G. VINNICOMBE

PART IIA 2003 MODULE 3F2 SOLUTIONS

A1 (a) $\dot{\underline{x}} = A\underline{x} + [B_1 \;\; B_2]\begin{bmatrix} u \\ d \end{bmatrix}$ ①

$\underline{y} = C\underline{x} + Dd$ ②

Take Laplace transform of ①:

$s\underline{x}(s) - \underline{x}(0) = A\underline{x}(s) + [B_1 \;\; B_2]\begin{bmatrix} u(s) \\ d(s) \end{bmatrix}$

or $(sI-A)\underline{x}(s) = [B_1 \;\; B_2]\begin{bmatrix} u(s) \\ d(s) \end{bmatrix} + \underline{x}(0)$

$\therefore \underline{x}(s) = (sI-A)^{-1}[B_1 \;\; B_2]\begin{bmatrix} u(s) \\ d(s) \end{bmatrix} + (sI-A)^{-1}\underline{x}(0)$

from ② $\underline{y}(s) = C(sI-A)^{-1}[B_1, B_2]\begin{bmatrix} u(s) \\ d(s) \end{bmatrix} + C(sI-A)^{-1}\underline{x}(0) + Dd(s)$

$= \underbrace{C(sI-A)^{-1}B_1}_{u\to y \text{ transfer fn.}} u(s) + \underbrace{[C(sI-A)^{-1}B_2 + D]}_{d\to y \text{ transfer fn.}} d(s) + C(sI-A)^{-1}\underline{x}(0)$

(b) By definition, $e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots$

Suppose $\underline{x}(t) = e^{At}\underline{x}(0) = \left(I + At + \frac{A^2t^2}{2!} + \cdots\right)\underline{x}(0)$

Then $\dot{\underline{x}}(0) = \left(0 + A + A^2t + \frac{A^3t^2}{2!} + \cdots\right)\underline{x}(0)$

$= A\left(I + At + \frac{A^2t^2}{2!} + \cdots\right)\underline{x}(0)$

$= A\,\underline{x}(t)$. Hence verified.

If $u(t) \equiv 0$ but $d(t) \neq 0$ then

$\underline{x}(t) = e^{At}\underline{x}(0) + \int_0^t e^{A(t-\tau)} B_2\, d(\tau)\, d\tau$

$= e^{At}\underline{x}(0) + \int_0^t e^{A\tau} B_2\, d(t-\tau)\, d\tau$

Both forms correct. (Convolution integral.)

(c) If $\underline{x}(0) = 0$ and $d(t) = 1$ for $t > 0$

then $\underline{x}(t) = \int_0^t e^{A\tau} B_2\, d\tau$ [since $t-\tau > 0$ for $0 \le \tau < t$]

$\approx \int_0^t (I + A\tau) B_2\, d\tau$ for small enough t.

$= \left(It + \frac{At^2}{2}\right) B_2$ ①

A3 cont'd. (c) contd.

and hence $\underline{y}(t) \approx C\left(It + \frac{At^2}{2}\right)B_2 + D$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -100 & 100 & 0 & 0 \end{bmatrix} \left[\underbrace{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \end{matrix}}_{} \begin{matrix} 0 \\ 0 \\ 0 \\ t \end{matrix} \right] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

1st 3 columns not important because of form of B_2.

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -100 & 100 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix}}}$$

(d) Eigenvalues of A:

$$\det(sI-A) = \det \begin{bmatrix} s+1 & 0 & -1 & 0 \\ 0 & s & -1 & 0 \\ 5 & 0 & s+2 & 0 \\ 100 & -100 & 0 & s \end{bmatrix}$$

$$= s \det \begin{bmatrix} s+1 & 0 & -1 \\ 0 & s & -1 \\ 5 & 0 & s+2 \end{bmatrix}$$

$$= s^2 \det \begin{bmatrix} s+1 & -1 \\ 5 & s+2 \end{bmatrix}$$

$$= s^2\left[(s+1)(s+2)+5\right] = s^2\left(s^2+3s+7\right)$$

$$= s^2\left[\left(s+\tfrac{3}{2}\right)^2 + \left(7-\tfrac{9}{4}\right)\right]$$

$$= s^2\left[\left(s+\tfrac{3}{2}\right)^2 + \tfrac{19}{4}\right]$$

Hence eigenvalues of A are $\underline{\underline{0,\ 0,\ -\frac{3}{2} \pm \frac{\sqrt{19}}{2}j}}$

