N.G. KINGSBURY

PART IIA, 3F4 – Solutions 2003.

1.) a) ISI occurs when the system response $h(t)$ exceeds one symbol period.

ISI reduces the eye openings consequently worsening the BER performance.

An equaliser is used to reduce the effect of ISI thus improving the BER.

Zero forcing is not often employed in practical systems since it will introduce high gain values at dips in the system response. Thus noise will be magnified, lowering potential performance gains yielded due to ISI reduction

The MMSE equaliser aims to minimise the error between the equalised symbol values and their true values. Thus the design is able to explicitly trade-off noise enhancement against ISI reduction. The disadvantage is that the noise variance needs to be known in advance.

b) From Fig. 2,

$p_0 = 0, \quad p_1 = 0.9, \quad p_2 = -0.45, \quad p_3 = 0.3$

The equalised output at time n is given by,

$$y_n = \sum_{i=0}^{q} p_{n-i}\, b_i$$

where $q = 2$ in our case (i.e. a 3-tap equaliser).

So,

$$y_n = p_n b_0 + p_{n-1} b_1 + p_{n-2} b_2$$

(2)

The zero forcing design constraint is,

$y_0 = 0, \quad y_1 = 1, \quad y_2 = 0, \quad y_3 = 0$

So,

$n = 0$, $\quad y_0 = p_0 b_0$

$0 = 0\, b_0 \quad \therefore$ can't say anything about b_0

$n = 1$, $\quad y_1 = p_1 b_0 + p_0 b_1$

$1 = 0.9 b_0 + 0\, b_1$

$\therefore \quad b_0 = 1/0.9 = \underline{\underline{1.11111}}$

$n = 2$, $\quad y_2 = p_2 b_0 + p_1 b_1 + p_0 b_2$

$0 = -0.45 \times 1.11111 + 0.9 b_1 + 0 b_2$

$\therefore \quad \underline{\underline{b_1 = 0.55555}}$

$n = 3$ $\quad y_3 = p_3 b_0 + p_2 b_1 + p_1 b_2$

$0 = 0.3 \times 1.1111 + (-0.45) \times 0.55555 + 0.9\, b_2$

$\therefore \quad \underline{\underline{b_2 = -0.0926}}$

<u>Without equalisation</u>

Worst case '1' $= 0.9 - 0.45 = 0.45$

Worst case '0' $= 0 + 0.3 = 0.3$

$\therefore$ eye opening, $h = 0.45 - 0.3$

$h = 0.15$

$BER = Q\left(\frac{h}{2\sigma}\right)$

$BER = Q\left(\frac{0.15}{2 \times 0.1}\right) = Q(0.75)$

Using equation (approx) yields

$\underline{\underline{BER = 0.2267}}$

3

With equalisation.

First need to calculate the residuals caused by truncation of filter. Use previous equation for p_n, this time for $n = 4$ and 5, ie,

$$y_4 = p_3 b_1 + p_2 b_2$$
$$= 0.3 \times 0.5555 + (-0.45) \times (-0.0926)$$
$$= 0.2083$$

$$y_5 = p_3 b_2$$
$$= 0.3 \times (-0.0926)$$
$$= -0.02778$$

Worst case '1' = 0.9 − 0.02778 = 0.8722

Worst case '0' = 0 + 0.2083 = 0.2083

$$\therefore h = 0.8722 - 0.2083$$
$$h = 0.6639$$

$$BER = Q\left(\frac{0.6639}{2\sigma_w}\right)$$

Need to calculate σ_w (noise at equaliser output)

$$\sigma_w = 0.1\sqrt{(1.1111)^2 + (0.5555)^2 + (-0.0926)^2}$$
$$\sigma = 0.1246\text{V}$$

$$\therefore BER = Q\left(\frac{0.6639}{2 \times 0.1246}\right) = Q(2.664)$$

$$\underline{BER \approx 3.87 \times 10^{-3}}$$

c) With a DFE – no noise enhancement and full eye opening. Ignore error propagation.

$$\therefore h = 0.9$$
$$BER = Q\left(\frac{0.9}{2 \times 0.1}\right) = Q(4.5)$$
$$BER \approx 3.4 \times 10^{-6}$$

(1)

2.) a) For timing synchronisation. - There should be sufficient information in the signal transitions and zero-crossings to permit symbol timing clock regeneration at the receiver.

The PSD of the transmitted signal should be compatible with the frequency response of the channel.

With reference to the copper channel with a.c. coupled amplifiers

- Such a channel cannot pass d.c. and has a poor response at low frequencies. Line coding should ensure zero d.c. content and low power density at low frequencies.

