ENGINEERING TRIPOS PART IIA

Friday 9 May 2003 9 to 10.30

Module 3C6

VIBRATION

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 A stretched string of length L has tension P and mass per unit length m. It is fixed at both ends.

(a) By considering the motion of a small element of the string, show that the differential equation governing small transverse vibration is:

$$m\frac{\partial^2 w}{\partial t^2} = P\frac{\partial^2 w}{\partial x^2},$$ [25%]

where w is the transverse displacement at position x and time t.

(b) Find expressions for the vibration modes and natural frequencies of the string. [25%]

(c) The string is held at rest with transverse displacement

$$w = \begin{cases} a\sin(2\pi x/L) & 0 \le x \le L/2 \\ 0 & L/2 \le x \le L \end{cases}$$

and at time $t = 0$ it is released. Find an expression for the subsequent vibration of the string, and hence show that all odd-numbered modes are excited, but only one even-numbered mode is excited. [50%]

2 (a) A xylophone bar can be represented by an Euler beam of length L, undergoing bending vibration satisfying the differential equation

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = 0$$

where w is the lateral displacement at position x and time t, EI is the bending stiffness, A is the cross-sectional area and ρ is the density. Both ends of the bar are free. Write down the appropriate boundary conditions, and hence show that the natural frequencies ω of the bar satisfy the equation

$$\cos \alpha L \cosh \alpha L = 1 \qquad \text{(Equation 1)}$$

where

$$\alpha = \left(\frac{\rho A}{EI}\right)^{1/4} \omega^{1/2}.$$

[45%]

(b) Sketch the first three mode shapes of the bar. How should the xylophone bar be supported so that it can give a long-ringing musical note when struck? [20%]

(c) Two adjacent notes on the xylophone are to be one semitone apart. If the two bars are of the same material and the same cross-section, what should be the ratio of lengths of the two bars? (A semitone is one-twelfth of an octave.) [15%]

(d) Given that the first three roots of equation Equation 1 are $\alpha L = 4 \cdot 73, 7 \cdot 85, 11 \cdot 00$, express the second and third frequencies as multiples of the fundamental frequency. The xylophone produces the most satisfactory musical tone if these frequencies are in exact whole-number ratios. Suggest how the design of the bar could be modified to achieve this effect. Explain briefly, without calculations, how Rayleigh's principle might be used to guide the detailed design modification. [20%]

3 Figure 1 shows a simple model of a tall chimney. It consists of three thin, uniform, rigid sections, which are freely pinned together. They each have length L and the masses are $3m$, $2m$, and m as shown. Torsional springs of stiffness k act at the top two joints, and one of stiffness S acts between the bottom section and the ground. Motion is assumed to be confined to one vertical plane, and to be small. The transverse displacements at the tops of the three sections are y_1, y_2, and y_3, as shown.

(a) Show that the kinetic energy of this system is given by:

$$T = \frac{m}{2}\left[\frac{5}{3}\dot{y}_1^2 + \dot{y}_2^2 + \frac{1}{3}\dot{y}_3^2 + \frac{2}{3}\dot{y}_1\dot{y}_2 + \frac{1}{3}\dot{y}_2\dot{y}_3\right],$$

and obtain an expression for the potential energy of the system. Neglect the effects of gravity. [30%]

(b) Sketch the mode shape you would expect for the fundamental mode for the two cases: $S >> k$ and $S << k$. [20%]

(c) Assume a suitable approximate mode shape for the fundamental mode in the case $S >> k$, and use Rayleigh's principle to estimate the fundamental frequency. [15%]

(d) Explain how you could modify the method used in part (c) to obtain a more accurate estimate of the fundamental frequency. Comment on the accuracy of this solution. [20%]

(e) Indicate briefly how the frequency of the lowest mode would change if gravity was allowed for. [15%]

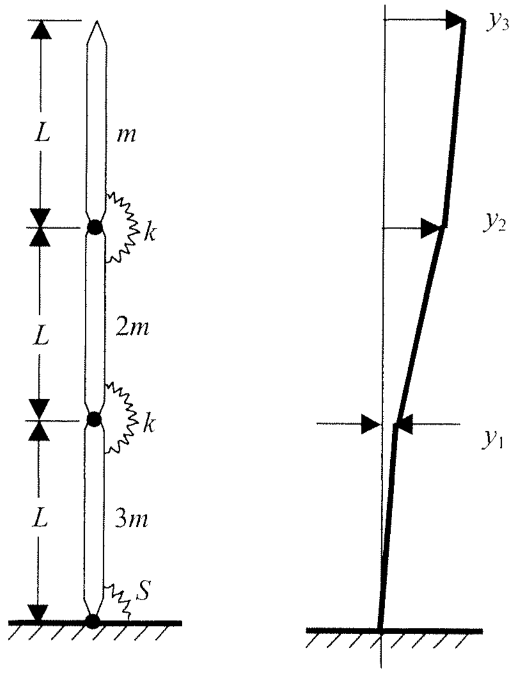

Fig. 1.

