ENGINEERING TRIPOS PART IIA 2004

Solutions to Module 3C3
Machine Design - Tribology
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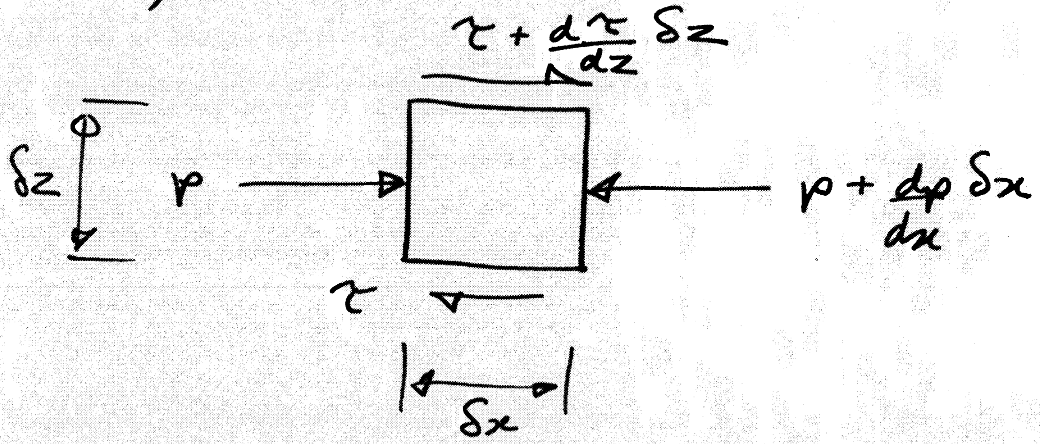
3C3 2004

1. a)



sum forces $\xrightarrow{+}$

$$p\,\delta z - \left(p + \frac{dp}{dx}\delta x\right)\delta z - \tau\,\delta x + \left(\tau + \frac{d\tau}{dz}\delta z\right)\delta x = 0$$

