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3F1 – 30 April 2003 – Solutions

1. (soln) (a)(i) $$U(z) = \sum_{k=0}^{\infty} \beta^k z^{-k} = \sum_{k=0}^{\infty} (\beta z^{-1})^k$$

$$= 1 + \beta z^{-1} + (\beta z^{-1})^2 + \cdots\cdots$$

$$= \frac{1}{1-\beta z^{-1}} \quad \text{(infinite geometric series)}$$

(ii) transfer function $H(z) = \dfrac{Y(z)}{U(z)} = \dfrac{1}{1-\alpha z^{-1}} \left(\dfrac{1}{1-\beta z^{-1}} \right)^{-1}$

$$= \frac{z-\beta}{z-\alpha}$$

(iii) $(z-\beta)U(z) = (z-\alpha)Y(z)$

$$\Rightarrow \quad y_{k+1} - \alpha y_k = u_{k+1} - \beta u_k$$

(b) (i) pulse response: $U(z) = 1 \Rightarrow Y(z) = H(z) = \dfrac{z-\beta}{z-\alpha} = \dfrac{z}{z-\alpha} - \dfrac{\beta}{z-\alpha}$

$$\Rightarrow \quad H(z) = \frac{1}{1-\alpha z^{-1}} - \frac{\beta z^{-1}}{1-\alpha z^{-1}}$$

$$\Rightarrow \quad h_k = \begin{cases} 1 & k = 0 \\ \alpha^k - \beta \alpha^{k-1} & k \geq 1 \end{cases}$$

step response $= \displaystyle\sum_{i=0}^{k} h(i)$

$$= \begin{cases} 1 & k=0 \\ \dfrac{1-\alpha^{k+1}}{1-\alpha} - \beta \dfrac{1-\alpha^k}{1-\alpha} & k \geq 1 \end{cases}$$

alter:
$$Y(z) = \frac{z-\beta}{z-\alpha} \cdot \frac{z}{z-1} = \frac{z}{z-\alpha} \cdot \frac{\alpha-\beta}{\alpha-1} + \frac{z}{z-1} \cdot \frac{1-\beta}{1-\alpha}$$

$$\Rightarrow \quad y_k = \frac{\alpha-\beta}{\alpha-1}\,\alpha^k + \frac{1-\beta}{1-\alpha}$$

$$= \frac{1}{1-\alpha}\left((\beta-\alpha)\alpha^k + 1-\beta\right)$$

(ii) $$y_1 = \alpha y_0 + u_1 - \beta u_0$$
$$y_2 = \alpha y_1 + u_2 - \beta u_1$$

pulse: $y_0 = 1$ ✓
$y_1 = \alpha - \beta$ ✓
$y_2 = \alpha(\alpha-\beta)$ ✓

step: $y_0 = 1$ ✓
$y_1 = \alpha + 1 - \beta$
$y_2 = \alpha(\alpha+1-\beta) + 1 - \beta$

check: $(\alpha+1-\beta)(1-\alpha) = \cancel{\alpha} + 1 - \beta - \alpha^2 - \cancel{\alpha} + \alpha\beta = \alpha(\beta-\alpha) + 1 - \beta$ ✓
$(\alpha^2+\alpha-\alpha\beta+1-\beta)(1-\alpha) = \cancel{\alpha^2} + \cancel{\alpha} - \cancel{\alpha\beta} + 1 - \beta - \alpha^3 - \cancel{\alpha^2} + \alpha^2\beta - \cancel{\alpha} + \cancel{\alpha\beta}$
$= \alpha^2(\beta-\alpha) + 1 - \beta)$ ✓

