ENGINEERING TRIPOS PART IIA

Saturday 10 May 2003 9 to 10.30

Module 3D7

CONTINUUM MECHANICS NUMERICAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 The statically indeterminate, two-dimensional structure shown in Fig. 1 consists of four pin-jointed bars which meet at joint M. All members are linear elastic and have the same cross-sectional area A and Young's Modulus E. The structure is initially unstressed. A vertical load W is applied to M.

(a) Set up the equilibrium equations relating the applied forces to the internal stress resultants $\mathbf{r}$ in matrix form. [20%]

(b) Obtain their general solution in the form

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{S}\boldsymbol{\alpha}$$

and explain the physical significance of $\mathbf{r}_0$ and $\mathbf{S}$. [30%]

(c) Evaluate $\mathbf{r}$. [30%]

(d) Explain, without carrying out any detailed calculations, how your solution would be modified if bars I and II were initially made 1% too long. [20%]

Use may be made, without proof, of the compatibility equation

$$\mathbf{S}^T\mathbf{F}\mathbf{S}\boldsymbol{\alpha} = -\left(\mathbf{S}^T\mathbf{e}_0 + \mathbf{S}^T\mathbf{F}\mathbf{r}_0\right)$$

where $\mathbf{F}$ is the flexibility matrix and $\mathbf{e_0}$ is the initial extension vector.

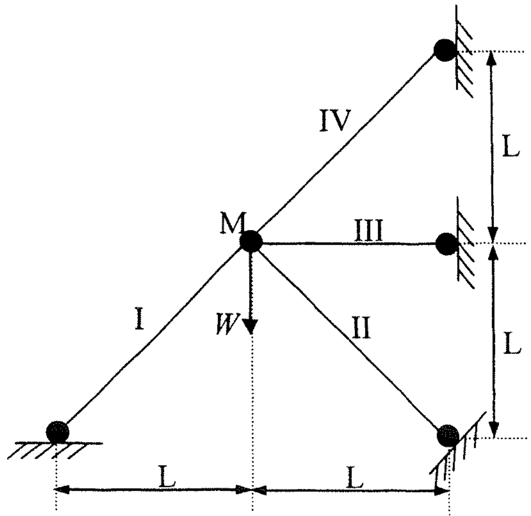


Fig. 1

2 The three-node finite element shown in Fig. 2 is to be used for the analysis of uniaxial stress. The element represents a linear elastic bar of uniform cross-sectional area A and Young's modulus E. Nodes 2 and 3 are unrestrained in the x direction, while Node 1 is fixed to a rigid support.

(a) Sketch the quadratic shape functions that correspond to unit displacements of Nodes 2 and 3. Hence obtain the shape function matrix **N** such that :

$$u(x) = \mathbf{N}(x)\begin{bmatrix} d_2 \\ d_3 \end{bmatrix}$$ [50%]

(b) Derive the two-by-two stiffness matrix which relates the displacements of Nodes 2 and 3 to the corresponding nodal loads **p**. [50%]

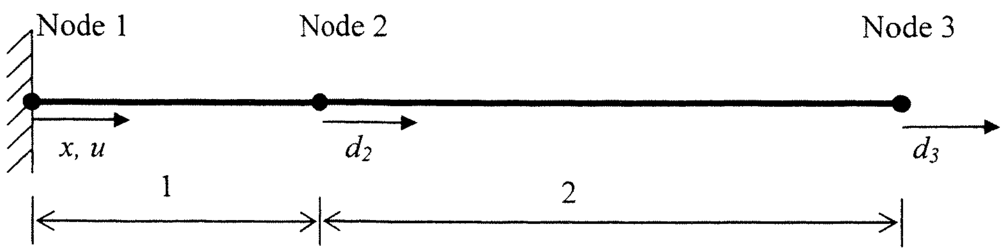


Fig. 2

3 (a) Using a flow chart, explain how stresses are computed by the finite element method for elastic stress analysis of a solid material. Body forces within the solid volume and forces at the boundaries are to be applied. Isoparametric elements are to be used. Distinguish between global, element and Gauss point levels in the flow chart. [30%]

(b) Stresses (**σ**) and displacements (**u**) in a dam are to be evaluated using the finite element method. The dam has a height of h, crown width of w_1 and bottom width of w_2, as shown in Fig. 3. The analysis is performed in plane strain conditions. Determine the boundary conditions and the body force load (**b**) for the dam using **σ**, **u**, the coordinate system (x_1 and x_2) and the unit vectors (**e** and **β**) given in the figure. Classify the boundary conditions. [30%]

(c) The governing equation of one dimensional fluid flow in a porous medium is given as follows.