$$-\frac{dp}{dx}\delta x\,\delta z + \frac{d\tau}{dz}\delta z\,\delta x = 0$$

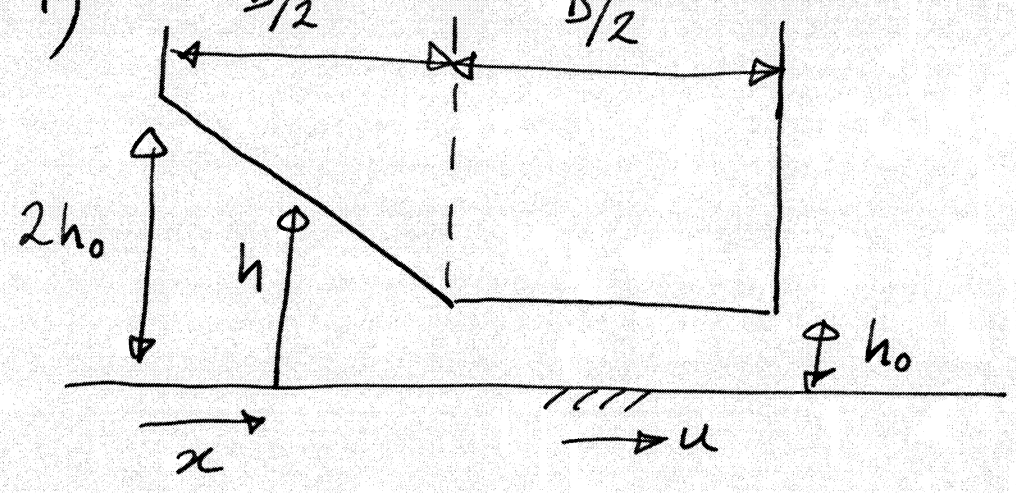
$$\frac{dp}{dx} = \frac{d\tau}{dz}$$

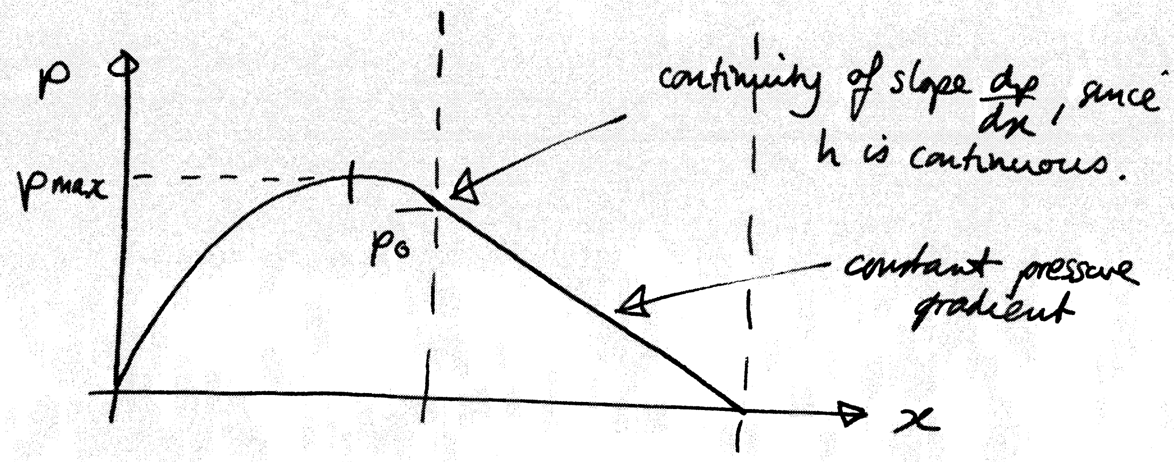
but given $\quad \tau = \eta\dot{\gamma} = \eta\frac{du}{dz}$

$$\therefore \frac{d\tau}{dz} = \eta\frac{d^2u}{dz^2}$$

hence $\quad \underline{\underline{\dfrac{dp}{dx} = \eta\dfrac{d^2u}{dz^2}}}$

b) i)





ii) Need to find h^*, to put in $\frac{Q}{L} = \bar{u} h^*$

$$\frac{dp}{dx} \text{ in second half is } -\frac{2p_0}{B} = 12\eta\,\bar{u}\,\frac{h_0 - h^*}{h_0^3} \quad (1)$$

In first half: $$\frac{dp}{dx} = 12\eta\,\bar{u}\,\frac{h-h^*}{h^3}$$

where $h = 2h_0 - 2x\frac{h_0}{B}$ $\quad \therefore \; dx = -dh\,\frac{B}{2h_0}$

$$\therefore \frac{dp}{dh} = -12\eta\,\bar{u}\,\frac{h-h^*}{h^3}\,\frac{B}{2h_0}$$

$$\int_0^{p_0} dp = \int_{2h_0}^{h_0} -12\eta\,\bar{u}\,\frac{h-h^*}{h^3}\,\frac{B}{2h_0}\,dh$$

$$p_0 = -12\eta \bar{u} \frac{B}{2h_0} \int_{2h_0}^{h_0} \frac{1}{h^2} - \frac{h^*}{h^3} \, dh$$

$$= -12\eta \bar{u} \frac{B}{2h_0} \left[-\frac{1}{h} + \frac{h^*}{2h^2} \right]_{2h_0}^{h_0}$$

$$= -12\eta \bar{u} \frac{B}{2h_0} \left[-\frac{1}{h_0} + \frac{h^*}{2h_0^2} + \frac{1}{2h_0} - \frac{h^*}{8h_0^2} \right] \quad (2)$$

Equating (1) and (2)

$$-12\eta \bar{u} \frac{B}{2} \frac{h_0 - h^*}{h_0^3} = -12\eta \bar{u} \frac{B}{2h_0} \left[\frac{-1}{h_0} + \frac{h^*}{2h_0^2} + \frac{1}{2h_0} - \frac{h^*}{8h_0^2} \right]$$

$$-\frac{1}{h_0^2} + \frac{h^*}{h_0^3} = -\frac{1}{2h_0^2} + \frac{h^*}{8h_0^3} + \frac{1}{h_0^2} - \frac{h^*}{2h_0^3}$$