- Copper channels are bandwidth limited therefore line coding should limit PSD at high frequencies in order to maximise transmission rate.

a b)

$$S_x(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{T}$$

The truncated signal $x_T(t)$ is

$$x_T(t) = \sum_{n=-N}^{N} a_n \delta(t - nT_s), \quad T = (2N+1)T_s$$

Now,

$$X_T(\omega) = \int_{-\infty}^{\infty} x_T(t) e^{-j\omega t} dt$$

$$X_T(\omega) = \int_{-\infty}^{\infty} \sum_{n=-N}^{N} a_n \delta(t - nT_s) e^{j\omega t} dt$$

(2)

$$X_T(\omega) = \sum_{n=-N}^{N} a_n \int_{-\infty}^{\infty} \delta(t - nT_s)\, e^{j\omega t}\, dt$$

$$= \sum_{n=-N}^{N} a_n\, e^{-jn\omega T_s}$$

Now,

$$|X_T(\omega)|^2 = X_T(\omega)\, X_T^*(\omega)$$

$$= \sum_{n=-N}^{N} a_n e^{-jn\omega T_s} \sum_{k=-N}^{N} a_k e^{jk\omega T_s}$$

$$= \sum_{n=-N}^{N} \sum_{k=-N}^{N} a_n a_k\, e^{j(k-n)\omega T_s}$$

Now,

$$E[|X_T(\omega)|^2] = E\left[\sum_{n=-N}^{N} \sum_{k=-N}^{N} a_n a_k\, e^{j(k-n)\omega T_s} \right]$$

$$= \sum_{n=-N}^{N} \sum_{k=-N}^{N} E[a_n a_k]\, e^{j(k-n)\omega T_s}$$

now let $k = n + m$ and so $m = k - n$, so,

$$E[|X_T(\omega)|^2] = \sum_{n=-N}^{N} \sum_{m=-N-n}^{N-n} E[a_n a_{n+m}]\, e^{j(n+m-n)\omega T_s}$$

$$= \sum_{n=-N}^{N} \sum_{m=-N-n}^{N-n} R(m)\, e^{jm\omega T_s}$$

Replace the outer sum over index n by $2N+1$,

$$E[|X_T(\omega)|^2] = (2N+1) \sum_{m=-N-n}^{N-n} R(m)\, e^{jm\omega T_s}$$

So,

$$S_x(\omega) = \lim_{N\to\infty} \frac{1}{(2N+1)T_s} (2N+1) \sum_{m=-N-n}^{N-n} R(m)\, e^{jm\omega T_s}$$

$$= \frac{1}{T_s} \sum_{m=-\infty}^{\infty} R(m)\, e^{jm\omega T_s}.$$

(3)

b) nominal levels are −1, 0, 1. With +0.2V offset actual levels are −0.8, 0.2 and 1.2.

m	b_k b_{k+m}	a_k a_{k+m}	R_i	p_i	$R(m)$
0	0 0	0.2 0.2	0.04	0.5	
	1 1	−0.8 −0.8	0.64	0.25	0.54
		1.2 1.2	1.44	0.25	
1	0 0	0.2 0.2	0.04	0.25	
	0 1	0.2 −0.8	−0.16	0.125	
		0.2 1.2	0.24	0.125	
	1 0	−0.8 0.2	−0.16	0.125	−0.21
		1.2 0.2	0.24	0.125	
	1 1	−0.8 1.2	−0.96	0.25	
		1.2 −0.8			
2	0 0	0.2 0.2	0.04	0.25	
	0 1	0.2 −0.8	−0.16	0.125	
		0.2 1.2	0.24	0.125	
	1 0	−0.8 0.2	−0.16	0.125	
		1.2 0.2	0.24	0.125	0.04
	1 1	1.2 (0.2) −0.8	−0.96	0.0625	
		−0.8 (0.2) 1.2	−0.96	0.0625	
		1.2 (−0.8) 1.2	1.44	0.0625	
		−0.8 (1.2) −0.8	0.64	0.0625	
>3					0.04

So,

$$S_x(\omega) = \frac{1}{T_s}\left\{ \dots + 0.04\, e^{-j2\omega T_s} - 0.21\, e^{-j\omega T_s} + 0.54 - 0.21\, e^{j\omega T_s} + 0.04\, e^{j2\omega T_s} + \dots \right\}$$

$$S_x(\omega) = \frac{1}{T_s}\left\{ \dots + 0.04\, e^{-j2\omega T_s} + 0.04\, e^{-j\omega T_s} + 0.04 + 0.04\, e^{j\omega T_s} + 0.04\, e^{j2\omega T_s} + \dots + (-0.25)e^{-j\omega T_s} + 0.5 + (-0.25)e^{j\omega T_s} \right\}$$

$$S_x(\omega) = \frac{1}{T_s}\left\{ \sum_{m=-\infty}^{\infty} 0.04\, e^{jm\omega T_s} + (-0.25)e^{-j\omega T_s} + 0.5 + (-0.25)e^{j\omega T_s} \right\}$$

$$S_x(\omega) = \frac{1}{T_s}\left\{ \sum_{m=-\infty}^{\infty} 0.04\, e^{jm\omega T_s} + 0.5\,(1 - \cos \omega T) \right\}$$

