ENGINEERING TRIPOS PART IIA

Thursday 8 May 2003 9 to 10.30

Module 3F3

SIGNAL AND PATTERN PROCESSING

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 (a) A radix-2 FFT of size $N = 2^M$ points is computed in a number of stages, in each of which a number of 'butterfly' computations is performed. State the number of stages, and number of butterflies per stage, in relation to the transform size N. State also what computations are required to compute one butterfly operation, and hence show that the total number of real operations (that is, real multiplications, additions or subtractions) required is $5N\log_2(N)$. [Include multiplications by 1 or j in the operation count.] [30%]

(b) A 48-point DFT is defined by the following equation

$$X(k) = \sum_{n=0}^{47} x(n) \exp\left(-j\frac{2\pi nk}{48}\right), \quad k = 0, 1, \ldots, 47$$

Show how this may be computed using a first stage of 3-point DFTs, followed by a stage of 16-point DFTs.

Assuming the 16-point DFTs are computed using FFTs, derive the total real operation count for the resulting transform and compare it with that for a direct implementation of the 48-point DFT. [30%]

(c) Two length-N real signal vectors $\mathbf{x}$ and $\mathbf{z}$ have DFT $\mathbf{X}$ and $\mathbf{Z}$ respectively. The vector $\mathbf{Y}$ is defined as the element-by-element product of $\mathbf{X}$ and $\mathbf{Z}$, i.e. $Y(m) = X(m)Z(m)$. Show that the inverse DFT of $\mathbf{Y}$ is the circular convolution of $\mathbf{x}$ and $\mathbf{z}$, that is

$$y(m) = \sum_{n=0}^{N-1} x(n)z(m-n)$$

where the index $(m-n)$ is interpreted modulo N.

Describe how in the Fast Convolution method, the FFT is used to implement filtering of a signal $x(n)$ by an FIR filter with length $M+1$ impulse response $h(n)$, $n = 0, 1, \ldots, M$. Equations are not required, but explain the overall operation of the algorithm. What advantage can this method offer? [40%]

2 A length-4 FIR filter was designed using the Remez algorithm, and the resulting coefficients were

$$[0.1340,\ 0.4641,\ 0.4641,\ 0.1340]$$

The design specification had two frequency bands, one (band A) from $\Omega = 0$ to $\Omega = \pi/3$ rad/sample, and the other (band B) from $\Omega = 2\pi/3$ to $\Omega = \pi$ rad/sample. In one of the bands the desired gain was 1.0 and in the other, 0.

(a) By computing the frequency response at zero frequency and at half the sample frequency, deduce whether the filter is a highpass (gain = 1 in band B) or lowpass (gain = 1 in band A) filter. Explain whether or not the filter has linear phase. [15%]

(b) By additionally computing the frequency response at frequencies $\Omega = \pi/3$ and $\Omega = 2\pi/3$ rad/sample, sketch the filter's frequency response and show that the filter has an 'equiripple' (or 'minimax') response. Deduce what relative weighting was applied to the errors in the two bands. [15%]

(c) Show that the mapping $z \rightarrow -z$ converts a lowpass response $H(z)$ to a highpass response $G(z) = H(-z)$, or alternatively converts a highpass response to a lowpass one. Deduce the coefficients of the length-4 FIR filter obtained by applying this mapping to the filter described above. Explain whether or not the resulting filter has linear phase. [20%]

(d) Define the terms power spectrum and autocorrelation function for a wide-sense stationary discrete time random process. Explain the interpretation of the power spectrum in terms of signal power. [20%]

(e) The filter designed above is applied with zero mean white noise at its input, having variance $\sigma^2 = 3$. What type of random process is the output of the filter? Show that the values of the output process are uncorrelated for time lags $|l| > L$ where L is a constant which should be stated. Determine and sketch the power spectrum at the output of the filter over the frequency range $\Omega = -2\pi$ to $\Omega = 2\pi$ rad/sample. Determine also the autocorrelation function for the process. [30%]

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3 In a navigation system it is desired to predict the distance of a vehicle from its starting position. This is to be achieved by measuring the distances up to the current time t and using these measured values to predict the next value x_{t+1}. A two-tap FIR digital filter is to be designed for the task, based upon the mean-square error between the true (but unknown) distance x_{t+1} and the filter output $\hat{x}_{t+1}$.

