ENGINEERING TRIPOS PART IIA 2004

Solutions to Module 3A3
Fluid Mechanics II

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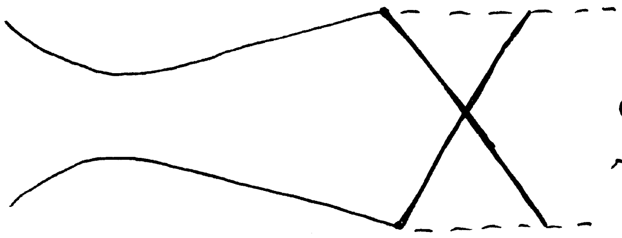
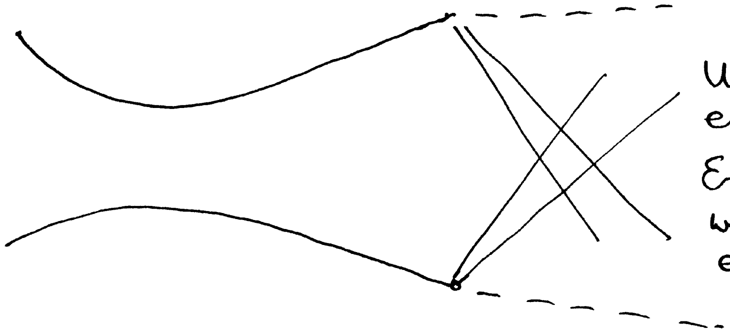
PAPER 3A3 - 2004

Q1

a)



Overexpanded
shocks at exit



Under-
expanded.
Expansion
waves at
exit.

b) $\dfrac{P_2}{P_{01}} = \dfrac{1}{4.25} = .2353 \rightarrow M_{2_{IS}} = 1.6$

The exit Mach No is 1.6 if the flow is isentropic.

This would have $\dfrac{A_2}{A^*} = \dfrac{1.281}{1.0246}$

$\rightarrow A_2 = 1.2502 \ A^*$

So the exit area and exit pressure are both compatible with isentropic flow at $M_2 = 1.6$.

Thrust $= \dot{m} V_2$ since exit pressure is ambient $= \dfrac{\dot{m}\sqrt{C_p T_0}}{A P_0} \times \dfrac{V}{\sqrt{C_p T_0}} \times A P_0$

Taking A at throat, $P_0 = P_{01}$

Q Cont

$$\text{Thrust} = 1.281 \times .8229 \times \frac{\pi \times .3^2}{4} \times 4.25 \times 10^5 \text{ N}$$

$$= \underline{\underline{31.66 \text{ KN}}}$$

c)

For nozzle 1, $A/A^* = 1.555$.

If flow is isentropic within the nozzle

$M_{exit} = 1.9$.

$\rightarrow \frac{P_e}{P_{01}} = .1492$

This is lower than the actual exit pressure so there must be a shock at exit or within the nozzle. For a normal shock at exit $P_2 / P_{01} = \frac{P_2}{P_1} \times \frac{P_1}{P_{01}} = \underbrace{4.025 \times .149}_{\text{Tables.}}$

$= .599$.

The actual exit pressure is lower than this so there is an oblique shock at exit.

$$\text{Thrust} = (P_e A_e + \dot{m} V_e) - P_a A_e$$

$$= \frac{F}{\dot{m}\sqrt{C_p T_0}} \times \frac{\dot{m}\sqrt{C_p T_0}}{A_e P_0} \times A_e P_0 - P_a A_e$$

$$= A^* P_0 \left[1.097 \times .824 \times \frac{A_e}{A^*} - \frac{P_a}{P_0} \frac{A_e}{A^*} \right]$$

$$= \frac{\pi .3^2}{4} \times 4.25 \times 10^5 \left[1.097 \times .824 \times 1.555 - \frac{1}{4.25} \times 1.555 \right]$$

$$= \underline{\underline{31.235 \text{ KN}}}$$

Q6 cont'd

c) For the second nozzle, $A/A^* = 1.066$

If isentropic flow $\rightarrow$ $M_{exit} = 1.3$

$$\rightarrow \frac{P_e}{P_{01}} = .3609.$$

This is higher than the actual exit pressure so there must be expansion waves at exit - the nozzle is underexpanded.

$$\text{Thrust} = \frac{F}{\dot m\sqrt{c_pT_0}} \times \frac{\dot m\sqrt{c_pT_0}}{A_e P_0} \times A_eP_0 - A_eP_a$$

$$= 1.066\ A^* \times P_0 \left[1.011 \times 1.201 - \frac{1}{4.25}\right]$$

$$= \underline{\underline{31.349}}\ \text{K.N}$$

d)

$$T = \dot m V_e + (P_e - P_a)A_e$$

$$\delta T = \dot m\,\delta V_e + (P_e - P_a)\,\delta A_e + A_e\,\delta P_e$$

Euler. $\delta V = -\frac{1}{\rho V}\,\delta P$

i.e. $\delta V_e = -\left(\frac{\dot m}{A_e}\right)^{-1} \delta P_e$

Substituting in eqn for δT

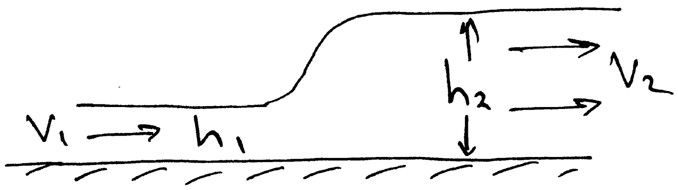
$\rightarrow$ $\delta T = (P_e - P_a)\,\delta A_e$

which $\rightarrow 0$ when $P_e = P_a$

So $P_e = P_a$ gives the maximum thrust.