(4)

$$S_x(\omega) = \frac{1}{T_s}\left\{0.04\sum_{m=-\infty}^{\infty} e^{jm\omega T_s} + \sin^2\left(\frac{\omega T_s}{2}\right)\right\}$$

Substitute sum of exponentials as a series of impulses in the frequency domain,

$$S_x(\omega) = \frac{1}{T_s}\left\{\frac{0.04\times 2\pi}{T_s}\sum_{m=-\infty}^{\infty}\delta\left(\omega - m\frac{2\pi}{T_s}\right) + \sin^2\frac{\omega T_s}{2}\right\}$$

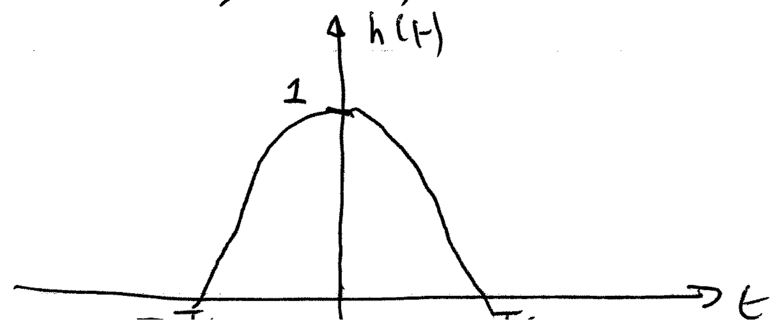
$$S_x(\omega) = \underbrace{\frac{0.04\times 2\pi}{T_s^2}\sum_{m=-\infty}^{\infty}\delta\left(\omega - m\frac{2\pi}{T_s}\right)}_{\text{set of impulses in freq domain due to d.c. offset.}} + \underbrace{\frac{1}{T_s}\sin^2\frac{\omega T_s}{2}}_{\text{Continuous PSD due to bipolar line coding.}}$$

An alternative solution could
i) Calculate continuous spectrum due to bipolar coding using nominal voltage levels.

ii) Calculate the psd ~~cause~~ due to a constant voltage of 0.2V.

iii) Sum the resulting spectra.

c) The defined impulse response $h(t)$ is actually the 'half-sine' pulse defined in the E and I Data Book i.e,



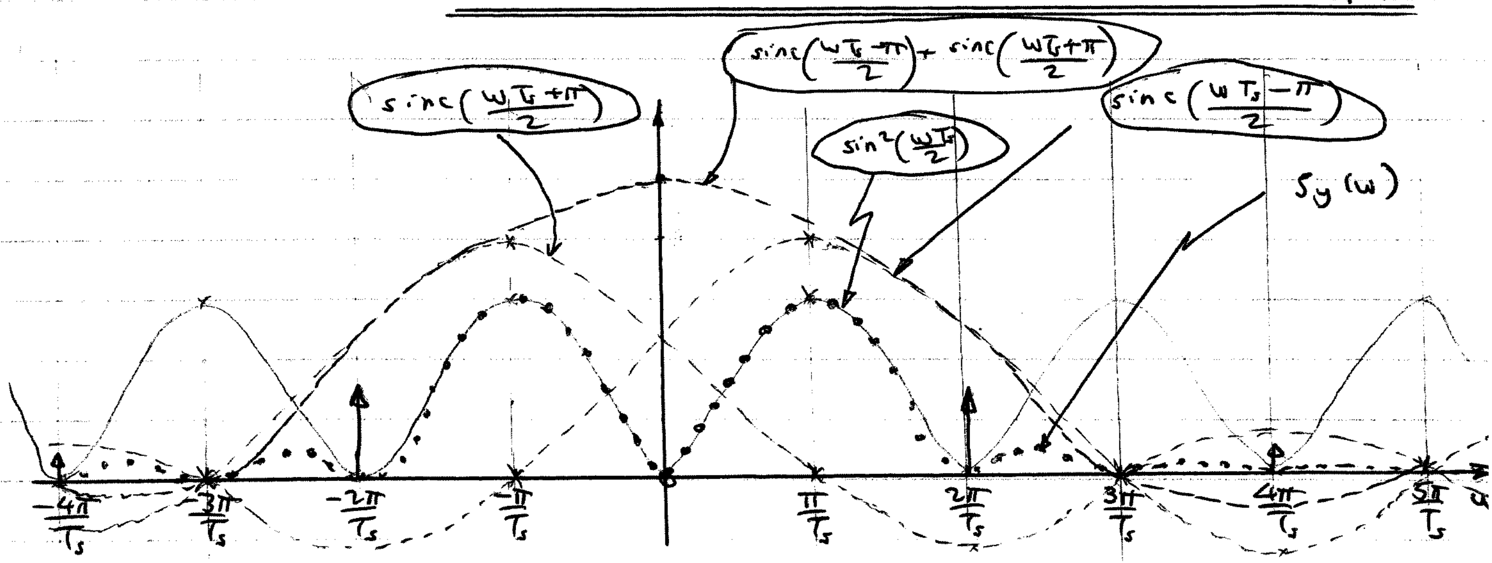
(5)