A8 cont'd. (d) cont'd. Suppose that $u = ky_1 = k[0\ 1\ 0\ 0]\underline{x}$

Then $\dot{\underline{x}} = A + B_1 k[0\ 1\ 0\ 0]\underline{x} + B_2 d$

$$= \underbrace{\left(A + kB_1[0\ 1\ 0\ 0]\right)}_{\text{Closed-loop stability depends on eigenvalues of this matrix.}}\underline{x} + B_2 d$$

Now $A + kB_1[0\ 1\ 0\ 0] =$

$$= \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & -2 & 0 \\ -100 & 100 & 0 & 0 \end{pmatrix} + k \begin{pmatrix} -1 \\ 0 \\ -20 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & -2 & 0 \\ -100 & 100 & 0 & 0 \end{pmatrix} + k \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \times & \times & \times & 0 \\ \times & \times & \times & 0 \\ \times & \times & \times & 0 \\ \times & \times & \times & 0 \end{pmatrix}$$

These don't matter. ← Column of zeros ⇒ singular ⇒ eigenvalue at 0 ⇒ not asymptotically stable.

If $u = ky_3 = k[-100\ \ 100\ \ 0\ \ 0]\underline{x} + kd$

the same argument holds, since this element is again 0.

The 'kd' term does not affect closed-loop stability since we have

$$\dot{\underline{x}} = \left(A + kB_1[-100\ \ 100\ \ 0\ \ 0]\right)\underline{x} + (B_1 k + B_2)d.$$

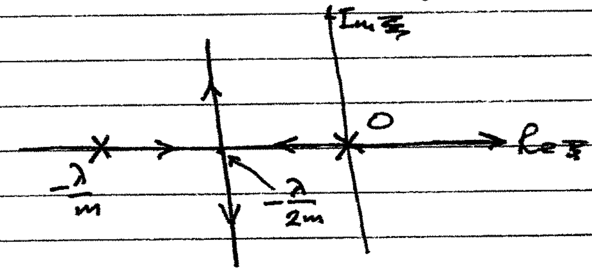
A.2 (a) $K(s) = k$, $G(s) = \dfrac{1}{s(ms+\lambda)}$ $\Rightarrow$ $K(s)G(s) = \dfrac{k}{s(ms+\lambda)}$

(This has monic denominator.) $= \dfrac{k/m}{s(s+\frac{\lambda}{m})}$

Root-locus is locus of roots of $1 + \dfrac{k/m}{s(s+\lambda/m)} = 0$.

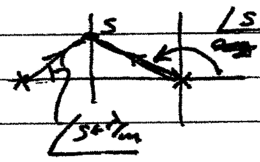
Root-locus plot:

Root-locus on real axis between the two poles.



Angle condition:

$-\angle s - \angle s+\lambda/m = -\pi$



is satisfied on $s = -\frac{\lambda}{2m} \pm j\gamma$ for any real γ (from diagram ↑).

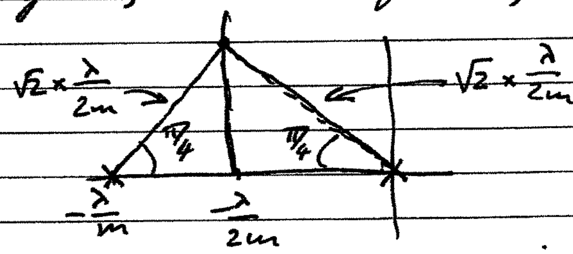
Closed-loop damping is $\frac{1}{\sqrt{2}}$ when $\angle s = 3\pi/4$.