Q2

a)



$V_1 h_1 = V_2 h_2$ — continuity 1

$V_1^2 h_1 + g\frac{h_1^2}{2} = V_2^2 h_2 + \frac{g h_2^2}{2}$ — momentum 2

$\rightarrow V_1^2 h_1 + \frac{g h_1^2}{2} = V_1^2 \frac{h_1^2}{h_2} + \frac{g h_2^2}{2}$

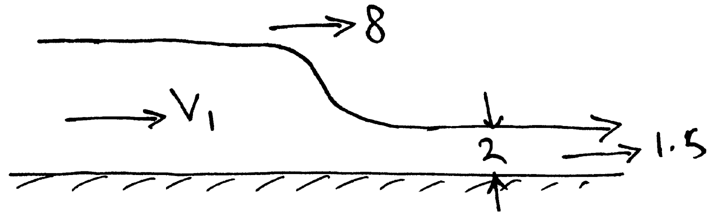
$V_1^2\left(h_1 - \frac{h_1^2}{h_2}\right) = \frac{g}{2}\left(h_2^2 - h_1^2\right)$

$\frac{V_1^2}{g h_1}\left(h_1^2 - \frac{h_1^3}{h_2}\right) = \frac{1}{2}\left(h_2^2 - h_1^2\right)$

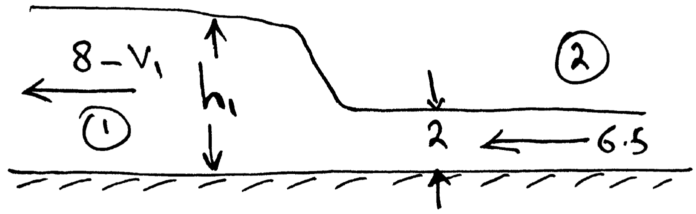
$F_1^2 \frac{h_1^2}{h_2}\left(h_2 - h_1\right) = \frac{1}{2}\left(h_2 - h_1\right)\left(h_2 + h_1\right)$

$$F_1^2 = \frac{1}{2}\left(\frac{h_2 + h_1}{h_1}\right)\frac{h_2}{h_1}$$ 3

b)



Absolute frame



Relative frame 3

$F_2 = \frac{6.5}{\sqrt{2g}} = 1.467$, $h_2 = 2$

Let $\frac{h_1}{h_2} = x$

$1.467^2 = \frac{1}{2} x(1 + x)$

Q cont'd

$$x^2 + x - 4.307 = 0$$

$$x = 1.6346$$

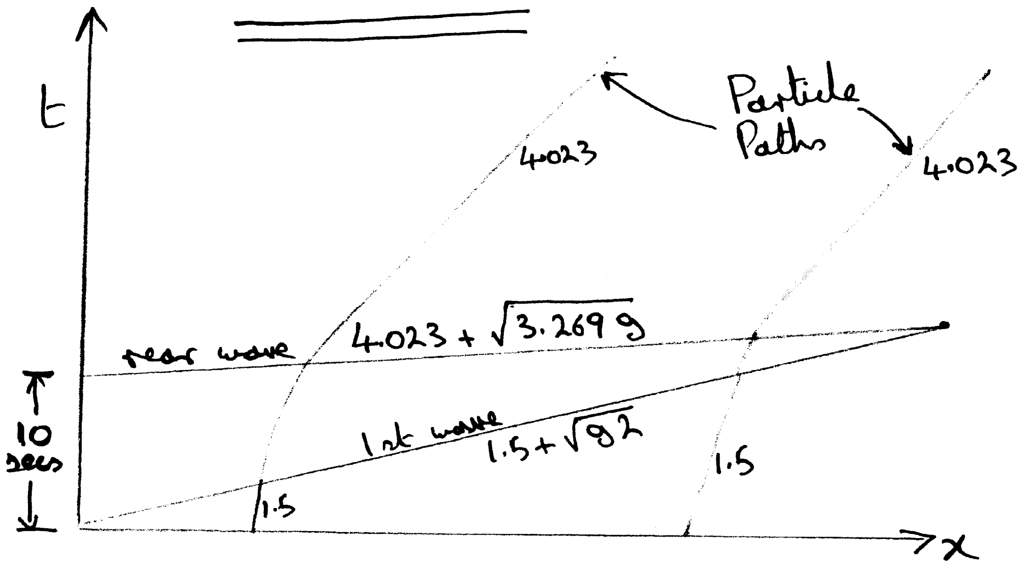
$$\rightarrow h_1 = 3.269 \text{ m}$$

$$V_1 h_1 = V_2 h_2 \rightarrow (8 - V_1) = \frac{6.5 \times 2}{3.269}$$

$$\rightarrow 8 - V_1 = 3.976$$

$$\underline{\underline{V_1 = 4.023}} \text{ m/s}$$

(c)



First wave moves at $1.5 + \sqrt{2g} = 5.929$ m/s

Rear wave moves at $4.023 + \sqrt{3.269g} = 9.686$ m/s.

