ENGINEERING TRIPOS PART IIA

Wednesday 30 April 2003 9 to 10.30

Module 3F1

SIGNALS AND SYSTEMS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 A linear, time-invariant discrete-time system gives an output sequence

$$y(k) = \alpha^k, \qquad k = 0, 1, 2, \ldots$$

in response to an input sequence

$$u(k) = \beta^k, \qquad k = 0, 1, 2, \ldots$$

with $u(k) = y(k) = 0$ for $k < 0$.

(a) (i) Calculate the z-transform of the input sequence from first principles. [10%]

(ii) Deduce the transfer function of the system. [10%]

(iii) Derive a difference equation that could implement this system. [10%]

(b) (i) Find the unit pulse and step responses of the system. [20%]

(ii) Check the first three values of these responses using your answer to part (a)(iii). [15%]

(c) For a sampling period of 1 ms, what values of the parameters α and β would give a low-pass filter with a 3 dB bandwidth of 5 Hz and a d.c. gain of 10 ? [35%]

2 (a) A continuous time system with transfer function $G(s)$ is connected to a digital-to-analogue converter (DAC) and an analogue-to-digital converter (ADC) as shown in Fig. 1. The DAC is a first order hold whose output consists of a linear extrapolation of the last two discrete inputs, and the sampling period is T seconds.

(i) Explain why the system relating $\{y(kT)\}$ to $\{u(kT)\}$ has a z-transfer function. [10%]

(ii) Show that the z-transfer function relating $\{y(kT)\}$ to $\{u(kT)\}$ is given by

$$H(z) = \frac{(z-1)^2}{Tz^2} \mathcal{Z}\left(\mathcal{L}^{-1}\left(G(s)V(s)\right)\big|_{t=kT}\right)$$

where $V(s) = \dfrac{Ts+1}{s^2}$. [40%]

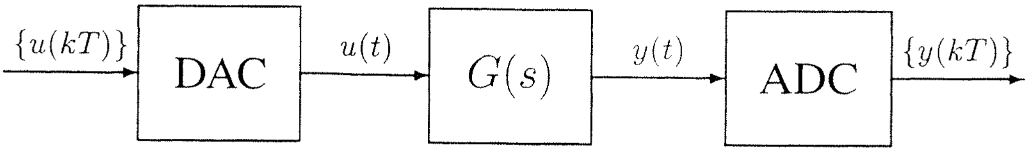

Fig. 1

(b) (i) From the definition of conditional probability

$$P(A|B) = P(A,B)/P(B)$$

derive Bayes' rule. [10%]

(ii) In a sampled data system, it is required to detect the most probable position of a pulse $v_k(n)$, embedded in noise $\varepsilon(n)$, where the measured signal at sample n is

$$x(n) = v_k(n) + \varepsilon(n)$$

and the pulse at location k is defined by

$$v_k(n) = \begin{cases} 0.5 & \text{if } n = k,\ k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Possible values for k are $\{0, 1, 2\}$ with prior probabilities of $\{0.25, 0.35, 0.4\}$ respectively.

Calculate the most probable pulse location k if the noise $\varepsilon(n)$ is unit variance Gaussian with pdf given by

$$f_\varepsilon(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$$

and the measured data values are [40%]

$$x(0) = 0.7,\ x(1) = 0.3,\ x(2) = 0.4,\ x(3) = -0.1\ .$$

3 Define the meaning of auto-correlation and cross-correlation functions, when applied to ergodic random processes. [10%]

A linear time-invariant system with impulse response $h(t)$ and frequency response $\mathcal{H}(\omega)$ is excited with a random noise process $X(t)$ that is ergodic and wide-sense stationary.

