ENGINEERING TRIPOS PART IIA

Wednesday 30 April 2003 2.30 to 4.00

Module 3E4

MODELLING CHOICE

*Answer not more than **two** questions.*

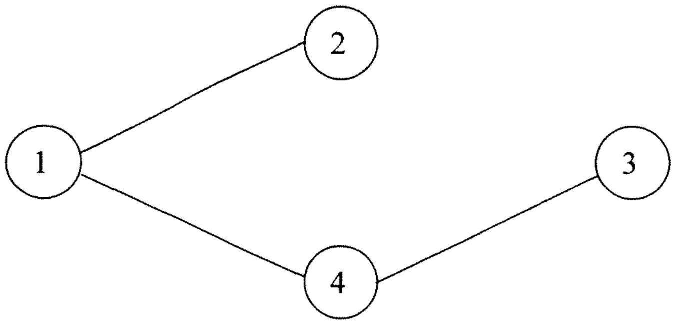
All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 (a) Consider an electrical network with a power station at each of nodes 1 and 3, and an electricity consumer (wholesaler & distributor) at each of nodes 2 and 4.



The cost of generating electrical energy at nodes 1 and 3 is £ c_1 and £ c_3 per kilowatt hour (kWh) respectively; no electricity is consumed at these nodes. Demand at node 2 is d_2 kWh at the fixed price £ p_2 per kWh, whereas at node 4, demand is d_4 kWh and the price is £ p_4 per kWh; no electricity is generated at these nodes.

(i) Formulate, but do not attempt to solve, a linear programming model for optimising electricity dispatch (distribution) in the above network, subject to satisfying demand. Your model should show the amount of energy flowing along each link, for example, x_{14} would be the amount of energy transmitted from node 1 to 4, and x_{41} would be the amount transmitted from node 4 to 1.
Note: no specific knowledge of electrical power is needed, simply think of a commodity like water that is pumped through the network, so that, for instance, the exact quantity of energy that flows out of node 3 must arrive at node 4. [25%]

(ii) Suppose a new link (power line) is added to join nodes 2 and 3. The loop or cycle created produces a new condition on energy flow: the sum of clockwise flows on links must equal the sum of counterclockwise flows. For example, if there is a 1 kWh flow along each of the links from node 1 to 2, node 1 to 4, and node 4 to 3, then 1 kWh must flow from node 2 to 3. Reformulate, without attempting to solve, the previous power dispatch problem. [25%]

(Cont.

(b) The following optimisation problem maximises profit for a production process:

$$
\begin{array}{llllllll}
\text{Max} & 15x_1 & + 10x_2 & - 10x_3 & & - 10x_5 & & \\
\text{subject to} & 5x_1 & + 3x_2 & + x_3 & & & & = 18 \\
& 3x_1 & + 2x_2 & + 2x_3 & - x_4 & & & = 9 \\
& 2x_1 & + x_2 & + x_3 & & + x_5 & & = 7 \\
& x_1 & + x_2 & + 4x_3 & & & + x_6 & = 5 \\
& \multicolumn{7}{l}{x_1, x_2, \ldots, x_6 \text{ non-negative}}
\end{array}
$$

(i) Verify that the optimal solution vector has basic variables x_1, x_2, x_4 and x_6. What are the optimal solution vector and optimal profit? [20%]

(ii) Consider reducing the profit coefficient of x_1 by an amount $\delta \geq 0$. Use the formula for the reduced cost vector to determine the maximum value that δ can take without jeopardising optimality of the solution vector found in part (i). If the maximum reduction δ is carried out, what will the optimal solution vector and optimal profit become? [20%]

(iii) It can be seen that an increase of any amount Δ of the right hand side of the third constraint, which currently has the value 7, can be matched by an increase of the value of x_5 by the same amount, thereby retaining feasibility and reducing the profit by 10Δ. Is it possible to do any better? [10%]

2 (a) A hi-tech firm is carrying out its annual planning for next year's research projects. An analysis of each of the possible projects provides its investment budget (£000s), projected revenue (£000s) and projected staffing level:

	PROJECTS AVAILABLE				
	A	B	C	D	E
Investment	72	59	113	44	71
Revenue	128	100	150	75	115
No. of Staff	16	9	15	8	10

The project planning team has identified several issues that need to be addressed. It is noted that any project that goes ahead must be fully financed and staffed, as there is little value in a partially completed project. Cash burn, i.e. total investment for all projects, must not exceed £250,000. Revenue should exceed £350,000 if possible; the exact figure isn't critical. Finally it is requested, as a low priority, that the total number of staff for all projects should remain roughly constant at this year's level of 35. Moreover, the Production Manager will oppose any proposal for which the total staffing level is outside the range of 30 to 40 staff.

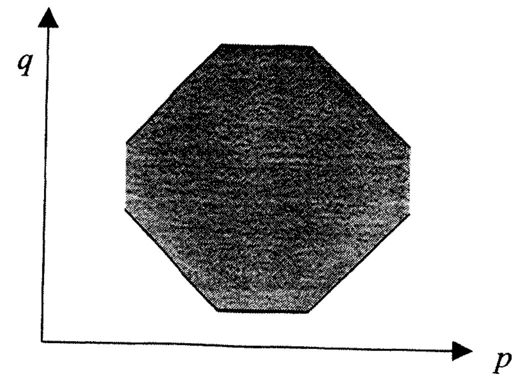
(i) Formulate, but do not attempt to solve, a goal programming (GP) model to help the project planning team, and suggest initial values for the weights. Use the percentage minimax framework. [30%]

(ii) Synergies between projects A, C and D mean that total investment can be reduced by £14,000 if at least two of these projects go ahead. Update the model in part (i) to accommodate this information. [10%]

(iii) The outputs of the GP model developed in part (i), namely the optimal value, optimal decision variables and optimal deviational variables, depend on the weights. Briefly discuss the relative importance of these three types of outputs in the decision making process. [15%]

(Cont.

