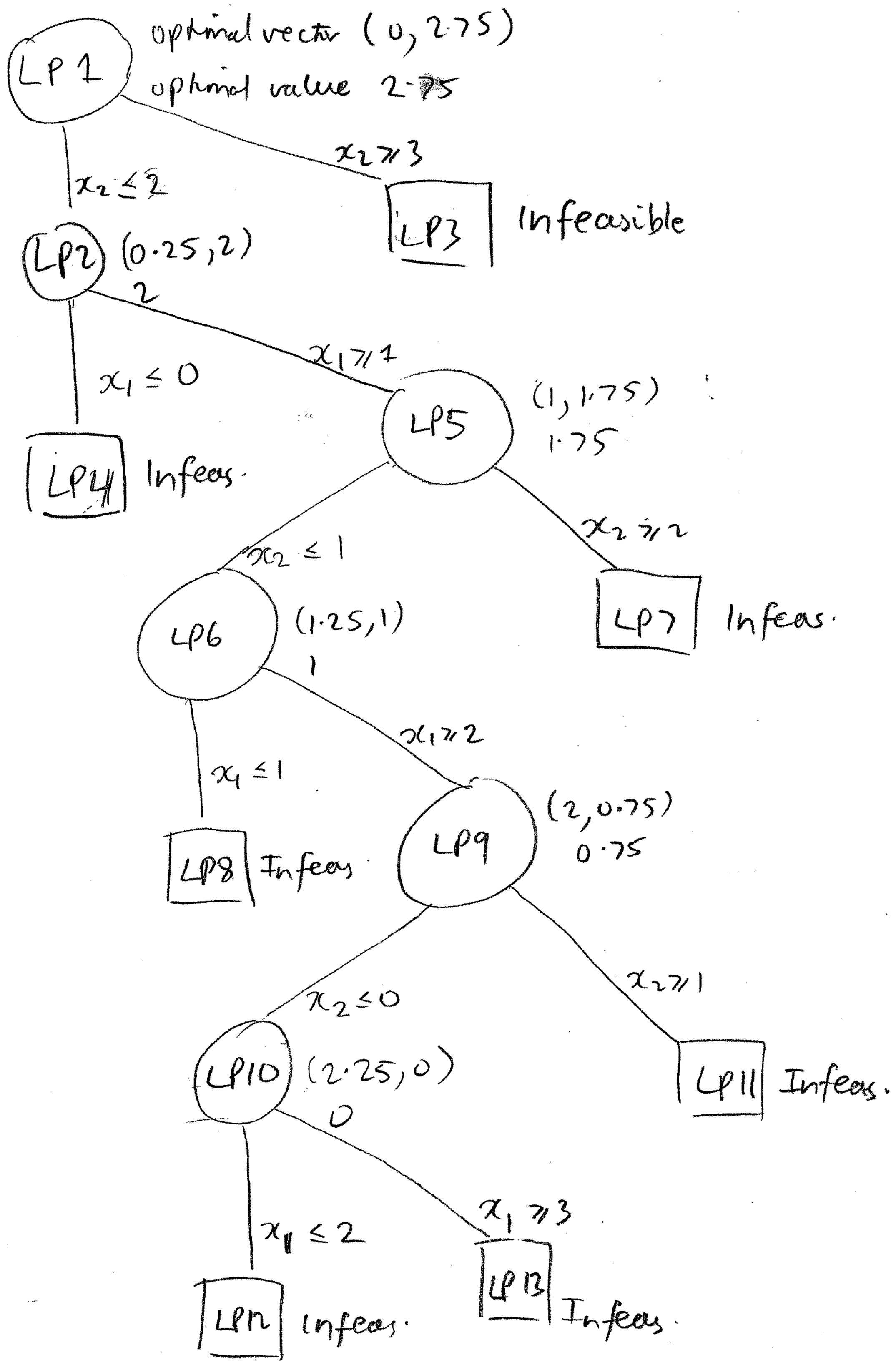
Solution $x = (0, 2.75)$
$x_2 = 2.75$ opt val

Branch on x_2:

LP2: add constraint $x_2 \le 2$ ..
→ solution vector $(0.25, 2)$
(any x_1 in $[0.25, 1]$)
optimal value 2

LP3: add constraint $x_3 \ge 3$ ... Infeasible.

4) c) cont.



[Marks given roughly in proportion to amount of tree correct] [33%]