The magnitude condition gives, at this point,

$$\frac{k}{m} = |s| \times \left|s + \frac{\lambda}{m}\right|$$

$$= \left(\sqrt{2} \times \frac{\lambda}{2m}\right)^2$$

$$= \frac{\lambda^2}{2m^2}$$



i.e. $k = \dfrac{\lambda^2}{2m}$

$\therefore$ closed-loop damping factor $\geq \frac{1}{\sqrt{2}}$ for $0 \leq k \leq \dfrac{\lambda^2}{2m}$

(Alternative solution:

Closed-loop characteristic polynomial is $s^2 + \frac{\lambda}{m}s + \frac{k}{m}$

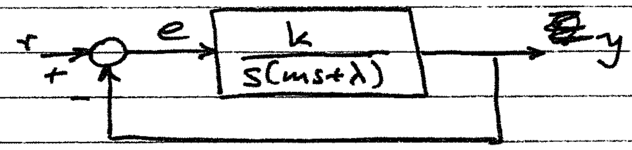
Compare this with standard form $s^2 + 2c\omega_n s + \omega_n^2$

so $c\omega_n = \frac{\lambda}{m}$ and $\omega_n = \sqrt{\frac{k}{m}}$, hence $c = \frac{\lambda}{m} \times \sqrt{\frac{m}{k}} = \frac{\lambda}{\sqrt{mk}}$

So $c = \frac{1}{\sqrt{2}} \Leftrightarrow \frac{1}{\sqrt{2}} = \frac{\lambda}{\sqrt{mk}} \Leftrightarrow k = \frac{\lambda^2}{2m}$.)

A.2 cont'd

(a) cont'd. $r(t) = \alpha t \Rightarrow \bar{r}(s) = \frac{\alpha}{s^2}$



Error : $e = r - y$

$$\frac{\bar{e}(s)}{\bar{r}(s)} = \frac{1}{1 + \frac{k}{s(ms+\lambda)}} = \frac{s(ms+\lambda)}{ms^2+\lambda s+k}$$

$$\therefore \bar{e}(s) = \frac{s(ms+\lambda)}{ms^2+\lambda s+k} \times \frac{\alpha}{s^2}$$

Hence, by Final Value Theorem :

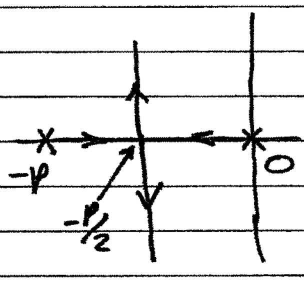
$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\bar{e}(s) = \lim_{s\to 0} \frac{(ms+\lambda)\alpha}{ms^2+\lambda s+k} = \frac{\lambda\alpha}{k}.$$

If $k \leq \frac{\lambda^2}{2m}$ then $\lim_{t\to\infty} e(t) \geq \lambda\alpha \times \frac{2m}{\lambda^2} = \frac{2m\alpha}{\lambda}$

(b) If $K(s) = \frac{k(s+\frac{\lambda}{m})}{s+p}$ then $K(s)G(s) = \frac{k(s+\frac{\lambda}{m})}{(s+p)} \times \frac{1}{s(ms+\lambda)}$

$$= \frac{k/m}{s(s+p)}$$

Now the root-locus has the same shape as before:



The closed-loop is critically-damped when both roots are at $-p/2$.
The value of gain required to achieve this is

$$\frac{k}{m} = \left(\frac{p}{2}\right)^2 \quad \frac{p^2}{2}$$ (using magnitude condition as before).

so $k \leq \frac{mp^2}{4}$

But the steady-state error is $\frac{\lambda\alpha}{k}$ mp $\leq \frac{m\alpha}{10\lambda}$ ← requirement

Hence $k \geq \frac{10\lambda^2}{m}$, so we need $\frac{mp^2}{4} \geq \frac{10\lambda^2}{m}$

or $p \geq \sqrt{40}\,\frac{\lambda}{m}$

$p \geq \frac{k}{10\lambda} \leq \frac{mp^2}{20\lambda}$ $\quad \frac{p}{k} \leq \frac{1}{10\lambda}$

$\Rightarrow p \geq 20\lambda/m$

A.2 cont'd.

(c) Suppose the nominal value of λ is λ_0.