(a) Express the cross-correlation function between the system output $Y(t)$ and the system input $X(t)$ as an expectation, and hence derive an expression relating it to the autocorrelation function of $X(t)$. [25%]

(b) If $S_X(\omega)$ is the power spectral density of $X(t)$, state how it relates to the autocorrelation function of $X(t)$. Hence derive an expression for the output power spectral density $S_Y(\omega)$ in terms of $S_X(\omega)$. [20%]

(c) When a plant is operating on-line, its input behaves as a zero-mean wide-sense stationary random process with auto-correlation function given by

$$r_{XX}(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T} & \text{for } |\tau| < T \\ 0 & \text{for } |\tau| \geq T \,. \end{cases}$$

If the cross-correlation function between input and output is measured to be

$$r_{XY}(\tau) = \begin{cases} 1 - \dfrac{|\tau - 2T|}{2T} & \text{for } 0 < \tau < 4T \\ 0 & \text{for } \tau \leq 0 \text{ or } \tau \geq 4T \,, \end{cases}$$

show graphically that $r_{XY}(\tau)$ can be generated from a linear combination of shifts of $r_{XX}(\tau)$, and hence show that $h(t)$ is a weighted sum of three impulses. [25%]

(d) Briefly discuss any problems that would be encountered if it was desired to estimate the plant impulse response from measurements of the input and output power spectral densities, instead of by measuring $r_{XY}(\tau)$. [20%]

4 Give definitions for the entropy of a discrete random variable X and the mutual information of two discrete random variables X and Y. [10%]

A source generates a stream of symbols S_n with $n = 1, 2, \ldots$, and each symbol takes one of two values, A and B. The probability $P(S_n | S_{n-1})$ that the source generates each value depends on the value of the previous symbol according to the following table

	$P(S_n\| S_{n-1})$	
	$S_{n-1} = A$	$S_{n-1} = B$
$S_n = A$	0.7	0.1
$S_n = B$	0.3	0.9

(a) Verify that the a priori probability distribution of the two symbols is given by

	$P(S_n)$
$S_n = A$	0.25
$S_n = B$	0.75

[10%]

(b) (i) Calculate the entropy of a symbol S_n taken at random from the stream. [10%]

(ii) Calculate the expected entropy of a symbol S_n given that the value of the previous symbol S_{n-1} is known. [20%]

(c) (i) Calculate the mutual information of two consecutive symbols S_n and S_{n+1} . [10%]

(ii) Calculate the mutual information of S_n and S_{n+2} . [20%]

(d) A sequence of M symbols from this stream is transmitted over a radio link. The link fails temporarily during transmission such that all the symbols are received correctly apart from a single symbol in the middle of the sequence whose value is unknown (i.e. the values of $S_1, \ldots, S_M$ are known apart from the symbol S_l for some unknown l with $2 \leq l \leq M - 1$). What is the expected loss of information due to the radio failure? [20%]

END OF PAPER

Engineering Tripos Part IIA
2003
Paper 3F1: Signals and Systems

Answers

1. (a)(i) $\dfrac{1}{1-\beta z^{-1}}$

   (a)(ii) $\dfrac{z-\beta}{z-\alpha}$

   (a)(iii)

$$y_{k+1} - \alpha y_k = u_{k+1} - \beta u_k$$

   (b)

$$\begin{aligned}
\text{pulse response: } h_k &= \begin{cases} 1 & k=0 \\ \alpha^k - \beta\alpha^{k-1} & k \geq 1 \end{cases} \\
\text{step response} &= \begin{cases} 1 & k=0 \\ \frac{1}{1-\alpha}((\beta-\alpha)\alpha^k + 1 - \beta) & k \geq 1 \end{cases}
\end{aligned}$$

   (c) $\alpha = 0.969$, $\beta = 0.689$.

2. (b) $k = 1$ gives the most probable position.

3. (a) $r_{XY}(\tau) = h(\tau) \star r_{XX}(\tau)$
   (b) $S_Y(\omega) = S_X(\omega)|H(\omega)|^2$
   (c) $h(t) = \frac{1}{2}\delta(t-T) + \delta(t-2T) + \frac{1}{2}\delta(t-3T)$
   (d) Formula in (b) doesn't tell us the phase of $H(\omega)$.

4. (b)(i) 0.8113
   (b)(ii) 0.5721
   (c)(i) 0.2392
   (c)(ii) 0.0859
   (d) 0.4188