(c) d.c. gain $= 10 \iff \dfrac{1-\beta}{1-\alpha} = 10$

at 5 Hz $\quad |H(z)| = \dfrac{10}{\sqrt{2}} \quad$ i.e. $\left|\dfrac{e^{j\omega T}-\beta}{e^{j\omega T}-\alpha}\right| = \dfrac{10}{\sqrt{2}}$

where $T = 10^{-3}$, $\omega = 10\pi$

$\omega T = \pi/100$ (small) $\qquad e^{j\omega T} \simeq 1 + j\omega T$

Need $\quad |1-\beta+j\omega T| = \dfrac{10}{\sqrt{2}}|1-\alpha+j\omega T|$

$$(1-\beta)^2 + \omega^2T^2 = 50\left((1-\alpha)^2 + \omega^2T^2\right)$$

$$\underset{}{\overset{100(1-\alpha)^2}{}}$$

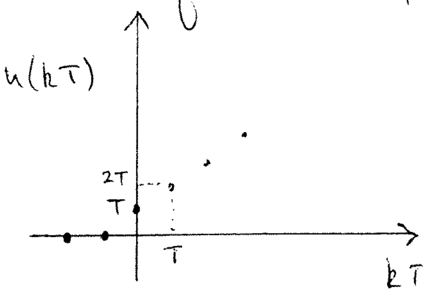
$\Rightarrow \quad 50(1-\alpha)^2 = 49\,\omega^2T^2 \quad \Rightarrow \quad 1-\alpha = \sqrt{\dfrac{49}{50}}\,\omega T$

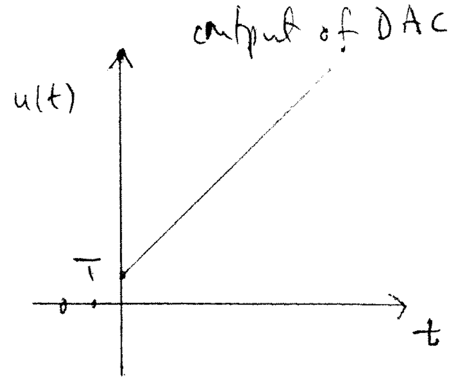
$\Rightarrow \alpha = 0.969$
$\beta = 0.689$

2. (soln.) (i) All elements are linear and time-invariant

(a)(ii)

Consider the following input:





This gives a convenient signal $u(t)$ and hence a useful formula for $H(z)$.

$$u(t) = T + t \Rightarrow U(s) = \frac{T}{s} + \frac{1}{s^2} = \frac{Ts+1}{s^2}$$

$$u(kT) = T + kT \Rightarrow U(z) = \frac{T}{1-z^{-1}} + \frac{Tz^{-1}}{(1-z^{-1})^2} = \frac{T}{(1-z^{-1})^2} = \frac{Tz^2}{(z-1)^2}$$

Then $\{y(kT)\} = \mathcal{L}^{-1}\left(G(s)\,\frac{Ts+1}{s^2}\right)\Big|_{t=kT}$

and the result now follows from: $H(z) = Y(z)/U(z)$.

(b) (i) $P(A,B) = P(A|B)P(B) = P(B|A)P(A)$

Hence $P(B|A) = \dfrac{P(A|B)P(B)}{P(A)}$ Bayes' rule.

(ii)

$k=0 \Rightarrow \varepsilon(n) = 0.2, -0.2, 0.4, -0.1$

$k=1 \Rightarrow \varepsilon(n) = 0.7, -0.2, -0.1, -0.1$

$k=2 \Rightarrow \varepsilon(n) = 0.7, 0.3, -0.1, -0.6$

Need to find $P(k|\underline{x}) = \dfrac{P(\underline{x}|k)P(k)}{P(\underline{x})}$ using Bayes' rule