This is used for the compensator design:

$$K(s) = \frac{k(s+\frac{\lambda_0}{m})}{s+p}$$

but the actual value is $\lambda \neq \lambda_0$:

$$G(s) = \frac{1}{s(ms+\lambda)}$$

so $$G(s)K(s) = \frac{\frac{k}{m}(s+\frac{\lambda_0}{m})}{s(s+\frac{\lambda}{m})(s+p)}$$

Closed-loop characteristic equation:

$$1 + G(s)K(s) = 0$$

$$\Rightarrow s(s+\frac{\lambda}{m})(s+p) + \frac{k}{m}(s+\frac{\lambda_0}{m}) = 0$$

~~$\Rightarrow s^2 + \frac{k}{m}s + \frac{k\lambda_0}{m^2} + \frac{\lambda}{m}s = 0$~~

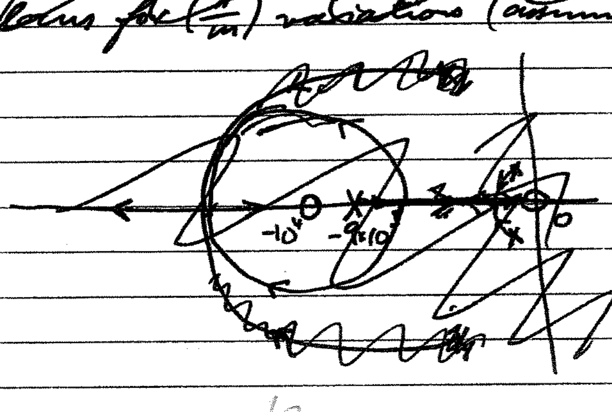
~~$\Rightarrow 1 + \lambda\frac{s}{ms^2+ks+k\lambda_0} = 0$~~

$$\Rightarrow s^2(s+p) + \frac{k}{m}(s+\frac{\lambda_0}{m}) + \frac{\lambda}{m}s(s+p) = 0$$

$$\Rightarrow 1 + \frac{\lambda}{m}\frac{s(s+p)}{s^2(s+p)+\frac{k}{m}(s+\frac{\lambda_0}{m})} = 0$$

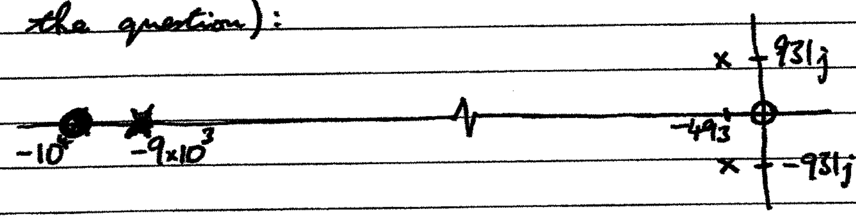
$$\Rightarrow 1 + \frac{\lambda}{m}\frac{s(s+10^4)}{s^3+10^4s^2+10^7s+10^{10}} = 0$$

~~Using the pole locations given in the question, the root-locus for $(\frac{\lambda}{m})$ variations (assuming m fixed) is approximately~~



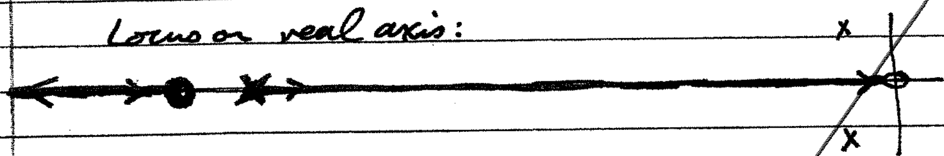
A.2 cont. (c) cont.

The pole-zero pattern is (using the pole locations given in the question):

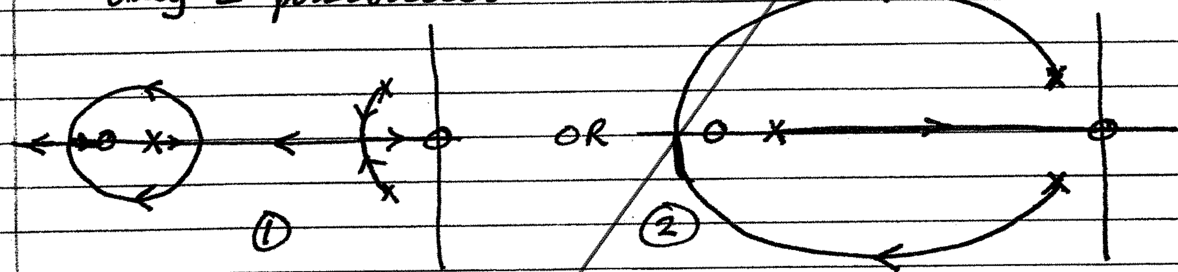


3 poles, 2 zeros ⇒ 1 asymptote, along -ve real axis.