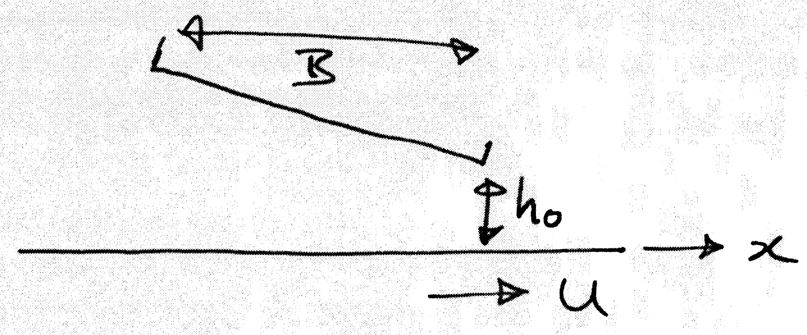
$$\frac{h^*}{h_0^3} \left(\frac{1}{h_0^3} - \frac{1}{8h_0^3} + \frac{1}{2h_0^3} \right) = \frac{1}{h_0^2} \left(-\frac{1}{2} + 1 + 1 \right)$$

$$\frac{h^*}{h_0} \quad \frac{8 - 1 + 4}{8} = \frac{3}{2}$$

$$\frac{h^*}{h_0} = \frac{3}{2} \cdot \frac{8}{11} = \frac{12}{11}$$

$$\therefore \quad \frac{Q}{L} = \bar{u} h^* = \frac{12 h_0}{11} \frac{U}{2} = \underline{\underline{\frac{6}{11} U h_0}}$$

2. a)



$$\frac{dp}{dx} = 12\eta\bar{u}\,\frac{h-h^*}{h^3}$$

$$\bar{u} = \frac{U}{2} \quad \therefore \quad \frac{dp}{dx} = 6\eta U \cdot \frac{1}{h_0^2}\left(\frac{\frac{h}{h_0} - \frac{h^*}{h_0}}{\frac{h^3}{h_0^3}}\right)$$

$$\therefore \quad \frac{h_0^2}{\eta U B}\,\frac{dp}{d\left(\frac{x}{B}\right)} = 6\left(\frac{\frac{h}{h_0} - \frac{h^*}{h_0}}{\frac{h^3}{h_0^3}}\right)$$

$$\therefore \quad \underline{\underline{P = \frac{p h_0^2}{\eta U B}}}$$

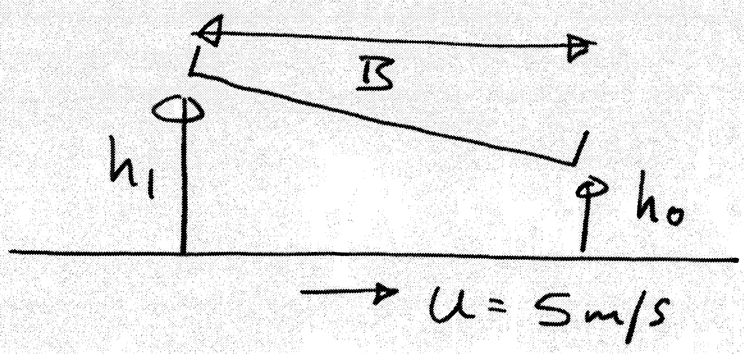
$$\frac{W}{L} = \int_0^B p\,dx = \frac{\eta U B}{h_0^2}\int_0^B P\,dx$$

let $\quad X = \frac{x}{B} \quad \rightarrow \quad dx = B\,dX$

$$\therefore \quad \frac{W}{L}\,\frac{h_0^2}{\eta U B} = \int_0^1 P \,.\, B\,dX$$

$$\underline{\underline{\frac{W}{L}\,\frac{h_0^2}{\eta U B^2} = \int_0^1 P\,dx = W^*}}$$

b)



$\eta = 0.2$ Pa s

$\bar{p} = \dfrac{W}{LB} = 5$ MPa

$h_0 = 15\,\mu m$

$L = B$

i) To minimise B at fixed W, η, U, maximise W^*

From table, W^* is max when $H = 2.2$

$$H = \frac{h_1}{h_0} = 2.2$$

$$\frac{h_1}{h_0} - \frac{h_0}{h_0} = 1.2$$

$$h_1 - h_0 = 1.2 h_0 = 1.2 \times 15\,\mu m = \underline{\underline{18\,\mu m}}$$

$$W^* = \frac{W h_0^2}{L B^2 \eta U} \qquad \therefore \quad B = \frac{W}{LB}\,\frac{h_0^2}{\eta U}\,\frac{1}{W^*}$$

$$= \bar{p}\,\frac{h_0^2}{\eta U}\,\frac{1}{W^*}$$

$$= \frac{5.10^6\,(15.10^{-6})^2}{0.2\,.\,5\,.\,0.07052}$$

$$\underline{\underline{B = 15.95\text{ mm}}}$$

ii) $h_1 - h_0$ as in (i), but h_0 varies, so H varies.