The measured distances can be assumed to be drawn from a stationary random process with known autocorrelation sequence $r_{xx}[q]$, $-\infty < q < +\infty$, and zero mean.

The FIR filter takes as its input the two *most recent* measured values of the distance, so that

$$\hat{x}_{t+1} = b_0 x_t + b_1 x_{t-1}$$

where b_0 and b_1 are the filter coefficients and $\hat{x}_{t+1}$ is the predicted distance at time $t+1$.

(a) Define for this problem the prediction error and the mean-squared prediction error. [10%]

(b) Show that the optimal filter which minimises the mean-square prediction error satisfies the following equations:

$$\begin{aligned} r_{xx}[0]b_0 + r_{xx}[1]b_1 &= r_{xx}[1] \\ r_{xx}[1]b_0 + r_{xx}[0]b_1 &= r_{xx}[2] \end{aligned}$$

[40%]

(c) Show that the minimum mean-square error corresponding to this solution is:

$$J = r_{xx}[0] - b_0 r_{xx}[1] - b_1 r_{xx}[2]$$

Your answer should derive this result from first principles based on the solutions to (a) and (b) above. [20%]

(d) It is thought that $\{x_t\}$ can be modelled as a first-order zero-mean autoregressive process with autocorrelation function $r_{xx}[q] = \alpha^{|q|}$, where α is a known constant lying between -1 and +1. Determine the filter coefficients and minimum mean-square error in this case. Comment on the significance of this result. You should consider the particular form of the solution in this case and also the error performance as α varies between -1 and +1. [30%]

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4 A multi-class classification problem is to be solved using a Bayes' minimum error rate classifier. There are K classes $\omega_1, \ldots, \omega_K$. The class-conditional probability density function for each of the K classes is assumed known and equal to $p(\mathbf{x}|\omega_k)$. The prior probability for each class is also known and equal to $P(\omega_k)$.

(a) Derive an expression for the posterior probabilty of class ω_1 given an observation $\mathbf{x}$ in terms of the class-conditional probability density functions and class priors. For this multi-class problem state the Bayes' minimum error rate decision rule. [20%]

(b) What are the limitations of Bayes' decision rule for building classifiers in complicated real-life problems? [10%]

(c) The equation of the decision boundary between two of the classes, ω_1 and ω_2 , is required. The class conditional probability density functions for class ω_1 and class ω_2 are multivariate Gaussian distributions with means μ_1 and μ_2 , and covariance matrices Σ_1 and Σ_2 respectively. The class prior probabilities for the two classes are equal.

(i) Show that a point $\mathbf{x}$ on the decision boundary satisfies an equation of the form

$$\mathbf{x}'\mathbf{A}\mathbf{x} + \mathbf{x}'\mathbf{b} + c = 0$$

where $\mathbf{x}'$ is the transpose of the vector $\mathbf{x}$. Give expressions for $\mathbf{A}$, $\mathbf{b}$ and c in terms of the parameters of the two class-conditional probability density functions. [30%]

(ii) Show that the posterior probability of class ω_1 can be written as

$$P(\omega_1|\mathbf{x}) = \frac{1}{1+\exp(-a)}$$

Using the result of section (c)(i), find an expression for a in terms of the class-conditional probability density function parameters. [20%]

(d) Rather than using the Bayes' minimum error rate classifier to perform the multi-class classification, multiple pairwise classifiers are used. Each of these pairwise classifiers is generated by applying Bayes decision between two classes. Contrast the use and performance of these multiple pair-wise classifiers with the multi-class Bayes' minimum error rate classifier. [20%]

END OF PAPER

Module 3F3 Answers – Part IIA 2003

1 (a) log2 (N) stages; N/2 butterflies per stage
(b) 2016 real operations, compared with 18336 for full 48 point DFT

2 (a) Low pass, linear phase
(b) Equirepple, hence relative weighting in each band is 1.0
(c) Modified coefficients: [0.1340, -0.4641, 0.4641, - 0.3440]. Coefficients are antisymmetric, hence strictly not linear phase
(d) L = 3. r[0] = 1.4, r[+/-1] = 1.02, r[+/-2] = 0.37, r[+/-3] = 0.05

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