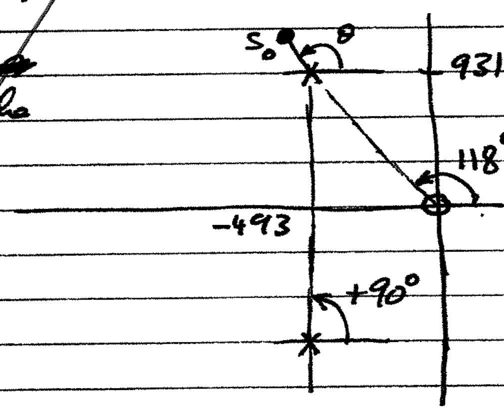
Locus on real axis:



Only 2 possibilities:



To determine which is correct, consider the root locus in the neighbourhood of $-493+931j$. The angle contributions from the pole and zero at -9×10^3 and -10^4 almost cancel, so for a point s_0 we have:



$\left(\tan^{-1}\left(\frac{493}{931}\right)=28^\circ\right)$

So if s_0 is on the root locus then the angle condition gives

$90+\theta-118 \approx 180$

$\Rightarrow \theta \approx 208^\circ$

which indicates that diagram ① above is correct.

A3 (a) With $x_1 = \theta$, $x_2 = \theta_a$, $x_3 = \dot{\theta}_a$, $x_4 = B$:

$T\frac{d\theta}{dt} = ku - (\theta - \theta_a)$ is the same as $T\dot{x}_1 = -x_1 + x_2 + ku$

or $\dot{x}_1 = -\frac{1}{T}x_1 + \frac{1}{T}x_2 + \frac{k}{T}u$

If $\theta_a = M\sin(\omega t + \phi) + N$ then $\dot{\theta}_a = M\omega\cos(\omega t + \phi)$,

and $\ddot{\theta}_a = -M\omega^2\sin(\omega t + \phi)$.

Hence $\ddot{\theta}_a = -\omega^2\theta_a + \omega^2 N$, or $\dot{x}_3 = -\omega^2 x_2 + \omega^2 x_4$.

By definition we have $\dot{x}_2 = x_3$.

Since N $(= x_4)$ is assumed constant, we have $\dot{x}_4 = 0$.

Hence
$$\dot{\underline{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \underbrace{\begin{pmatrix} -1/T & 1/T & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\omega^2 & 0 & \omega^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} k/T \\ 0 \\ 0 \\ 0 \end{pmatrix} u$$

(b) ~~If only the state~~ We have $\theta = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}}_{C} \underline{x}$

if only θ is measured.

The condition for observability is $\text{rank}(Q) = \text{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = 4$

since $\underline{x} \in \mathbb{R}^4$. (A is defined in (a).)

Now $CA = \left(-\frac{1}{T}, \frac{1}{T}, 0, 0\right)$

$CA^2 = \left(\frac{1}{T^2}, -\frac{1}{T^2}, \frac{1}{T}, 0\right)$

$CA^3 = \left(-\frac{1}{T^3}, \frac{1}{T^3} - \frac{\omega^2}{T}, -\frac{1}{T^2}, \frac{\omega^2}{T}\right)$

and hence
$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/T & 1/T & 0 & 0 \\ 1/T^2 & -1/T^2 & 1/T & 0 \\ -1/T^3 & \frac{1}{T^3} - \frac{\omega^2}{T} & -1/T^2 & \omega^2/T \end{bmatrix}$$

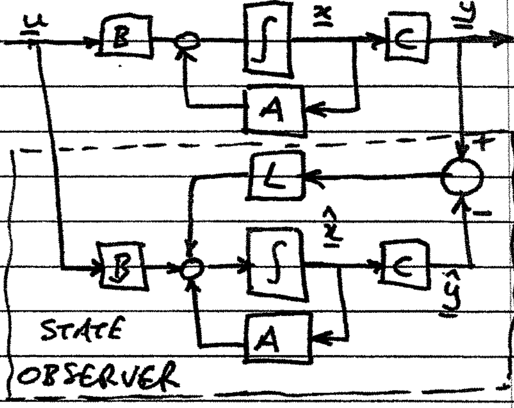
Since this is triangular, rank$(Q) = 4$ if $\omega \neq 0$.
Hence $\underline{x}$ is observable.