Define $\tilde{W} = \dfrac{W(h_1-h_0)^2}{LB^2\eta U} = W^*(H-1)^2$

$\eta \rightarrow 0.2\eta$

$\therefore$ $\tilde{W}$ increases by $\frac{1}{0.2} = 5$

H	2.2	3.5	4
W^*	0.07052	0.06339	0.05903
$\tilde{W} = W^*(H-1)^2$	0.10155	0.3962	0.53127

$\times 5$: 0.10155 → 0.50775

Interpolate to find $H = 3.913$
and $W^* = 0.0598$

Now $h_0 = \sqrt{\dfrac{W^* LB^2\eta U}{W}}$

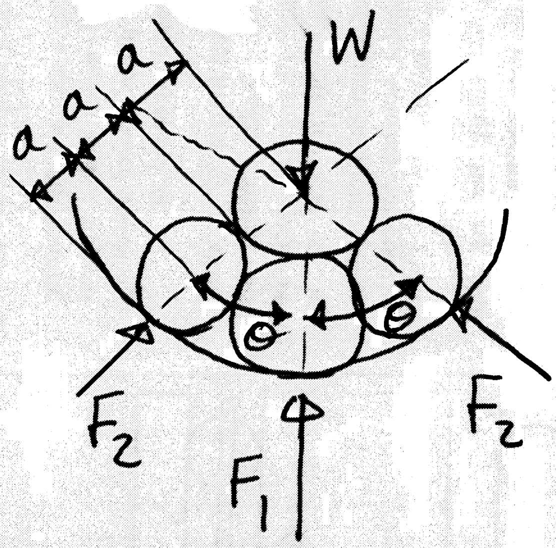
so $h_0 \propto \sqrt{W^*\eta}$

$$\frac{h_{0new}}{h_{0old}} = \sqrt{\frac{W^*_{new}\,\eta_{new}}{W^*_{old}\,\eta_{old}}}$$

$$= \sqrt{\frac{0.0598}{0.07052}\,0.2}$$

$$h_{0new} = 0.412\, h_{0old}$$

3 (a)



When $N=6$ and $a = \frac{R_o}{3}$

then $R_i = a$, $\theta = 60^\circ$

Three balls carry the load,

$$W = F_1 + 2F_2 \cos 60^\circ$$

assume contact force varies with $\cos\theta$, so

$$F_2 = F_1 \cos 60$$

$$\therefore W = F_1 + 2F_1 \cos^2 60$$

$$W = F_1 (1 + 2 \cdot 0.5^2)$$

$$\underline{\underline{F_1 = \frac{2}{3} W}}$$

if $W = \frac{N F_1}{4}$ and $N=6$, $W = \frac{6}{4} F_1$

$$\therefore \underline{\underline{F_1 = \frac{2}{3} W}}$$

(b) max contact pressure is between inner track and most heavily loaded ball.

Data sheet $\quad p_o = \left\{ \frac{W' E^*}{\pi R} \right\}^{\frac{1}{2}}$

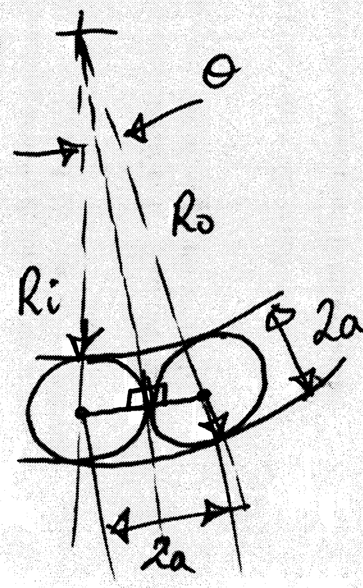
where $\quad W' = \frac{F_1}{b} = \frac{2}{3} \frac{W}{b}$

and $\frac{1}{R} = \frac{1}{R_i} + \frac{1}{a}$, where $R_i = a$ and $a = \frac{R_o}{3}$

so $R = \frac{R_o}{6}$

thus $$p_{(N=6)} = \left\{ \frac{2}{3} \frac{W}{b} \frac{E^*}{\pi} \frac{6}{R_o} \right\}^{\frac{1}{2}} = \underline{\underline{\sqrt{\frac{4 \, W E^*}{R_o b \pi}}}}$$