The waves coalese when

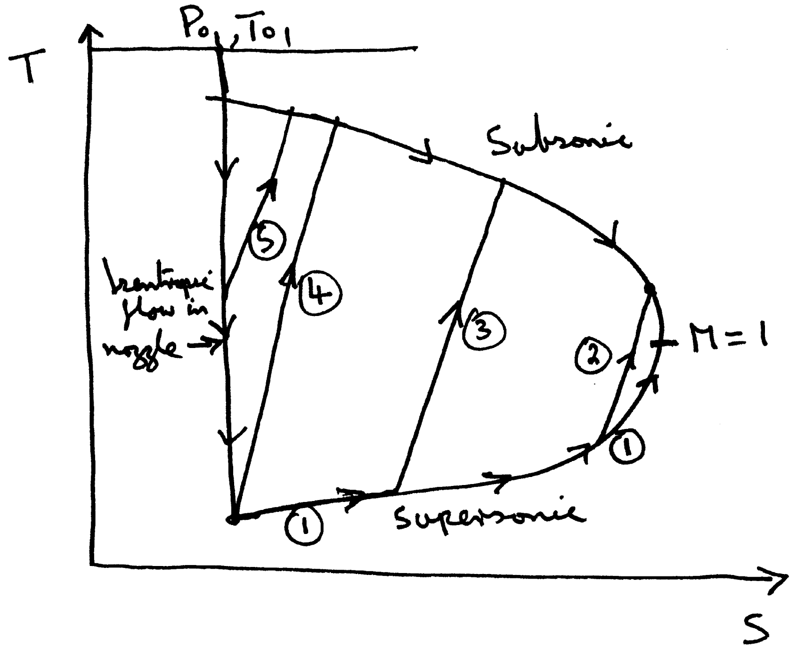
$$5.929\, t = 9.686(t - 10)$$

$$t = 25.78 \text{ secs.}$$

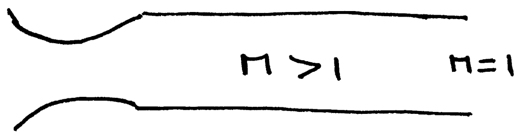
$$\underline{\underline{x = 152.8 \text{ m}}}$$

Q3

a)



1

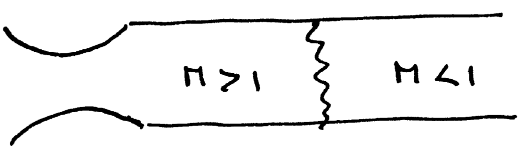


Low C_f

2

higher C_f
shock at exit

3

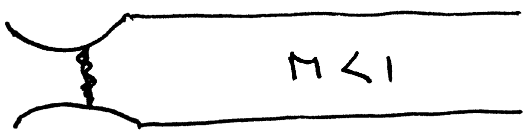


higher C_f
Shock in pipe

4

higher C_f
Shock at end of nozzle

5



high C_f
Shock in nozzle.

b) At exit $\dfrac{\dot{m}\sqrt{C_p T_0}}{A\,P} = 2.425$ and $P = 1$ bar.

At exit of nozzle $\dfrac{\dot{m}\sqrt{C_p T_0}}{A\,P_0} = 2.425 \times \dfrac{1}{2.725}$

$= \underline{.890}$

Q3 Cont'd

b) Tables → $M_2 = 1.8$ or $.454$ 3

→ $\frac{4fL_{max}}{D} = 0.242$ or 1.5

Since the pipe is choked at exit $L_{max} = 6.05$ m.

$\therefore\ f = \underline{\underline{.005}}$ or $\underline{\underline{0.031}}$ 3

c) After shock $p = 1.49$ bar.

So $\frac{\dot{m}\sqrt{C_pT_0}}{AP} = 2.425 \times \frac{1}{1.49} = 1.6275$

→ $M = 0.70$ after shock

→ $M_s = 1.50$ before shock.

Downstream of shock $\frac{4fL}{D} = .208$

$L = (6.05 - 2.03) = 4.02$

→ $\underline{f = .00647}$ 5

Upstream of shock $\frac{4fL}{D} = 0.136$

At exit of nozzle $\frac{4fL}{D} = .242$.

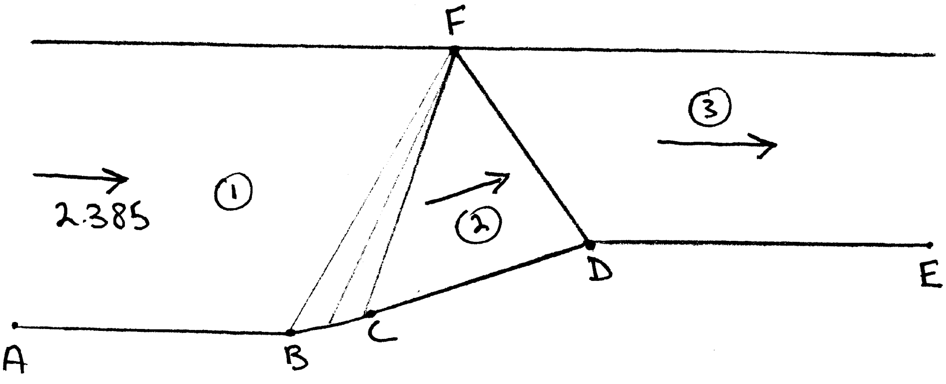
$\frac{4f\Delta L}{D} = .242 - .136 = .106$

and $\Delta L = 2.03$ m

→ $\underline{\underline{f = .00652}}$ 3

Values of f are effectively the same upstream and downstream of the shock.

Q4



a) The compression waves from BC are weak and isentropic and turn the flow by 10°

$\nu + \theta = \text{const}$

Tables → $\nu_1 = 36.28°$

$\therefore \quad \nu_2 = 26.28° \rightarrow M_2 = 2.0$ v closely

Across the shock FD flow turns by 10° from $M_2 = 2.0$.