$P(\underline{x}|k) = P(\underline{x}|v_k) = P(\underline{\varepsilon}_k) \propto f_\varepsilon(\varepsilon_k(0)) \times f_\varepsilon(\varepsilon_k(1)) \times f_\varepsilon(\varepsilon_k(2)) \times f_\varepsilon(\varepsilon_k(3))$

proportional to

Hence $P(\underline{n}\,|\,k)$ is proportional to

$$\exp\left(-\sum_{i=0}^{3}(q_k(i))^2/2\right) = S_k$$

$$S_0 = (0.2)^2 + (0.2)^2 + (0.4)^2 + (0.1)^2 = 0.25$$

$$S_1 = (0.7)^2 + (0.2)^2 + (0.1)^2 + (0.1)^2 = 0.55$$

$$S_2 = (0.7)^2 + (0.3)^2 + (0.1)^2 + (0.6)^2 = 0.95$$

$$P(k=0\,|\,\underline{n}) \propto P(\underline{n}\,|\,k=0)\,P(k=0) \propto e^{-0.25/2}\,0.25 = 0.2206$$

$$\ldots \quad e^{-0.55/2}\,0.35 = 0.2659$$

$$\ldots \quad e^{-0.95/2}\,0.4 = 0.2488$$

Hence $k=1$ given most probable position.

For any process

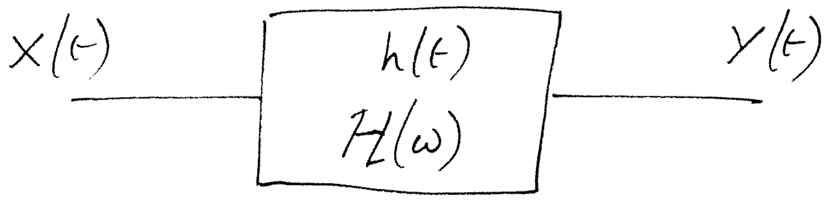
3. $r_{xx}(t_1, t_2) = E[x(t_1, \alpha)\, x(t_2, \alpha)]$

For an ergodic process, α & t are interchangeable so

$$r_{xx}(\tau) = E[x(t)\, x(t+\tau)]$$
$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt$$

Similarly for cross-correlation:

$$r_{xy}(\tau) = E[x(t)\, y(t+\tau)]$$



(a) $r_{xy}(\tau) = E[x(t)\, y(t+\tau)]$

$$= E\left[x(t) \cdot \int_{-\infty}^{\infty} h(\alpha)\, x(t+\tau-\alpha)\, d\alpha\right]$$

$$= E\left[\int h(\alpha)\, x(t)\, x(t+\tau-\alpha)\, d\alpha\right]$$

$$= \int h(\alpha)\, E[x(t)\, x(t+\tau-\alpha)]\, d\alpha$$ 06

$$= \int_{-\infty}^{\infty} h(\alpha)\, r_{xx}(\tau-\alpha)\, d\alpha = \underline{\underline{h(\tau) * r_{xx}(\tau)}}$$

3. (b) $S_x(\omega)$ is the Fourier Transform of $r_{xx}(\tau)$

ie $S_x(\omega) = \displaystyle\int_{-\infty}^{\infty} r_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau$

$$\therefore S_y(\omega) = \int r_{yy}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$r_{yy}(\tau) = E\left[y(t)\,y(t+\tau)\right]$$

$$= E\left[\left(\int h(\alpha_1)\,x(t-\alpha_1)\,d\alpha_1\right)\left(\int h(\alpha_2)\,x(t+\tau-\alpha_2)\,d\alpha_2\right)\right]$$

$$= \iint h(\alpha_1)\,h(\alpha_2)\; E\left[x(t-\alpha_1)\,x(t+\tau-\alpha_2)\right] d\alpha_1\, d\alpha_2$$

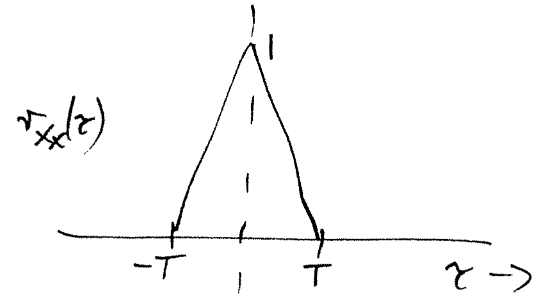
$$= \iint h(\alpha_1)\,h(\alpha_2)\; r_{xx}(\tau+\alpha_1-\alpha_2)\, d\alpha_1\, d\alpha_2$$