From the E and I tables,

$$H(\omega) = \frac{T_s}{2}\left[\text{sinc}\left(\frac{\omega T_s - \pi}{2}\right) + \text{sinc}\left(\frac{\omega T_s + \pi}{2}\right)\right]$$

The transmitted PSD is given by

$$S_y(\omega) = S_x(\omega)\,|H(\omega)|^2$$
$$= S_x(\omega)\left|\frac{T_s}{2}\left[\text{sinc}\left(\frac{\omega T_s - \pi}{2}\right) + \text{sinc}\left(\frac{\omega T_s + \pi}{2}\right)\right]\right|^2$$



Note large reduction in sidelobes in excess of $\pm\frac{3\pi}{T_s}$ (ie, 1.5 times the symbol rate)

Also note impulses remain in spite of filtering.

If rectangular pulses are used the resulting sinc frequency response has nulls at multiples of $\frac{2\pi}{T_s}$

Owing to the slow decay of the sinc function the sidelobe levels of $S_y(\omega)$ will remain significant compared ~~with~~ with those when using the ½ sine pulse shape.
Note however the impulses will be eliminated

3. (a) Bookwork from notes:

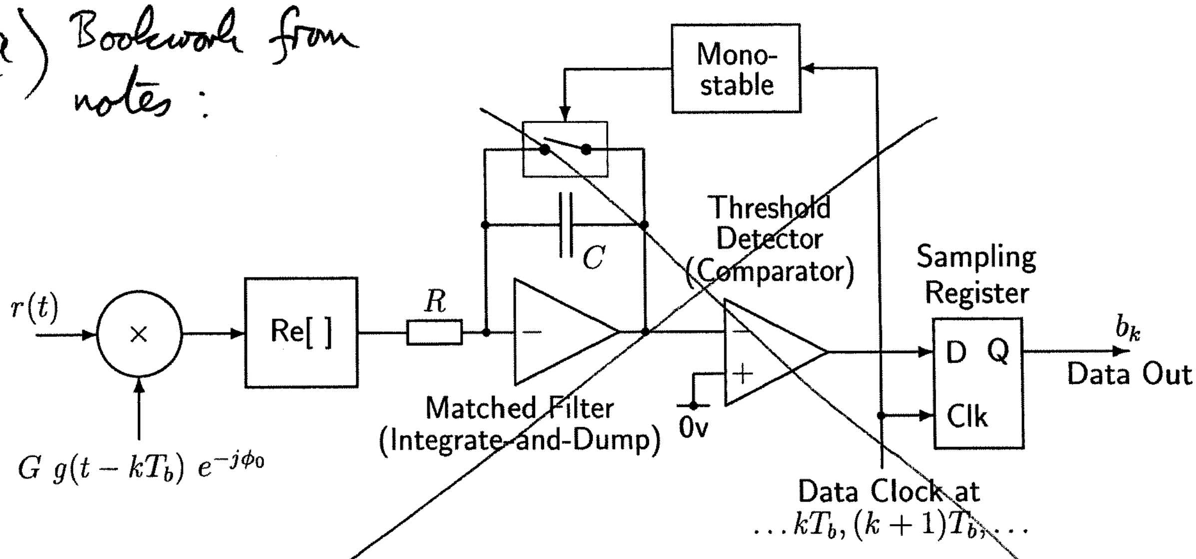


Fig 3.2: Matched correlator demodulator for BPSK.

3.3 Optimum Demodulator:

An optimum demodulator for equiprobable signals in gaussian noise is one which selects the data value that results in the minimium mean square error between the received signal plus noise and the signal corresponding to that data value.