Tables → $\underline{\underline{M_3 = 1.6395}}$ and $\frac{\Delta S}{C_v} = .0062$

$$\frac{\dot{m}\sqrt{C_pT_0}}{A_3\, P_{03}} = .9984 \quad \text{at } M_3 = 1.6395$$

$$\frac{\dot{m}\sqrt{C_pT_0}}{A_1\, P_{01}} = .5404 \quad \text{at } M_1 = 2.385.$$

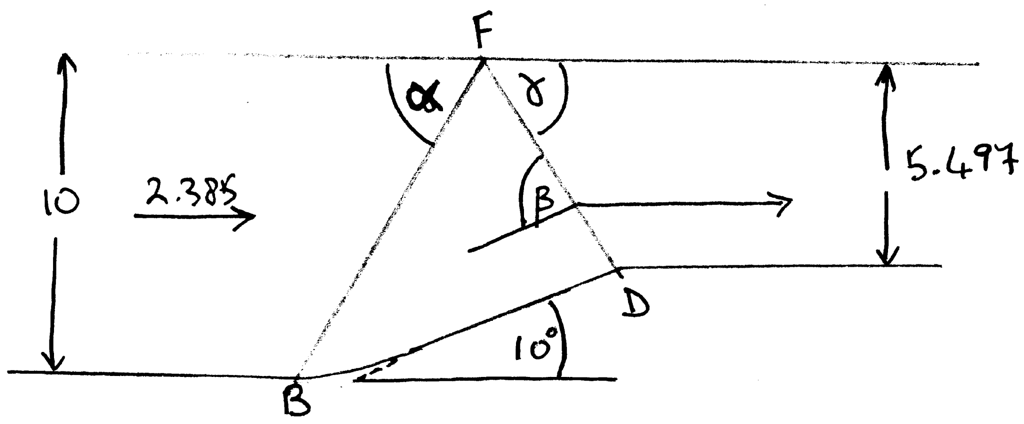
$$\therefore \quad \frac{A_3}{A_1} = \frac{P_{01}}{P_{03}} \times \frac{.5404}{.9984}$$

$$\Delta S = -R \ln \frac{P_{03}}{P_{01}} = .0062\, C_v$$

$$\rightarrow P_{03}/P_{01} = e^{-.0062\left(\frac{1}{\gamma-1}\right)} = .9846$$

$$\rightarrow \underline{\underline{A_3/A_1}} = \underline{\underline{0.5497}}$$

Q b)



Angle $\alpha = \sin^{-1}\frac{1}{M_1} = 24.81$

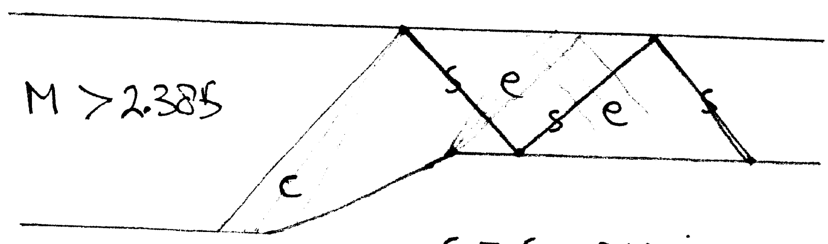
Angle $\beta = 39.33°$ from Tables.

$\therefore$ Angle $\gamma = 29.33°$.

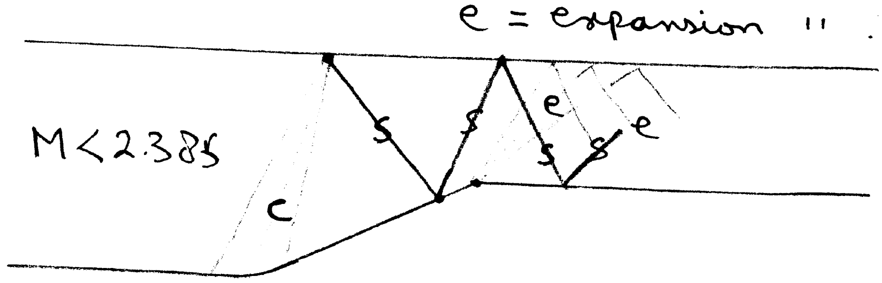
$BF = \left(\frac{10}{\tan\alpha}, 10\right) = (21.63, 10)$