A3 contd.

(c) Given a system $\dot{\underline{x}} = A\underline{x} + B\underline{u}$, $\underline{y} = C\underline{x}$, a state observer is a system of the form

$$\dot{\hat{\underline{x}}} = A\hat{\underline{x}} + B\underline{u} + L(\underline{y} - C\hat{\underline{x}})$$

$$= (A-LC)\hat{\underline{x}} + B\underline{u} + L\underline{y}$$



If all the eigenvalues of $(A-LC)$ have negative real parts then the observer is stable, in the sense that the error $\underline{e} = \underline{x} - \hat{\underline{x}}$ decays to $\underline{0}$, so $\hat{\underline{x}}(t) \to \underline{x}(t)$.

(i) Only θ measured: $C = [1 \quad 0 \quad 0 \quad 0]$

LC must be a 4×4 matrix, so L is a 4×1 matrix (column vector).

(ii) Both θ and θ_a measured: $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Again LC must be a 4×4 matrix, so L is a 4×2 matrix.

(d) Only θ measured $\Rightarrow$ L is 4×1 matrix $\Rightarrow$ 4 entries.

Let $L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix}$.

Then $A - LC = A - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix} [1 \quad 0 \quad 0 \quad 0]$

$$= \begin{bmatrix} -\frac{1}{T} - l_1 & \frac{1}{T} & 0 & 0 \\ -l_2 & 0 & 1 & 0 \\ -l_3 & -\omega^2 & 0 & \omega^2 \\ -l_4 & 0 & 0 & 0 \end{bmatrix}$$

Let the eigenvalues of $(A-LC)$ be λ_i $(i = 1,2,3,4)$.

Then $\sum_i \lambda_i = \text{trace}(A-LC) = -\frac{1}{T} - l_1$.

Since $\lambda_i = -1$ then $\sum_i \lambda_i = -4$ and hence $\underline{\underline{l_1 = 3}}$ (since $T=1$).

A3 contd (d) contd.

$$\prod_i \lambda_i = \det(A - LC)$$ (Easiest evaluation: expand determinant along 4'th row.)

$$= l_4 \times \begin{vmatrix} 1/T & 0 & 0 \\ 0 & 1 & 0 \\ -\omega^2 & 0 & \omega^2 \end{vmatrix} = \frac{\omega^2 l_4}{T}$$

Since $\lambda_i = -1$ then $\prod_i \lambda_i = +1$, hence $l_4 = \frac{T}{\omega^2} = \left(\frac{24}{2\pi}\right)^2$

$= 14.59$

(e) Comment: Since each $\lambda_i = -1$, the time constants associated with estimation of the state vector are $-1/\lambda_i = 1$ hour. Hence the estimation time is of the order of 3 hours (taking 3× time constant). Assuming that the parameters M, N, ϕ remain approximately constant over one 24-hour period, this is fast enough* to get good control most of the time — assuming that enough heater power is available.

However, the water temperature measurement is available, so the estimation of x_1 could be much faster (unless the measurement is very noisy) than that of the other elements of $\underline{x}$.

An overall comment is that the temperature of a swimming pool does not usually have to be controlled very precisely, so the estimation of x_2, x_3, x_4 is not likely to be critical in this application.

*Note: Although M and ϕ are not estimated directly, knowledge of x_2 and x_3 is equivalent to knowing M and ϕ (if N is known).

Numbers wrong.