If the received signal+noise phasor is

$$r(t) = p(t) + p_N(t)$$

the optimum demodulator for binary data during the k^{th} bit period measures

$$\int_{kT_b}^{(k+1)T_b} |r(t) - \underbrace{g(t-kT_b)e^{j\phi_0}}_{+1 \text{ phasor}}|^2 dt \qquad \text{and} \qquad \int_{kT_b}^{(k+1)T_b} |r(t) - \underbrace{(-g(t-kT_b)e^{j\phi_0})}_{-1 \text{ phasor}}|^2 dt$$

and selects whichever gives the smaller result.

But we can simplify this.

$$|r-g|^2 = (r-g)(r^*-g^*) = |r|^2 - 2Re[rg^*] + |g|^2$$

If $g = \pm g(t-kT_b)e^{j\phi_0}$, $|r|^2$ and $|g|^2$ are the same for both integrals, so we may ignore these terms and simply calculate

$$y(b,k) = \int_{kT_b}^{(k+1)T_b} Re[r(t)\ b\ g(t-kT_b)e^{-j\phi_0}]dt$$

for $b = +1$ and -1, and select the value of b according to which b gives the larger result.

Since b can be taken out of the integral, detecting the larger of $y(1,k)$ and $y(-1,k)$, is equivalent to detecting the polarity of just $y(1,k)$. This is equivalent to correlating $r(t)$ with $g(t-kT_b)e^{j\phi_0}$ over the bit period, ~~& is implemented as~~

3. (b) V_s ~~...~~ = signal voltage at the detector

σ = std. deviation of noise at the detector

E_b = energy per bit of the received signal phasor waveform.

N_0 = ~~noise~~ power spectral density of the noise phasors.

(alternatively E_b & N_0 could both be measured using the ~~...~~ R.F. signal at the receiver input & their ratio would be the same)

Bit error rate $= Q\left(\frac{V_s}{\sigma}\right)$ (assuming Gaussian noise pdf)

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$E_b = V^2(\text{bit period}) = \frac{2^2}{56.10^3}$$

$$N_0 = 4.10^{-5}\ V^2\,Hz^{-1}$$

$$\therefore \sqrt{\frac{2E_b}{N_0}} = \sqrt{\frac{2.2^2}{56.10^3.4.10^{-5}}} = \sqrt{3.5714} = 1.8898$$

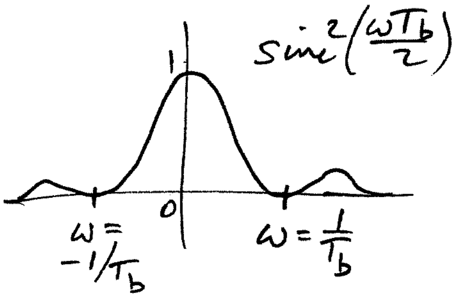
$\therefore$ BER $= Q(1.8898) = \underline{\underline{0.0295}}$ using formula given.

3. (c) $g(t)$ is a rectangular pulse of width $T_b = \frac{1}{56.10^3}$ s.

$$\therefore\ G(\omega) = a_0 T_b \operatorname{sinc}\left(\frac{\omega T_b}{2}\right)$$

& Power spectral density $\propto |G(\omega)|^2 = a_0^2 T_b^2 \operatorname{sinc}^2\left(\frac{\omega T_b}{2}\right)$

$\operatorname{sinc}^2\left(\frac{\omega T_b}{2}\right) = 1$ when $\omega = 0$ & this is the max.



At −3dB point ($\frac{1}{2}$ power):

$$\operatorname{sinc}^2\left(\frac{\omega T_b}{2}\right) = 0.5$$

Solve for $\sin\theta = \frac{1}{\sqrt{2}}.\theta$

gives $\theta \simeq 1.392$

$$\therefore\ \omega_{-3} = \frac{2}{T_b}.1.392$$

$$\& \ f_{-3} = \frac{1}{2\pi}.\frac{2}{T_b}.1.392 = 0.4431\ (\text{bit rate})$$

$$= 24.8\ \text{kHz}$$

Hence −3dB bandwidth extends from f_{-3} below the carrier to f_{-3} above the carrier & is $2 \times f_{-3} = \underline{49.6\ \text{kHz}}$

3(d) 15 kHz is ~~only~~ less than $\frac{1}{3}$ of the −3dB bandwidth of a BPSK signal, so we need a modulation scheme that has $\sim \frac{1}{4}$ of the symbol rate of BPSK, ie it needs $2^4 = 16$ levels of modulation. 16-QAM is the most error-robust form of 16-level modulation and is an appropriate choice. However the bit error rate would be significantly degraded (typically 4dB degradation in SNR performance).

4.(a)

1.3 Phasor Representation of Modulated Signals

It is useful to handle AM, PM and FM in a unified way.

Let the modulated wave be:

$$s(t) = a(t)\cos(\omega_C t + \phi(t)) \tag{1.1}$$

Note $a(t)$ and $\phi(t)$ are difficult to combine.