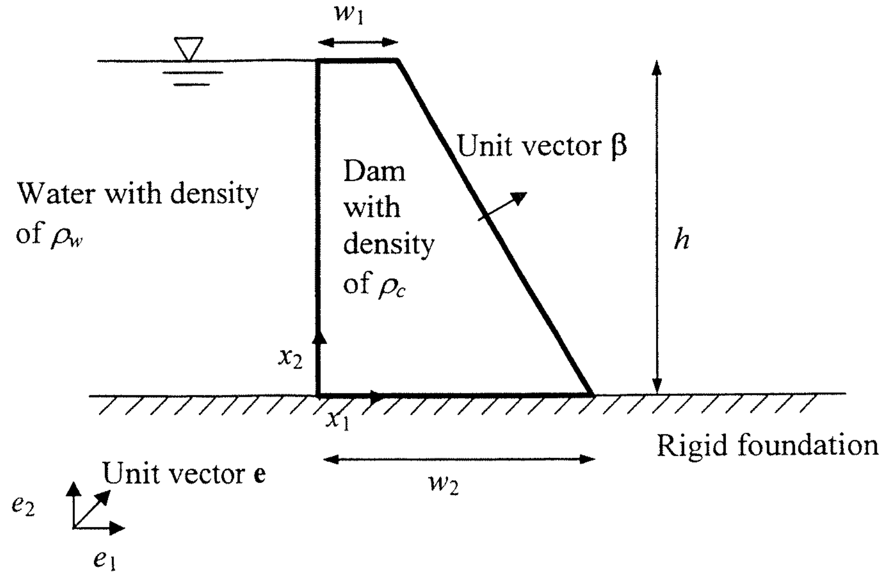
$$-\frac{d}{dx}(AV)+Q=0$$

where A = cross sectional area
Q = volumetric rate of water supply per unit length
$V = -k(dH/dx)$ = fluid Darcy's velocity
H = hydraulic head
k = hydraulic conductivity

The boundary conditions are given as follows.

$V=\overline{V}$ at $x = 0$ (Neumann boundary)
$H=\overline{H}$ at $x = L$ (Dirichlet boundary)

For the finite element formulation, derive the weak form of this boundary value problem. [40%]



w_1
Unit vector β
Water with density of ρ_w
Dam with density of ρ_c
h
x_2
x_1
Rigid foundation
Unit vector **e**
e_2
e_1
w_2

Fig. 3

4 Fig. 4 shows an iso-parametric element *QRST* and its 'parent', a standard four-node quadrilateral, *qrst*.

(a) Find the bi-linear shape functions for the nodes of the parent element. [10%]

(b) Set up a linear geometric mapping between points in the 'parent' element and the corresponding points in the iso-parametric element. [25%]

(c) Verify that point P = (4.047, 2.625) corresponds to point p = (0.75, 0.75) [20%]

(d) Sketch the line to which *q p s* is transformed in the iso-parametric element. Is it a straight line? [20%]

(e) Find the displacement components at point P due to nodal displacements:

$$\mathbf{d} = [d_{QX} \quad d_{QY} \quad d_{RX} \quad d_{RY} \quad d_{SX} \quad d_{SY} \quad d_{TX} \quad d_{TY}]$$
$$= [\,0.1 \quad 0.2 \quad 0 \quad 0 \quad 0.15 \quad 0 \quad -0.05 \quad 0\,]$$
[25%]

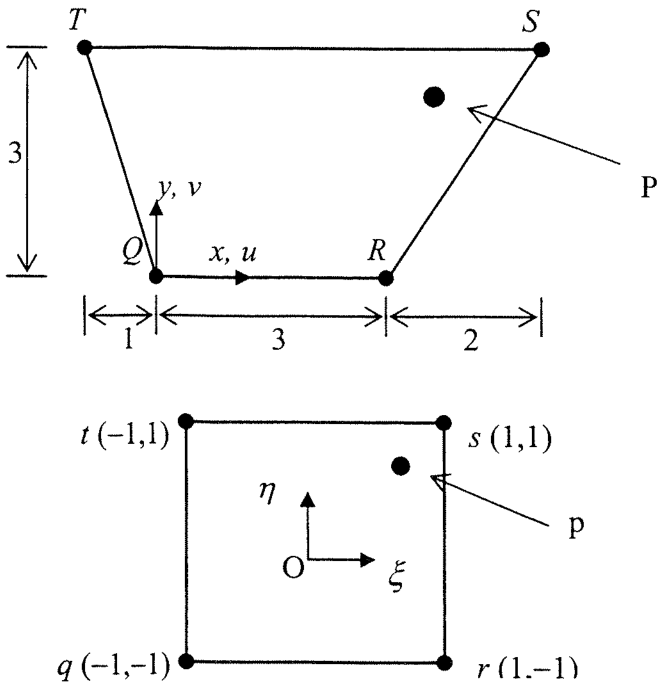


Fig. 4

END OF PAPER

Part IIA Module 3D7 Continuum Mechanics Numerical Methods 2003 Solutions

1

(a) $$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & -1 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} t_I \\ t_{II} \\ t_{III} \\ t_{IV} \end{bmatrix} = \begin{bmatrix} 0 \\ -W \end{bmatrix}$$

(b) $$\begin{bmatrix} -W/\sqrt{2} \\ -W/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/\sqrt{2} & 1 \\ -1/\sqrt{2} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

(c) $$\begin{bmatrix} -0.4142 \\ -0.5858 \\ -0.1716 \\ 0.4142 \end{bmatrix} W$$

(d) –

2.

(a) $$\begin{bmatrix} (3x - x^2)/2 & (-x + x^2)/6 \end{bmatrix} \begin{bmatrix} d_2 \\ d_3 \end{bmatrix}$$

(b) $$EA \begin{bmatrix} 9/4 & -3/4 \\ -3/4 & 7/12 \end{bmatrix}$$

3.

(a) –

(b) –

(c) $$\int_0^L \frac{dv}{dx} Ak \frac{dH}{dx} dx = v(0) A\overline{V} + \int_0^L vQdx$$

$$H(x=L) = \overline{H}$$

4.

(a) $n_q = (1-\xi)(1-\eta)/4$, $n_r = (1+\xi)(1-\eta)/4$, $n_s = (1+\xi)(1+\eta)/4$, $n_t = (1-\xi)(1+\eta)/4$

(b) $$\begin{bmatrix} x \\ y \end{bmatrix} = 1/4 \begin{bmatrix} 7 + 9\xi + \eta + 3\xi\eta \\ 6 + 6\eta \end{bmatrix}$$

(c) –

(d) –

(e) [0.116, 0.0125]