(c)



First, find general expression for F, without N. Consider geometry:

$$(R_o - a) \sin\frac{\theta}{2} = a$$

but $N = \frac{2\pi}{\theta}$, since a is as large as possible for given N

assume N is large $\rightarrow$ θ is small

so $\sin\frac{\theta}{2} \rightarrow \frac{\theta}{2}$

$$\therefore (R_o - a)\frac{\theta}{2} = a \quad , \quad \frac{\theta}{2} = \frac{a}{R_o - a} = \frac{\pi}{N}$$

from (a) $F_1 = \frac{1}{4} \frac{W}{N} = \frac{4W}{\pi} \frac{a}{R_o - a}$

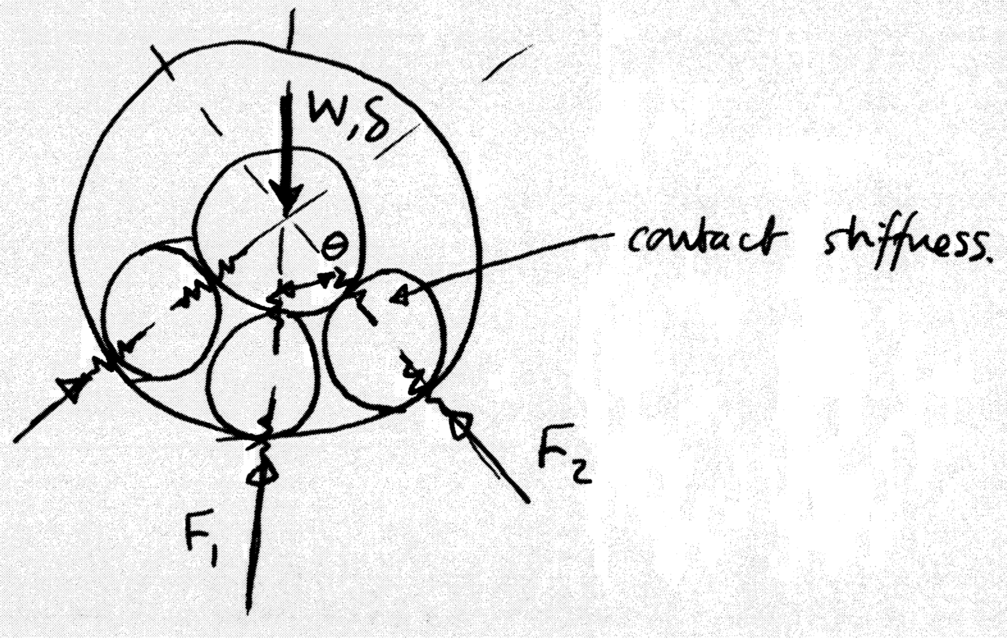
Substitute for F_1 into max contact pres. equation, with $\frac{1}{R} = \frac{1}{R_i} + \frac{1}{a} = \frac{a + R_i}{a R_i}$

$$p_o = \left\{ \frac{4W}{b\pi} \frac{a}{R_o - a} \frac{E^*}{\pi} \frac{a + R_i}{a R_i} \right\}^{\frac{1}{2}}$$

but $R_i + a = R_o - a$, and $R_i \simeq R_o$ when N is large,

$$p_o = \left\{ \frac{4W}{b\pi^2} \frac{E^*}{R_o} \right\}^{\frac{1}{2}} = \underline{\underline{p_{(N=6)} \sqrt{\frac{1}{\pi}}}}$$

d)