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4 The torsional system shown in Fig. 2 consists of three disks, each with polar moment of inertia J, connected by two light elastic shafts, each with torsional stiffness k. The system is supported in frictionless bearings and gravity may be neglected. The absolute angular displacements of the disks are θ_1, θ_2, θ_3. Torque input T is applied to disk 1.

(a) Determine, by inspection or otherwise, the natural frequencies and natural mode shapes of oscillation of the system. [30%]

(b) For the case when T is a sinusoidal torque input, sketch a graph of the steady-state angular displacement response of disk 2, measured in dB, as a function of the frequency of excitation. Show salient values. [30%]

(c) The system is initially at rest and in equilibrium. Disk 1 is given a torsional impulse $T = I\,\delta(t)$ at time $t = 0$. Determine the angular displacement of disk 2 at time $t = \sqrt{(J/k)}$. [40%]

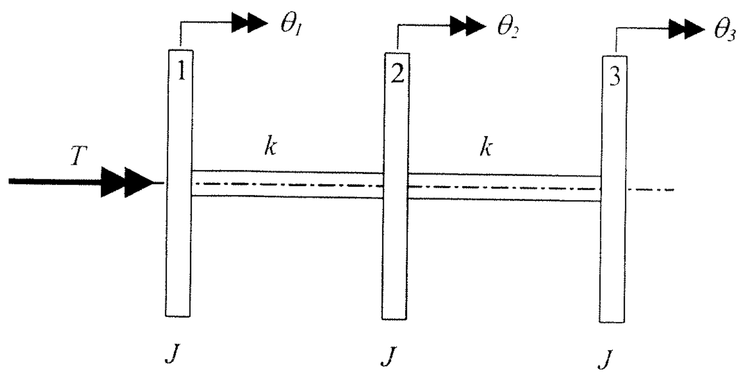

Fig. 2.

END OF PAPER

ENGINEERING TRIPOS PART IIA

Module 3C6 Examination, 2003

Answers

1. (b) $u^{(n)} = \sin\left(\frac{n\pi x}{L}\right), \quad \omega_n = \frac{n\pi}{L}\sqrt{\frac{P}{m}}$;

   (c) $w(x,t) = \sum_n a_n \cos\omega_n t \, \sin\left(\frac{n\pi x}{L}\right)$, with $a_n = \begin{cases} a/2, n = 2 \\ \dfrac{-4a\sin\left(\frac{n\pi}{2}\right)}{\pi\left(n^2-4\right)}, \text{otherwise} \end{cases}$

2. (c) $\frac{L_1}{L_2} = \left(\frac{\omega_2}{\omega_1}\right)^{1/2} = 2^{1/24} = 1.029$

   (d) $\omega \propto (\alpha L)^2, \quad 1:2.76:5.40$

3. (a) $V = \frac{k}{2L^2}\left[(y_3 + y_1 - 2y_2)^2 + (y_2 - 2y_1)^2\right] + \frac{Sy_1^2}{2L^2}$

   (c) $\mathbf{y} = [0 \quad 1 \quad 3]^T; \quad \omega = \frac{1}{L}\sqrt{\frac{2k}{5m}}$

   (d) lower

4. (a) $\begin{cases} \mathbf{u}^{(1)} = [1 \quad 1 \quad 1]^T, \omega_1 = 0 \\ \mathbf{u}^{(2)} = [1 \quad 0 \quad -1]^T, \omega_2 = \sqrt{\frac{k}{J}} \\ \mathbf{u}^{(3)} = [1 \quad -2 \quad 1]^T, \omega_3 = \sqrt{\frac{3k}{J}} \end{cases}$

   (b) $\theta_2\left(\sqrt{J/k}\right) = 0.143 I \sqrt{Jk}$