$FD = \left(\frac{5.497}{\tan\gamma}, -5.497\right) = (9.784, -5.497)$

$\therefore$ $\underline{\underline{BD = (31.41, 4.503)}}$

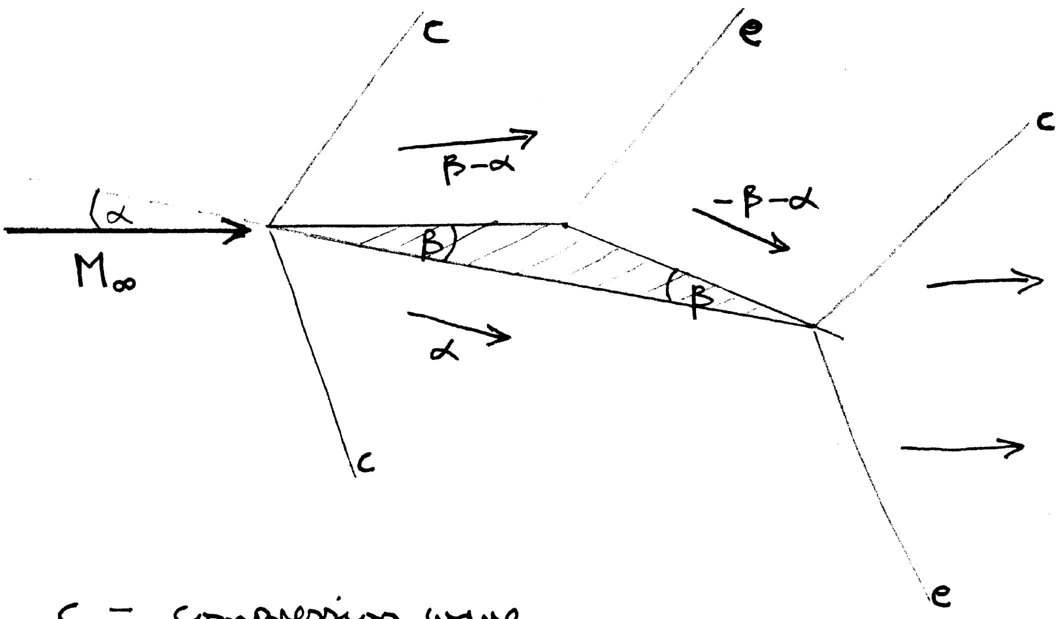


c = compression waves
s = shock "
e = expansion " .



1

Q5



c = compression wave
e = expansion wave

$$C_p = \pm \frac{2\theta}{\sqrt{M_\infty^2 - 1}} \qquad \begin{array}{l} + \text{ for compression} \\ - \text{ for expansion.} \end{array}$$

On lower surface $\quad C_p = \dfrac{+2\alpha}{\sqrt{M_\infty^2-1}}$

On front upper surface $\quad C_p = +\dfrac{2(\beta-\alpha)}{\sqrt{M_\infty^2-1}}$

On rear upper surface $\quad C_p = \dfrac{-2(\beta+\alpha)}{\sqrt{M_\infty^2-1}}$

Resolving perp. to lower surface of aerofoil

$$F_n = \frac{c}{\sqrt{M_\infty^2-1}}\left[2\alpha - (\beta-\alpha) + (\beta+\alpha)\right] \times \tfrac{1}{2}\rho_\infty V_\infty^2$$

$$= \frac{4\alpha c}{\sqrt{M_\infty^2-1}} \times \tfrac{1}{2}\rho_\infty V_\infty^2$$

Resolving parallel to the lower surface

$$F_D = \frac{2\frac{c}{2}\tan\beta}{\sqrt{M_\infty^2-1}}\left[(\beta-\alpha) + (\beta+\alpha)\right] \times \tfrac{1}{2}\rho_\infty V_\infty^2$$

Q
cont

$$\frac{F_D}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{2c\beta^2}{\sqrt{M_\infty^2-1}} \quad \text{for } \beta \text{ small.}$$

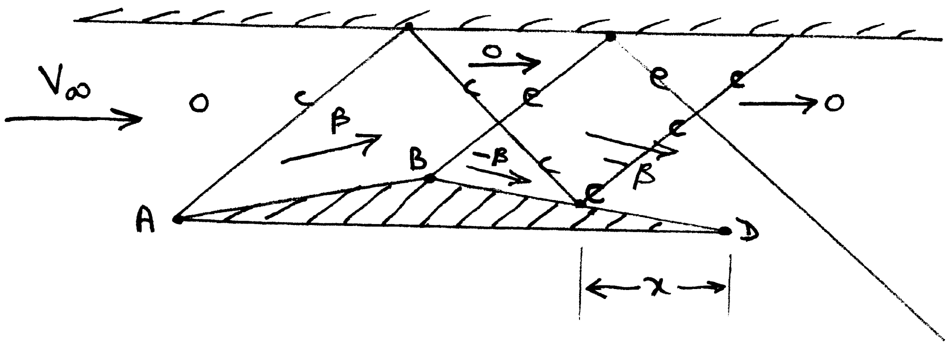
Resolving these parallel and perpendicular to the main stream

$$\frac{L}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{4\alpha c}{\sqrt{M_\infty^2-1}}\cos\alpha - \frac{2c\beta^2}{\sqrt{M_\infty^2-1}}\sin\alpha$$

$$\frac{D}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{2c\beta^2}{\sqrt{M_\infty^2-1}}\cos\alpha + \frac{4\alpha c}{\sqrt{M_\infty^2-1}}\sin\alpha$$

For small α

$$C_L = \frac{4\alpha}{\sqrt{M_\infty^2-1}}, \quad C_D = \frac{4\alpha^2+2\beta^2}{\sqrt{M_\infty^2-1}}$$



Along AB $\quad C_p = \dfrac{+2\beta}{\sqrt{M_\infty^2-1}}$

Along BC $\quad C_p = \dfrac{-2\beta}{\sqrt{M_\infty^2-1}}$

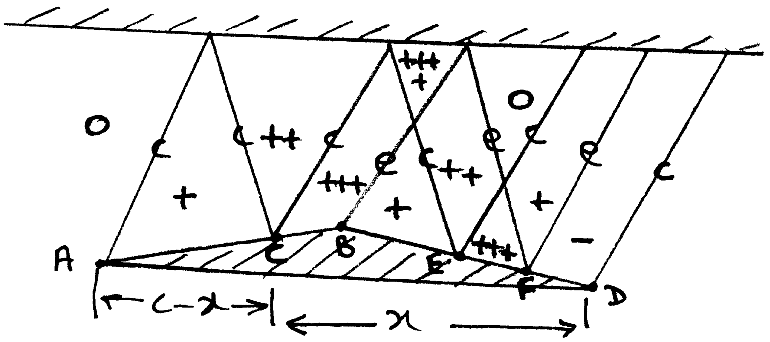
Along CD $\quad C_p = \dfrac{+2\beta}{\sqrt{M_\infty^2-1}}$

$$\therefore C_D = \frac{(c/2-x)\,4\beta}{c\sqrt{M_\infty^2-1}}\tan\beta$$

Q cont

For small β $\quad C_D = \frac{1}{\sqrt{M_\infty^2 - 1}}\beta^2\left(2 - \frac{4x}{c}\right)$ 9

When $x > c/2$ the wave pattern becomes.