(b) Suppose we plot all pairs of objective values (p,q) of a bi-criteria linear programming problem, over all possible feasible points, and obtain the polyhedron below.



The problem requires minimisation of the first objective and maximisation of the second.

(i) Sketch the set of Pareto optimal pairs (p,q). [5%]

(ii) Consider the weighted sum reformulation of the bi-criteria linear program. Sketch the set of all pairs (p,q) that correspond to solving the weighted sum reformulation with positive weights. Repeat this for the case when non-negative weights are examined. Repeat this for the case when positive weights are examined and the Simplex method is used to find solutions of the weighted sum problem. [15%]

(c) Is it possible to reformulate the problem below as a convex nonlinear program? Explain why or why not.

$$\begin{aligned} \text{Min} \quad & \exp(x_3^2) + u_1^2 + (u_2 - u_1)^2 \\ \text{subject to} \quad & x_1 = 1 \\ & x_2 = 0.95\, x_1 + 0.05\, u_1 \leq 0.75\, x_1 \\ & x_3 = 0.95\, x_2 + 0.05\, u_2 \leq 0.75\, x_2 \\ & 0 \leq 1 - (u_1^2 + u_2^2) \\ & x_1, x_2, x_3 \ \text{non-negative.} \end{aligned}$$

[25%]

3 (a) A supermarket chain has contracted a business software development firm to design and implement a stock management system that combines a product storage module, PSM, and a demand and forecasting module, DFM. Tasks B-G below relate to PSM, while tasks H-M relate to DFM.

Task	Description	Estimates of task durations (weeks)			Predecessor tasks
		Earliest	Likeliest	Latest	
A	product data assessment	11	13	15	none
B	specification of PSM	4	5	8	A
C	software implementation	20	24	32	B
D	data transfer from existing database to PSM database	4	6	8	B
E	integration of database into PSM	3	4	5	C, D
F	user documentation and training materials	3	3	3	E
G	staff training	2	2	2	F
H	specification of DFM	7	9	13	A
I	software implementation	13	15	18	H
J	data transfer from existing database to DFM database	3	4	5	H
K	calibration of DFM	3	3	4	I, J
L	user documentation and training materials	2	2	2	I, J
M	staff training	2	2	2	K, L
N	integration of PSM and DFM	2	4	6	G, M
O	staff training on combined system	1	2	3	N

(i) Draw an AON network for this project. Apply PERT, using the β-distribution for the duration of each task, to determine the latest time T that the project will finish with 95% probability. Which activities are critical? Recall that the β-distribution of a random variable with a most likely value m, a lowest value l and a highest value h, has a mean of $(l + 4m + h)/6$ and a variance of $(h-l)^2/36$.

[35%]

(ii) Briefly discuss the risk of not completing the project within T weeks.

[15%]

(b) (i) Carry out two iterations of the projected gradient method for the following problem, starting from $x^0 = (3/2, 3/2)$ and using stepsize 1 at each iteration. Determine the required projections by inspection of a graph of the feasible region.

$$\begin{array}{ll} \text{Min} & -x_1 - x_2 - 2x_1 x_2 + \tfrac{1}{2}(x_1^2 + 3x_2^2) \\ \text{subject to} & 1 \le x_1 \le 2 \\ & 1 \le x_2 \le 2 \end{array}$$

Is the first iteration vector x^1 a stationary point? Explain why or why not. [40%]

(ii) If you are given a stationary point x for the problem in part (i), can you tell if it is a local or global minimiser? Explain your reasoning. [10%]

4 (a) A garden centre uses optimisation as a monthly planning tool, to model the maximisation of profits given product demand and prices, while sharing raw materials, limited labour and other inputs across its various product lines. Currently it pays 20 pence per kilogram for sawdust mulch. Describe how the profit maximisation problem could be modified to allow for a 25% decrease in the unit cost of sawdust mulch, for every kilogram ordered in excess of 1000 kg in a month. Your answer should describe the necessary changes or additions to any aspect of the original optimisation problem. [34%]

(b) Use integer linear programming modelling to represent the one-dimensional feasible set $\{ x_1 : 0 \le x_1 \le 1 \text{ or } 2 \le x_1 \le 3 \}$ using two binary variables, and as many continuous variables as needed. [33%]

(c) Use the Branch and Bound method to prove that the following problem is infeasible:

$$
\begin{array}{llcccl}
\text{Max} & & & x_2 & & \\
\text{subject to} & x_1 & + & x_2 & \le & 2.75 \\
 & x_1 & + & x_2 & \ge & 2.25 \\
 & \multicolumn{5}{l}{x_1,\ x_2 \text{ are non-negative integers.}}
\end{array}
$$

If there is a choice, branch on x_1 before x_2. You may solve the linear programs that arise at each node graphically. Show the tree of subproblems that are generated by the Branch and Bound procedure, giving appropriate details at each node such as constraints, optimal value and optimal solution vector. [33%]

END OF PAPER