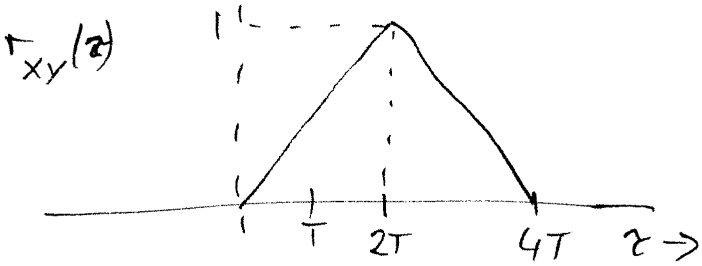
$$= r_{xx}(\tau) * h(-\tau) * h(\tau)$$

$$h(\tau) \rightleftharpoons H(\omega) \quad \& \quad h(-\tau) \rightleftharpoons H^*(\omega)$$

$$\therefore S_y(\omega) = S_x(\omega)\,.\,H^*(\omega)\,.\,H(\omega) = \underline{\underline{S_x(\omega)\,.\,|H(\omega)|^2}}$$

3. (c)

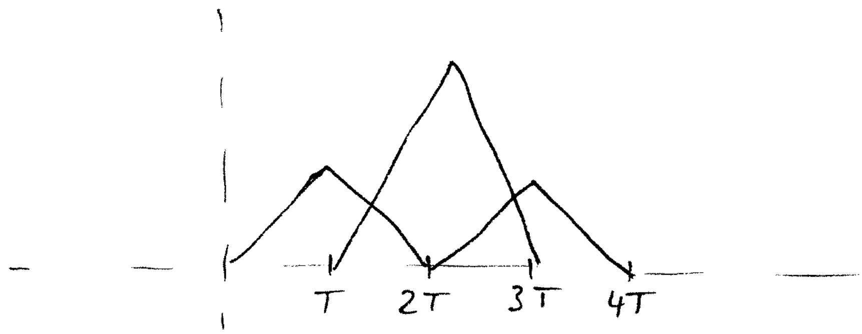




$$r_{xy}(\tau) = r_{xx}(\tau) * h(\tau)$$

~~We can generate r_{xy}~~

Both r_{xx} & r_{xy} comprise linear ramp functions so we can generate $r_{xy}(\tau)$ from a linear combination of shifted $r_{xx}(\tau)$ as follows:



$$r_{xy}(\tau) = \tfrac{1}{2} r_{xx}(\tau - T) + r_{xx}(\tau - 2T) + \tfrac{1}{2} r_{xx}(\tau - 3T)$$

This is equivalent to convolving $r_{xx}(\tau)$ with

$$h(t) = \tfrac{1}{2}\delta(t - T) + \delta(t - 2T) + \tfrac{1}{2}\delta(t - 3T)$$

Hence this is the impulse response of the plant.

(d) By taking the ratio ~~of~~ $\dfrac{S_y(\omega)}{S_x(\omega)}$ we may obtain $|H(\omega)|^2$, <u>but</u> this only tells about the <u>magnitude</u> of $H(\omega)$, not about its phase. Whereas $r_{xy}(\tau)$ and $r_{xx}(\tau)$ can tell us the phase as well as the magnitude via $h(\tau)$. Also if $S_x(\omega)$ has any ~~zeros~~ zeros, we could not estimate $|H(\omega)|$ at these ω.

4 If X has N values with probabilities P_i $(i = 1 \cdots N)$

Then $H(X) = -\sum_{i=1}^{N} P_i \log_2(P_i)$

$I(X, Y) = H(X) - H(X|Y)$

OR $= H(Y) - H(Y|X)$

OR $= H(X) + H(Y) - H(X, Y)$

a) True if $\begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$ is an Eigen vector of

$\begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{pmatrix}$ with Eigen value $= 1$.