Pressure increases are denoted by + and decreases by − where each +/− represents one multiple of $\frac{2\beta}{\sqrt{M_\infty^2 - 1}}$ on C_p.

The 2 regions of +++ cancel out.

Length $AC = (c - x)$

Length $BE = 2(c - x) - c/2 = (1.5c - 2x)$

Length $FD = \frac{c}{2} - (c - x) = (x - c/2)$

The drag is therefore proportional to

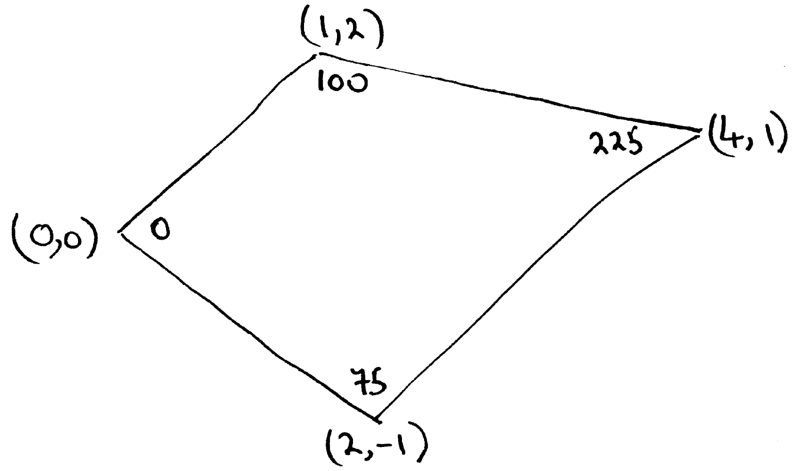
$(c - x) - (1.5c - 2x) + (x - c/2)$ 3

$= (-c + 2x)$

So the drag increases again with x when $x > c/2$.

Q6

d)



Gauss $\quad \dfrac{\partial\phi}{\partial x} = \dfrac{1}{A}\oint \phi\, dy\ , \quad \dfrac{\partial\phi}{\partial y} = \dfrac{-1}{A}\oint \phi\, dx$

$A =$ area of cell $= \frac{1}{2} \times$ vector product of diagonals

$$= \tfrac{1}{2}\,(4i + j)\times(-i + 3j)$$

$$= \tfrac{1}{2}\,(12 + 1) = 6.5$$

Assuming a linear variation of ϕ between the corners of the cell,

$$\oint \phi\, dy = \frac{75}{2}(-1) + \frac{75+225}{2}(2) + \frac{225+100}{2}(1) + \frac{100+0}{2}(-2)$$

$$= -37.5 + 300 + 162.5 - 100$$

$$= +325.$$

So $\dfrac{\partial\phi}{\partial x} = \dfrac{325}{6.5} = \underline{\underline{50\ m/s.}}$

$$\oint \phi\, dx = \frac{75+0}{2}(2) + \frac{75+225}{2}(2) + \frac{225+100}{2}(-3) + \frac{100+0}{2}(-1)$$

$$= 75 + 300 - 487.5 - 50 = -162.5$$

So $\dfrac{\partial\phi}{\partial y} = +\dfrac{162.5}{6.5} = \underline{\underline{25\ m/s.}}$

Q | Assuming a change $\Delta\phi$ at the (4,1) grid point,

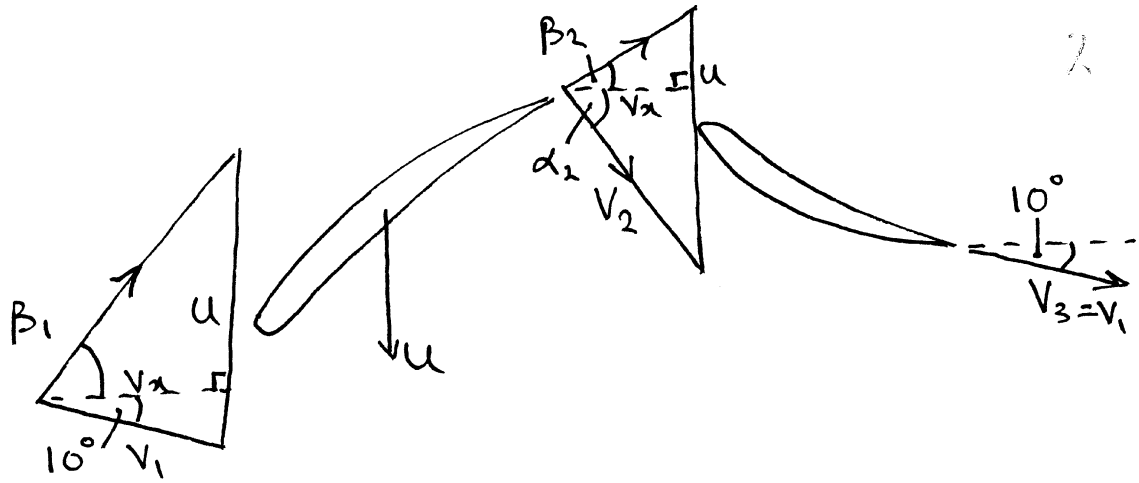
change in $\oint \phi \, dy = \frac{\Delta\phi}{2}(2+1) = 1.5\,\Delta\phi$

so change in $V_x = \frac{1.5}{6.5}\Delta\phi = \underline{\underline{.231\,\Delta\phi}}$

change in $\oint \phi \, dx = \frac{\Delta\phi}{2}(2-3) = -0.5\,\Delta\phi$

so change in $V_y = +\frac{0.5}{6.5}\Delta\phi = \underline{\underline{+.0769\,\Delta\phi}}$ 2

b)