A.4 (a) Let $v_e = \frac{1}{2}$ ~~$\frac{1}{8}$~~. At equilibrium $\frac{dv}{dt} = 0$

so $f_e - v_e^3 = 0$ where f_e is the equilibrium value of f

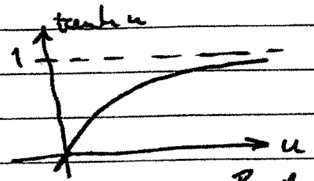
Hence $\underline{f_e = \frac{1}{2}}$ $\frac{1}{8}$

But also $\frac{df}{dt} = 0$ at equilibrium,

so $-2f_e + 2\tanh u_e = 0$

hence $u_e = \tanh^{-1}(\frac{1}{2}\ \frac{1}{8}) = $ ~~0.55~~ 0.1257

Maximum speed: $v_{max} = \sqrt[3]{f_{max}}$ (assuming $\frac{dv}{dt} = 0$).



$f_{max} = (\tanh u)_{max}$ (assuming $\frac{df}{dt} = 0$)

But $0 \leq \tanh u < 1$ if $0 \leq u < \infty$

(reasonable to assume $u \geq 0$)

Hence $f_{max} < 1$, and so $\underline{v_{max} < 1}$ ~~(This assumes unlimited travel of the throttle pedal!)~~

(b) Let $x_1 = v$, $x_2 = f$

Then $\dot{\underline{x}} = \phi(\underline{x}, u) = \begin{bmatrix} -x_1^3 + x_2 \\ -2x_2 + 2\tanh u \end{bmatrix}$

Let $\underline{x} = \underline{x}_e + \underline{\delta x}$, $u = u_e + \delta u$

Then $\dot{\underline{\delta x}} = A\,\underline{\delta x} + \underline{b}\,\delta u$

where $A = \left.\frac{\partial \phi}{\partial \underline{x}}\right|_{\underline{x}_e, u_e} = \begin{bmatrix} -3x_1^2 & 1 \\ 0 & -2 \end{bmatrix}_{\underline{x}_e, u_e} = \begin{bmatrix} -3/4\ (\text{~~}-3/64\text{~~}) & 1 \\ 0 & -2 \end{bmatrix}$

and $\underline{b} = \left.\frac{\partial \phi}{\partial u}\right|_{\underline{x}_e, u_e} = \begin{bmatrix} 0 \\ 2\,\mathrm{sech}^2 u \end{bmatrix}_{\underline{x}_e, u_e} = \begin{bmatrix} 0 \\ 2\,\mathrm{sech}^2(0.55)\ (0.1257) \end{bmatrix} = \begin{bmatrix} 0 \\ \text{~~}1.5\text{~~} \end{bmatrix}$ $\frac{126}{64}$

[Note: $2\,\mathrm{sech}^2(u_e) = 2[1 - \tanh^2(u_e)]$

$= 2\left[1 - \frac{1}{64}\right] = \frac{126}{64}$ But this can also be obtained using calculator.]

A.4 cont'd. (c) ~~(b) Contd.~~

Controllability condition is rank$[\underline{b}, A\underline{b}] = 2$.

In our case we have

$$[\underline{b}, A\underline{b}] = \begin{bmatrix} 0 & 2\operatorname{sech}^2 u_e \\ 2\operatorname{sech}^2 u_e & -4\operatorname{sech}^2 u_e \end{bmatrix}$$

which has rank 2 for all $\operatorname{sech} u_e \neq 0$, hence for all $u_e < \infty$.

(Note OK to check this only for the specific value $2\operatorname{sech}^2 u_e = 1.969$)

State feedback:

$$\delta u = -[k_1, k_2]\,\delta x$$

$$\Rightarrow \dot{\underline{\delta x}} = \begin{pmatrix} -3/4 & 1 \\ 0 & -2 \end{pmatrix} \underline{\delta x} - \begin{pmatrix} 0 \\ \frac{126}{64} \end{pmatrix} (k_1, k_2)\, \underline{\delta x}$$

$$= \begin{pmatrix} -3/4 & 1 \\ -\frac{126k_1}{64} & -2-\frac{126k_2}{64} \end{pmatrix} \underline{\delta x}$$