Consider the cos term as the real part of a complex exponential:

$$\begin{aligned} s(t) &= Re[a(t)\ e^{j(\omega_C t+\phi(t))}] \\ &= Re[a(t)\ e^{j\phi(t)}\ e^{j\omega_C t}] \\ &= Re[\ \underbrace{p(t)}_{\substack{\text{modulation}\\ \text{phasor}}}\ \underbrace{e^{j\omega_C t}}_{\substack{\text{carrier}\\ \text{wave}}}\ ] \end{aligned} \tag{1.2}$$

$$\text{where } p(t) = \underbrace{a(t)}_{\substack{\text{ampl.}\\ \text{of } p(t)}}\ \underbrace{e^{j\phi(t)}}_{\substack{\text{phase}\\ \text{of } p(t)}} \tag{1.3}$$

Now the modulation is completely separated from the carrier wave, and a and ϕ have been combined into a single **phasor** waveform, $p(t)$.

Phasors are very useful for defining any form of modulation that is to be applied to a carrier wave, without involving the carrier itself. Phasors vary much more slowly than the modulated wave, $s(t)$, and are easier to analyse. For digital signals with combined ~~See figs 1.2, 1.3 and 1.4 (3D plots of AM, PM and FM).~~ phase and amplitude mod such as QAM, this is very helpful.

Significance of the Real and Imaginary Parts of Phasors:

If $p(t)$ has real and imaginary parts, $i(t)$ and $q(t)$, then:

$$\begin{aligned} \text{Let}\quad p(t) &= i(t) + jq(t) && (1.4)\\ \therefore \text{ from (1.3)}\quad i(t) &= a(t)\cos\phi(t) && (1.5)\\ \text{and}\quad q(t) &= a(t)\sin\phi(t) && (1.6) \end{aligned}$$

We may obtain $s(t)$ in terms of i and q by substituting (1.4) into (1.2):

$$s(t) = Re[\{i(t) + jq(t)\}\ e^{j\omega_C t}] = i(t)\cos(\omega_C t) - q(t)\sin(\omega_C t) \tag{1.7}$$

Hence $i(t)$ is the inphase component of $s(t)$ and $-q(t)$ is the quadrature component.

4. (a) (cont)

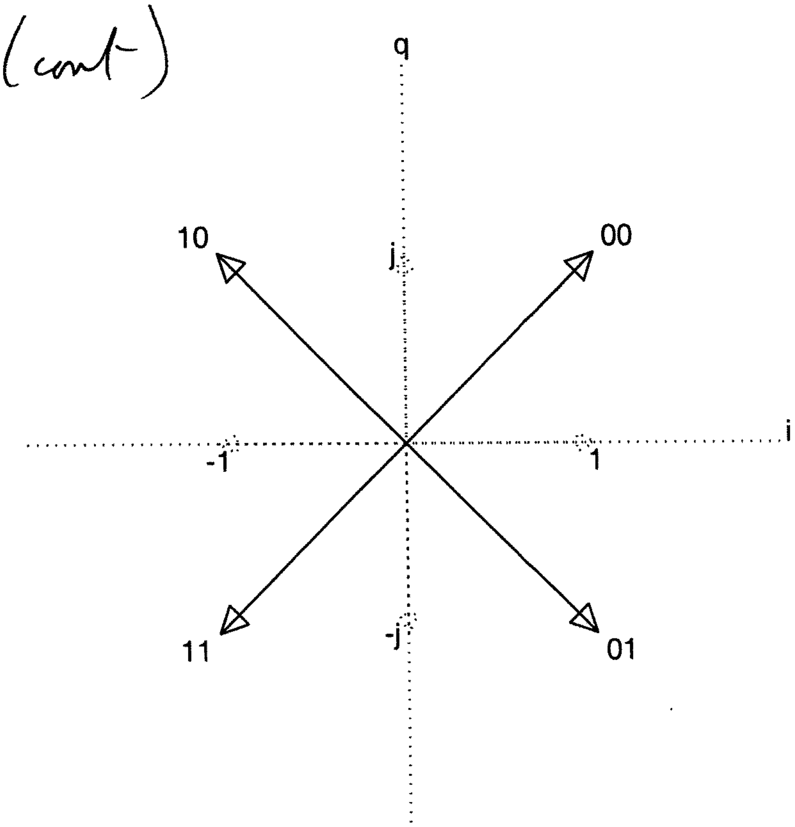


Fig 4.1: QPSK phasor diagram.

~~4 Other Binary Schemes~~

4.1 Quadrature PSK (QPSK):

QPSK is equivalent to BPSK on two quadrature carriers.