2

(i) $\tan\beta_1 = \frac{-U + V_x \tan 10^\circ}{V_x} = -\frac{320}{144} + \tan 10^\circ$

$\rightarrow \underline{\underline{\beta_1 = -63.9^\circ}}$

$\therefore \underline{\underline{\beta_2 = -33.9^\circ}}$ (because 30° turning)

$\tan\alpha_2 = \frac{U - V_x \tan\beta_2}{V_x} = \frac{320}{144} - \tan 33.9^\circ$

$\rightarrow \alpha_2 = +\underline{\underline{57.15^\circ}}$

$\Delta h_{stage} = U\,\Delta V_\theta$

$= 320 \times 144\,(\tan 63.9^\circ - \tan 33.9^\circ)$

$= \underline{\underline{63.1 \text{ KJ/Kg}}}$

Q

(ii)

$$\Delta h_{stator} = \frac{1}{2}\left(V_2^2 - V_3^2\right)$$

$$= \frac{1}{2}\, 144^2 \left(\tan^2 57.15^\circ - \tan^2 10^\circ\right)$$

$$= 24.55 \text{ KJ/Kg.}$$

$$\text{Reaction} = 1 - \frac{\Delta h_{stat}}{\Delta h_{stage}}$$

$$= 1 - \frac{24.55}{63.1} = \underline{\underline{61.1\ \%}}$$

2

Proof – see end of Q.

Q7 a) Radial equilibrium $\frac{dV}{dy} = -\frac{V}{r_c}$

Approximate by $V_{i+1} = V_i - \frac{\Delta y V_i}{r_{c_i}}$

Taylor Series $V_{i+1} = V_i + \frac{\partial V}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 V}{\partial y^2}\Delta y^2$
$+ \frac{1}{6}\frac{\partial^3 V}{\partial y^3}\Delta y^3 + O(\Delta y)^4$

so $\frac{\partial V}{\partial y} = \frac{V_{i+1} - V_i}{\Delta y} - \frac{1}{2}\frac{\partial^2 V}{\partial y^2}\Delta y - \frac{1}{6}\frac{\partial^3 V}{\partial y^3}\Delta y^2$
$+ O(\Delta y)^3$

The approximation used is the first term on the RHS and so the largest term neglected is $O(\Delta y)$ – the method is 1st order accurate.

The truncation error is $\frac{1}{2}\frac{\partial^2 V}{\partial y^2}\Delta y$ which can be written as $-\frac{\partial V}{\partial y}\frac{1}{2r_c} = +\frac{V}{2r_c^2}\Delta y$ if r_c is constant.

b) Let $V_i = \bar{V}_i + V_i'$, where $\bar{V}_i$ is the correct value

$\rightarrow V_{i+1} = \bar{V}_i + V_i' - \frac{\bar{V}_i + V_i'}{r_{c_i}}\Delta y$.

$\rightarrow V_{i+1} = \left[\bar{V}_i - \frac{\bar{V}_i \Delta y}{r_{c_i}}\right] + V_i'\left[1 - \frac{\Delta y}{r_{c_i}}\right]$

so an error $V_i' \rightarrow$ an error $V_i'\left[1 - \frac{\Delta y}{r_c}\right]$ at point $i+1$.

Q
Cont

For stability the numerical value of the error at $i+1$ must be less than that at i, ie. $\left|1 - \frac{\Delta y}{r_c}\right| < 1$ 2

so $\frac{\Delta y}{r_c} < 2$ for stability 2

c) Use $\frac{V_{i+1} - V_i}{\Delta y} = -\frac{(V_i + V_{i+1})}{2r_c}$ because $r_c \simeq$ const

$$V_{i+1}\left(1 + \frac{\Delta y}{2r_c}\right) = V_i\left(1 - \frac{\Delta y}{2r_c}\right)$$

Let $\frac{\Delta y}{2r_c} = f$

$$V_{i+1} = V_i\left(\frac{1-f}{1+f}\right)$$

$$= V_i(1-f)(1 - f + f^2 - f^3 + \cdots)$$

$$= V_i(1 - f + f^2 - f^3 - f + f^2 - f^3 + \cdots)$$

$$= V_i(1 - 2f + 2f^2 - 2f^3 + \cdots)$$ 2

and $\frac{dV}{dy} = -\frac{V}{r_c} = -\frac{2fV}{\Delta y}$

$$\frac{d^2V}{dy^2} = +\frac{V}{r_c^2} = +\frac{4f^2}{\Delta y^2}\cdot V$$

$$\frac{d^3V}{dy^3} = -\frac{V}{r_c^3} = -\frac{8f^3}{\Delta y^3}\cdot V$$

Substituting for f, f^2, f^3 -- in the above series

Q7 cont

$$V_{i+1} = V_i + \frac{dV}{dy}\Delta y + \frac{1}{2}\frac{d^2V}{dy^2}\Delta y^2 + \frac{1}{4}\frac{d^3V}{dy^3}\Delta y^3 + \dots$$

Compare this with the Taylor series – the 1st two terms are the same and the error in the $\frac{d^3V}{dy^3}$ term is

$$\left(\frac{1}{6} - \frac{1}{4}\right)\frac{d^3V}{dy^3}\Delta y^3$$

So the method is second order accurate and the truncation error is $-\frac{1}{12}\frac{d^3V}{dy^3}\cdot\Delta y^2$