Characteristic polynomial of this is

$$\det\begin{pmatrix} s+3/64 & -1 \\ \frac{126k_1}{64} & s+2+\frac{126k_2}{64} \end{pmatrix} = \left(s+\frac{3}{64}\right)\left(s+2+\frac{126k_2}{64}\right) + \frac{126k_1}{64}$$

$$= s^2 + \left(2+\frac{3}{64}+\frac{126k_2}{64}\right)s + \left(\frac{3}{32}+\frac{9k_2}{128}+\frac{3k_1}{2}\right)$$

$$\frac{3(128+126k_2)}{4\times 64} + \frac{126k_1}{64}$$

Change numbers

But we want this to be the same as

$$(s+0.3)(s+2) = s^2 + 2.3s + 0.6$$

Hence $2+\frac{3}{64}+\frac{3k_2}{2} = 2.3$, $\Rightarrow \frac{3k_2}{2} = 0.2531$ ~~2.0469~~ $\Rightarrow k_2 = 0.1687$ ~~1.3646~~

and $\frac{3}{32}+\frac{9k_2}{128}+\frac{3k_1}{2} = 0.6$, $\Rightarrow \frac{3k_1}{2} =$ ~~0.4103~~ ~~0.5041~~ $0.4944 \Rightarrow k_1 =$ ~~0.3296~~ ~~0.2735~~ ~~0.3360~~

Hence ... $k_2 = -0.2286$, $k_1 = -0.2857$

A.4 cont'd

(c) cont'd.

Comment on pole locations: (1) Both −ve real, so stable.

(2) One of the poles is concerned with ~~dy~~ dynamics of the car, the other with the dynamics of the ~~effects of the~~ engine ~~throttle~~. It would not be reasonable to try to make these the same. The slow pole here (−0.3) corresponds to a time constant of approx 3 sec, which is reasonable for a car. (Though it is a lot faster than the open-loop time constant of approx 20 sec — might be too ambitious.) The other pole is left unchanged from its open-loop value, ~~so there is no~~ (time constant 0.5 sec).

(d) To include integral action, define another state variable as the integral of the speed error:

$$x_3 = \int_0^t (v_{ref} - v)\, dt$$

Then $\dot{x}_3 = v_{ref} - v$ = ~~[illegible]~~

$$= v_{ref} - x_1 = v_{ref} - (x_1)_e - \delta x_1$$

so the linearised state equation becomes

$$\begin{pmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -3/64 & 1 & 0 \\ 0 & -2 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 3/2 \\ 0 \end{pmatrix} \delta u + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v_{ref}$$

Now setting $\delta u = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ x_3 \end{bmatrix}$

gives

$$\begin{pmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -3/64 & 1 & 0 \\ -\frac{3k_1}{2} & -2-\frac{3k_2}{2} & -\frac{3k_3}{2} \\ -1 & 0 & 0 \end{pmatrix}} \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v_{ref}$$

(k_1, k_2, k_3) can be chosen to place eigenvalues in any required locations.
(Controllability can be checked — not ~~required~~ expected here.)

A.4 cont'd. (d) cont'd.

Benefit of integral action:

Removes steady-state errors if constant disturbances are present, or the model is not accurate.
In the cruise-control application integral action compensates for the effects of hills, headwinds, passenger weight, and variations in engine and friction characteristics.

(e) If measurements of f were not available, then one could

(1) Use an observer. Observability from the velocity measurement can be checked:

$$\begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3/64 & 1 \end{pmatrix}$$ has rank 2, hence observable.

Then the estimate $\hat{f}$ could be used instead of f in the state feedback scheme.

(Only one expected.) OR

(2) $\hat{f}$ could be obtained by filtering u

~~Since $\dot{v} = f - v^3$, and hence $\delta\dot{v} = \delta f - \frac{3}{64}\delta v$ (if $v_e = 1/8$), we have $\frac{\overline{\delta f}(s)}{\overline{\delta v}(s)} = \frac{1}{s + 3/64}$~~

Since $\frac{3}{8}\dot{f} = -2f + 2\tanh u$, and hence

$$\delta\dot{f} = -2\delta f + \frac{3}{2}\delta u \quad (\text{if } v_e = \tfrac{1}{8} \text{ etc})$$

we have $$\frac{\overline{\delta f}(s)}{\overline{\delta u}(s)} = \frac{3/2}{s+2}$$