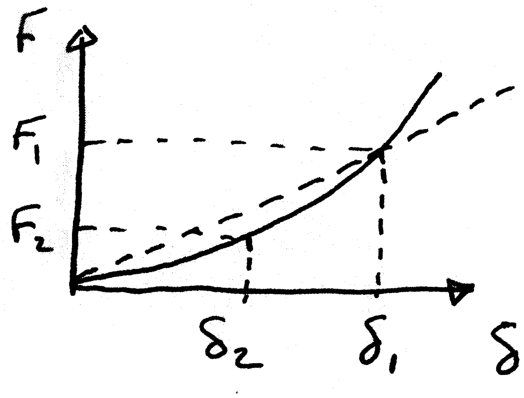


Contact forces are influenced by the contact stiffnesses. Displacement across each set of contact stiffnesses is $\delta\cos\theta$. Determine contact force using data sheet formulae:

$$\delta_i = \frac{2W'}{\pi}\left[\frac{1-\nu_1^2}{E_1}\left\{\ln\left(\frac{4R_1}{b}\right)-\frac{1}{2}\right\} + \frac{1-\nu_2^2}{E_2}\left\{\ln\left(\frac{4R_2}{b}\right)-\frac{1}{2}\right\}\right]$$

and $$b = 2\left\{\frac{W'R}{\pi E^*}\right\}^{\frac{1}{2}}$$

Thus relationship between contact force and contact displacement is best obtained numerically.