Q7

1st part – prove $\frac{dV}{dy} = -\frac{V}{r_c}$

Radial equilibrium $\frac{1}{\rho}\frac{\partial P}{\partial y} = +\frac{V^2}{r_c}$

$$Tds = dh - \frac{1}{\rho}dP = dh_o - VdV - \frac{1}{\rho}dP$$

$S = \text{const}$, $h_o = \text{const}$

$\rightarrow \frac{1}{\rho}dP = -VdV.$

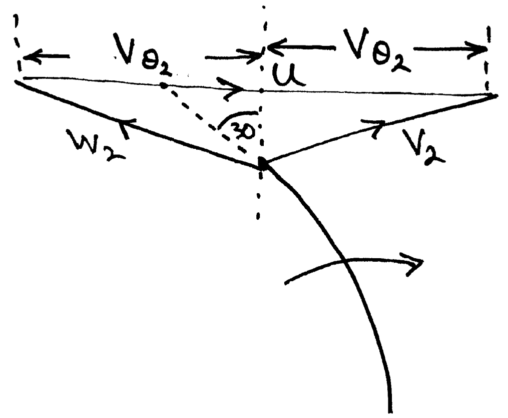
So R.E. eqn becomes $-V\frac{dV}{dy} = \frac{V^2}{r_c}$

or $\frac{dV}{dy} = -\frac{V}{r_c}.$

<u>Note</u> It is <u>not</u> correct to just quote the Euler eqn :- $dP = -\rho VdV$ as this is <u>along</u> a streamline and we need the pressure gradient $\perp$ to a streamline.

Q8

a)



(i) $V_{\theta_{2,I}} = V_{r_2}\tan 30^\circ + U_2$

$= -160\tan 30 + 380$

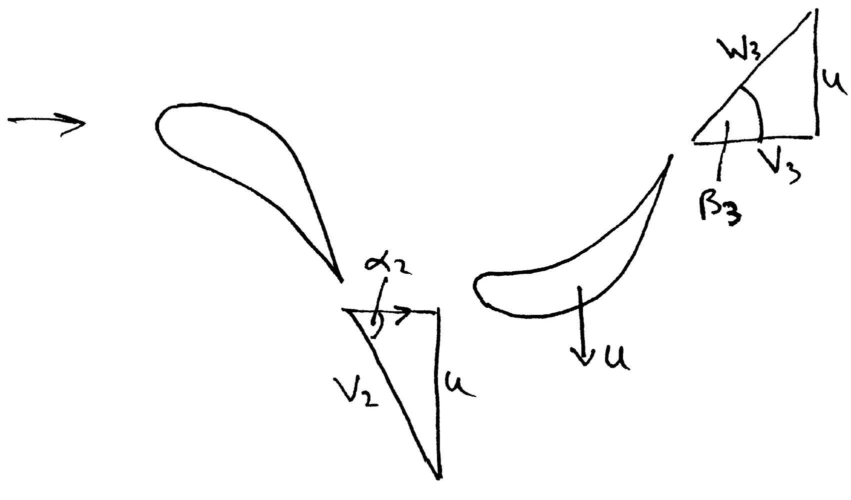
$= 287.6$ m/s.

$V_{\theta_2} = \sigma V_{\theta_{2,I}} = 0.88 \times 287.6 = \underline{\underline{253.1}}$ m/sec.

(ii) $\Delta h_0 = U_2 V_{\theta_2}$

Power $= \dot{m}\Delta h_0 = 2 \times 380 \times 253.1 = \underline{\underline{192.4 \text{ kW}}}$

b)



$\dot{m}\Delta h_0$ = same as compressor.

(i) $\rightarrow U_2 V_{\theta_2} = 192.4/2 \times 10^3$

$\rightarrow V_{\theta_2} = \underline{\underline{310.3}}$ m/sec,

$\rightarrow \alpha_2 = \tan^{-1}\dfrac{310.3}{95} = \underline{\underline{73^\circ}}$

$\Delta h_0/U^2 = \dfrac{V_{\theta_2}}{U} = \underline{\underline{1.0}}$

For M_2:

$V_2 = \sqrt{V_{\theta_2}^2 + V_x^2} = 324.5$ m/s

$\dfrac{V_2}{\sqrt{c_p T_{02}}} = .3086$

$\rightarrow \underline{\underline{M_2 = 0.5}}$

Q8 cont

$$T_{03} - T_{01} = \frac{\Delta h_0}{C_p} = \frac{192.4 \times 10^3}{2 \times 1005} = 95.72\,K.$$

(ii)

$$\Delta T_{0_{1S}} = \frac{1}{\eta_{TT}} \Delta T_0 = 112.61\,K.$$

$$T_{03_{1S}} = 1100 - 112.61 = 987.38\,K.$$

$$\frac{P_{03}}{P_{01}} = \left(\frac{T_{031S}}{T_{01}}\right)^{\frac{\gamma}{\gamma-1}} = 0.685$$

$$P_{01} = 4 \rightarrow P_{03} = \underline{\underline{2.741 \text{ bar.}}}$$

$$T_3 = T_{03} - \frac{1}{2} \times 95^2 / C_p = 999.79\,K$$

$$\frac{P_3}{P_{03}} = \left(\frac{T_3}{T_{03}}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{999.79}{1004.28}\right)^{35} = .9844$$

$$\rightarrow P_3 = \underline{\underline{2.698 \text{ bar.}}}$$

(c) See lecture notes.