Even bits b_{2k} modulate the inphase carrier.

Odd bits b_{2k+1} modulate the quadrature carrier.

$$\therefore \ p_k(t) = [b_{2k} + j\ b_{2k+1}]\ g(t - kT_s)\ e^{j\phi_0}$$

where $g(t)$ is as for BPSK except that it is now non-zero from $t = 0$ to T_s, the 2-bit symbol period ($T_s = 2T_b$).

Hence $p(t)$ can have one of 4 values:

$$(\pm 1 \pm j)\ g(t - kT_s)\ e^{j\phi_0}$$

See fig 4.1 for QPSK phasor diagram.

QPSK can be regarded as 4-level modulation, but it is usually easier to treat it as two independent 2-level (binary) processes.

See Fig. 4.3a for symbol timing.

4.(b) No. of carriers $= \frac{1.6.10^6}{800} = 2000$

QPSK sends 2 bits per symbol

symbol rate $= 600$ sym/sec on each carrier

$= 600 \times 2000 = 1.2.10^6$ sym/sec

$\therefore$ Bit rate over channel $= 2 \times 1.2.10^6$ bit/sec

But ECC rate $= 1:2$

$\therefore$ User bit rate $= \frac{1}{2}$ channel bit rate $= 1.2.10^6$ bit/s

$= \underline{\underline{1.2 \text{Mb/s}}}$.

QPSK is a good choice for reception in vehicles because:

1. It is robust to noise (same BER performance as BPSK).
2. It is constant amplitude, so detection can be achieved without needing to know the signal amplitude. Hence rapid fading can be well tolerated.
3. It is reasonably efficient spectrally (~~twice as good as~~ only half the bandwidth of BPSK).

4 (c) (very full answer)

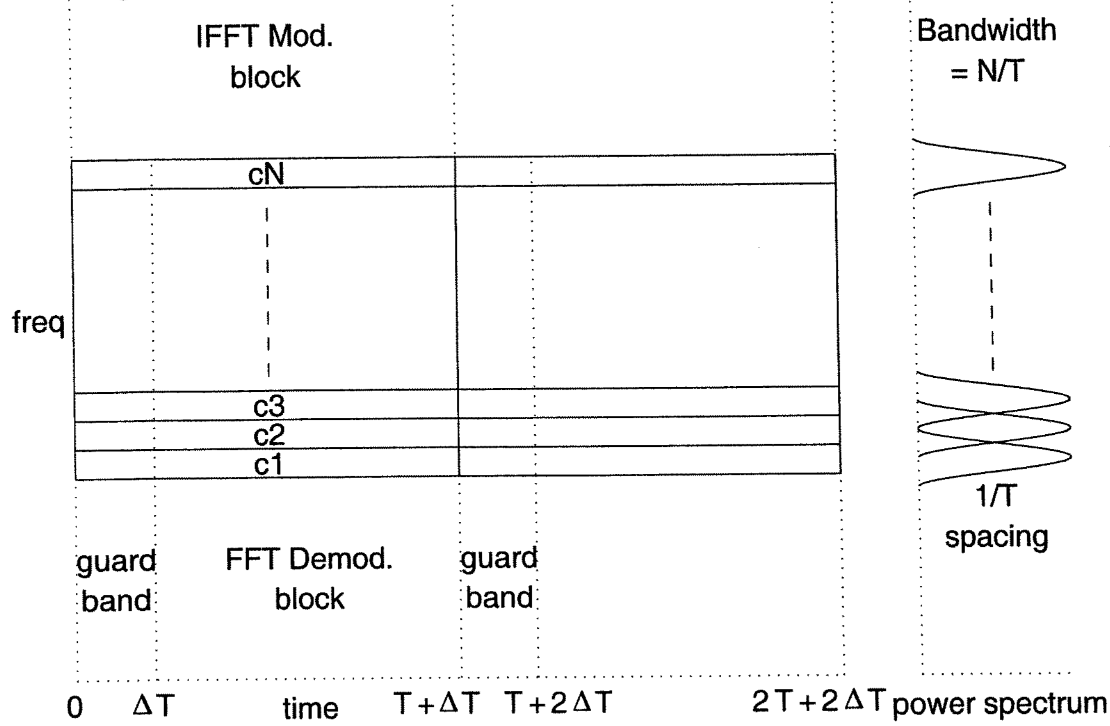


Fig 6.2: Orthogonal Frequency Division Multiplexing (OFDM) with N carriers.

Orthogonal Frequency Division Multiplexing

The aim of OFDM is to demultiplex the high-speed bit stream into N streams, each at $1/N$ of the original rate, which are then modulated onto N separate carrier waves, as shown in fig. 6.2. These have much improved resilience to typical multipath delays because of their lower modulation rates. Typically $N \approx 1000$ to 2000.