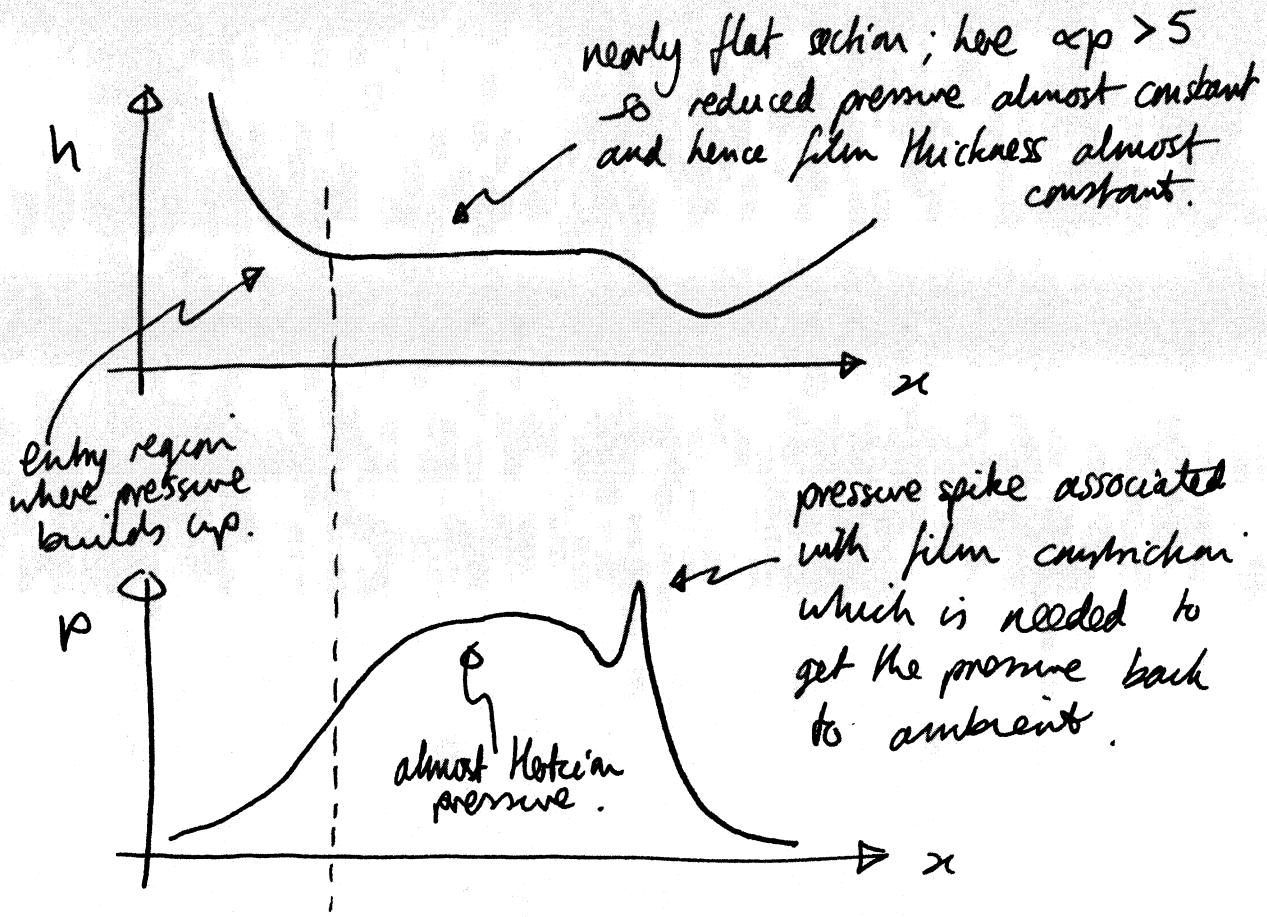


Expect a nonlinear relationship between F and δ, compared to linear assumption in part (a).

Then $W = F_1 + 2F_2\cos\theta$

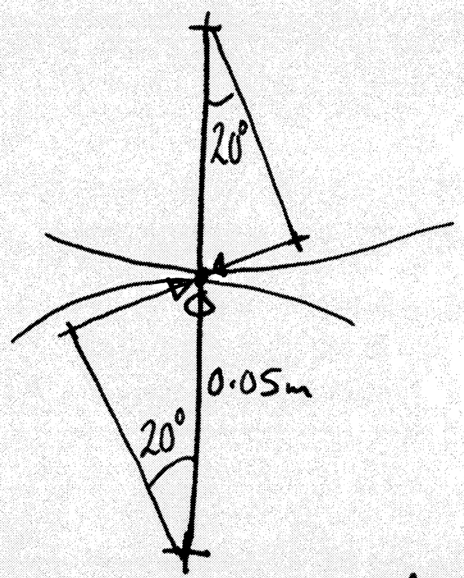
4 a)

- Need to consider elastic deformations of contacts associated with fluid pressure – this gives the deformed shape from the initial geometry.
- Include increase in viscosity with pressure e.g. $\eta = \eta_0 e^{\alpha p}$
- Then solve Reynolds' equation to find the variation in pressure through the contact.
- These form two sets of equations to be solved for p and h.



- Relevant to most lubricated Hertz-type contacts e.g. cams, gears, ball bearings.

b) i)



torque is 5Nm

pitch diameter is

$20 \times 5 = 100\,mm$

$\therefore$ contact force

is $\dfrac{5}{0.5 . 0.1 \cos 20} = 106.42\,N$

force per unit width

$$p' = \frac{106.42}{0.01} = 10.642\ kN/m$$

Hertz pressure for line contact

$$p_0 = \sqrt{\frac{p' E^*}{\pi R}}$$

where $R = \dfrac{0.05 . \sin 20}{2} = 0.00855\,m$

$$\therefore \quad p_0 = \sqrt{\frac{10642 . 115.10^9}{\pi \quad 0.00855}}$$

$$\underline{\underline{p_0 = 213.5\ MN/m^2}}$$

Stress is okay for steel gears.

c) $$\bar{h} = R \cdot 1.6 \left(2\alpha E^*\right)^{0.54} \left(\frac{\bar{u}\,\eta_0}{2E^* R}\right)^{0.7} \left(\frac{W}{2E^* RL}\right)^{-0.13}$$

$\eta_0 = 0.1$ Pa s

$\alpha = 2.10^{-8}$ m^2/N

$\bar{u} = 5$ m/s

$$\therefore \quad \bar{h} = 0.00855 . 1.6 \left(2.2.10^{-8}.115.10^{9}\right)^{0.54} \left(\frac{5.0.1}{2.115.10^{9}.0.00855}\right)^{0.7} \times \left(\frac{10642}{2.115.10^{9}.0.00855}\right)^{-0.13}$$

$$= 0.01368 \; . \; 95.036 \; . \; 192.10^{-9} \; . \; 4.838$$

$$\underline{\underline{\bar{h} = 1.207\,\mu m}}$$

This film thickness is of the order of a rough ground finish, so should be adequate to avoid lubrication problems if the teeth are more carefully ground.