Check: $\begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{pmatrix} \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$ ✓

b)(i) $H(S_n) = H(0.27, 0.75) = \underline{\underline{0.8113}}$ (4sf)

(ii) $H(S_n | S_{n-1}) = 0.25 \times H(0.7, 0.3) + 0.75 \times H(0.1, 0.9)$

$= 0.25 \times 0.8813 + 0.75 \times 0.4690$

$= \underline{\underline{0.5721}}$ (4sf) c)(i) $I(S_{n+1}; S_n) = H(S_{n+1}) - H(S_{n+1} \mid S_n)$

$= 0.8113 - 0.5721 = \underline{0.2392}$ (4s.f.)

$I(S_{n+2}; S_n) = H(S_{n+2}) - H(S_{n+2} \mid S_n)$

i) $H(S_{n+2} \mid S_n)$ depends on transition probabilities from S_n to S_{n+2}

If $S_n = A$

Then $P(S_{n+2} = A) = P(AAA) + P(ABA)$ (given $S_n = A$)

$= 0.7 \times 0.7 + 0.3 \times 0.1 = 0.52$

$P(S_{n+2} = B) = P(AAB) + P(ABB)$

$= 0.7 \times 0.3 + 0.3 \times 0.9 = 0.48$

If $S_n = B$

$P(S_{n+2} = A) = P(BAA) + P(BBA)$

$= 0.1 \times 0.7 + 0.9 \times 0.1 = 0.16$

$P(S_{n+2} = B) = P(BAB) + P(BBB)$

$= 0.1 \times 0.3 + 0.9 \times 0.9 = 0.84$

So transition matrix is:

S_n \ $P(S_{n+2})$	A	B
A	0.52	0.48
B	0.16	0.84

and $H(S_{n+2} \mid S_n) = 0.25 \times H(0.52, 0.48)$

$+ 0.75 \times H(0.16, 0.84)$

$= 0.25 \times 0.9988 + 0.75 \times 0.6343$

$= 0.7254$

So $I(S_{n+2}; S_n) = 0.8113 - 0.7254 = \underline{0.0859}$ (4sf)

This is less than $I(S_{n+1}; S_n)$ because all mutual information between S_n, S_{n+2} is conveyed via S_{n+1}.

(d) Easy Way: Loss of information

$$= H(S_{k-1}, S_k, S_{k+1}) - H(S_{k-1}, S_{k+1})$$

$$= H(S_{k-1}) + H(S_k \mid S_{k-1}) + H(S_{k+1} \mid S_k) - H(S_{k-1}) - H(S_{k+1} \mid S_{k-1})$$

$$= H(S_n) - 2I(S_{n+1}; S_n) + I(S_{n+2}; S_n)$$

$= 0.8113 - 2 \times 0.2392 + 0.0859$

$= 0.4188$ (4sf)

(4)

Hard Way:

write down probabilities for $S_{\ell-1}, S_\ell, S_{\ell+1}$

AAA	$0.25 \times 0.7 \times 0.7$	$= 0.1225$
ABA	$0.25 \times 0.3 \times 0.1$	$= 0.0075$
AAB	$0.25 \times 0.7 \times 0.3$	$= 0.0525$
ABB	$0.25 \times 0.3 \times 0.9$	$= 0.0675$
BAA	$0.75 \times 0.1 \times 0.7$	$= 0.0525$
BBA	$0.75 \times 0.9 \times 0.1$	$= 0.0675$
BAB	$0.75 \times 0.1 \times 0.3$	$= 0.0225$
BBB	$0.75 \times 0.9 \times 0.9$	$= 0.6075$

work out the entropy of this distribution $= 1.9554$

subtract $H(S_j S_{n+2})$

AA	$0.1225 + 0.0075$	$= 0.13$
AB	$0.0525 + 0.0675$	$= 0.12$
BA	$0.0525 + 0.0675$	$= 0.12$
BB	$0.0225 + 0.6075$	$= 0.63$

entropy $= 1.5367$

Information loss $= 1.9554 - 1.5367 = \underline{0.4187}$ (4sf)