The inverse FFT may be used to put QPSK (or QAM) data on each of N carriers, spaced by $1/T$ Hz, where T is the IFFT block period. Each carrier is an IFFT basis function which is multiplied by the modulation phasor ($\pm 1 \pm j$ in the case of QPSK). In this way the carriers are orthogonal to each other and may be demodulated by an equivalent FFT process without mutual interference at the receiver. The mutual orthogonality of the IFFT basis functions, means that there should be no interference between each modulated carrier and its neighbours. Orthogonality is not affected by the modulation process, because the modulation rate is no faster than once per FFT block period, so each modulated carrier is a pure tone for the duration of the block period T.

Figures 6.3 to 6.10 show the 2 input data bits and the modulated IFFT basis functions for QPSK modulation of frequency slots 1 to 8. Figure 6.11 shows the result of combining all 8 slots together into an OFDM signal. Since all the 8 slots are at positive frequencies, there is an overall positive bias to the rotation of the composite phasor waveform in fig 6.11. In practise negative frequency slots would be used as well, and fig 6.12 shows the result for 32 carriers occupying slots −16 to +15. The resulting phasor waveform looks much more random and noiselike.

Multipath delays tend to vary across the band of N carriers, and this could upset the orthogonality property because some modulation transitions would need to occur during the demodulator FFT analysis block if each block followed its predecessor immediately. To avoid this problem, a guard period ΔT is inserted between consecutive blocks in the modulator and demodulator. For optimum demodulator performance, the modulator should periodically extend the inverse FFT output waveform into the guard period before each block, so that the transmitted waveform is continuous from the point where the modulation transitions occur at the start of each guard period. The FFT demodulator analyses the interval from ΔT to $T + \Delta T$. In this way multipath delays varying from 0 to ΔT can be tolerated without any modulation transitions intruding into the FFT analysis interval and spoiling the orthogonality of the carrier waves. Unfortunately the guard periods either reduce the throughput of the system or increase its bandwidth in the ratio $T : (T + \Delta T)$.

The outputs of the FFT demodulator are N complex Fourier coefficients, each of which is a phasor representing the demodulated amplitude and phase of the carrier in that FFT slot. For each carrier, conventional demodulation methods may then be used to recover the binary data from the sequence of phasors from consecutive FFT analysis intervals.

Finally the data from the N QPSK decoders is multiplexed back into a single serial data stream which is passed on to the error correction decoder. This can correct errors which typically occur when multipath causes selective fading of some carriers. Improved error correction performance can often be achieved if soft decision information is passed to the error decoder from each QPSK decoder.

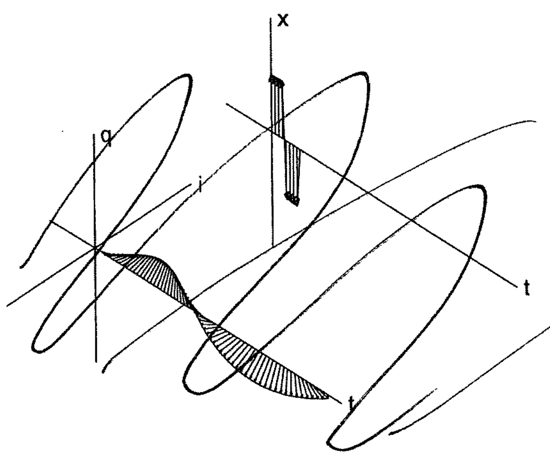


Fig 6.3: OFDM slot 1

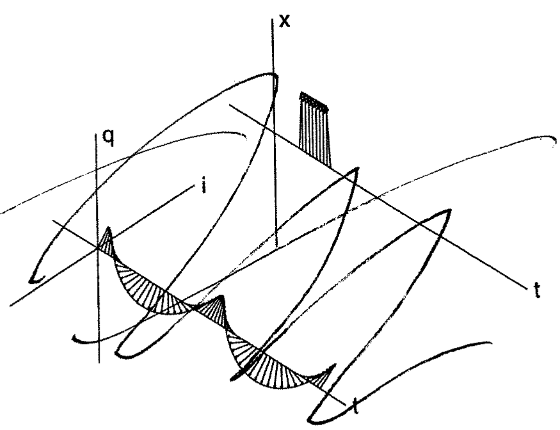


Fig 6.4: OFDM slot 2

$T + \Delta T = \frac{1}{600}$ s $= 1.667$ ms

$T = \frac{1}{800}$ s $= 1.25$ ms

$\therefore \Delta T = 1.667 - 1.25 = \underline{\underline{0.417}}$ ms

ΔT is the max delay variation before system degradation starts to occur (see above description).