# ENGINEERING TRIPOS PART IIA 2004

Solutions to Module 3C5
Dynamics
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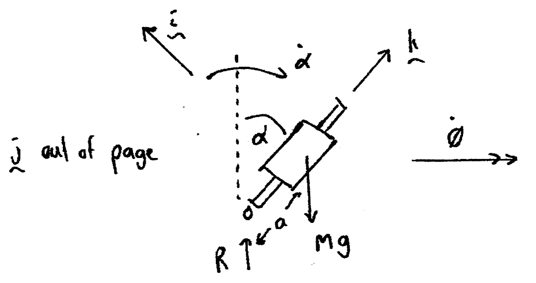
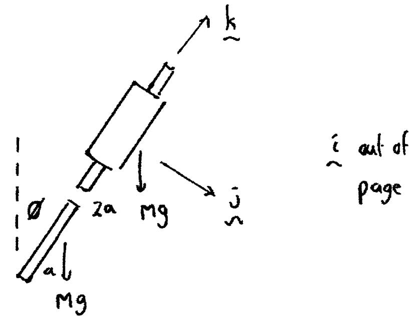
Second Assessor: Dr H E M Hunt

# 3C5 DYNAMICS 2004 CRIBS

a)




Side view




$Q_2 = -Mga\sin\alpha \simeq -Mga\alpha$ $\qquad$ $Q_1 = -Mga\sin\phi - 3Mga\sin\phi \simeq -4Mga\phi$

Now $\omega_2 = -\dot{\alpha}$ $\qquad$ $\omega_1 = -\dot{\phi}\cos\alpha \simeq -\dot{\phi}$ $\qquad$ [note $\alpha = \pi/2 - \theta$] $\qquad$ [40%]

b) Gyro equations

$$Q_1 = A\dot{\omega}_1 - (A\Omega_3 - C\omega_3)\omega_2$$
$$Q_2 = A\dot{\omega}_2 + (A\Omega_3 - C\omega_3)\omega_1$$
$$Q_3 = C\dot{\omega}_3$$

For fast spin and steady state : $Q_1 = CW\omega_2 \Rightarrow -4Mga\phi = -CW\dot{\alpha}$

$Q_2 = -CW\omega_1 \Rightarrow -Mga\alpha = CW\dot{\phi}$

$$\Rightarrow \underline{CW\dot{\alpha} - 4Mga\phi = 0} \qquad \underline{CW\dot{\phi} + Mga\alpha = 0}$$ [35%]

c) $CW\dot{\alpha} - 4Mga\phi = 0 \Rightarrow CW\ddot{\alpha} - 4Mga\dot{\phi} = 0$

$\Rightarrow CW\ddot{\alpha} - 4Mga(-Mga\alpha/CW) = 0$

$\Rightarrow \underline{\ddot{\alpha} + 4(Mga/CW)^2\alpha = 0}$

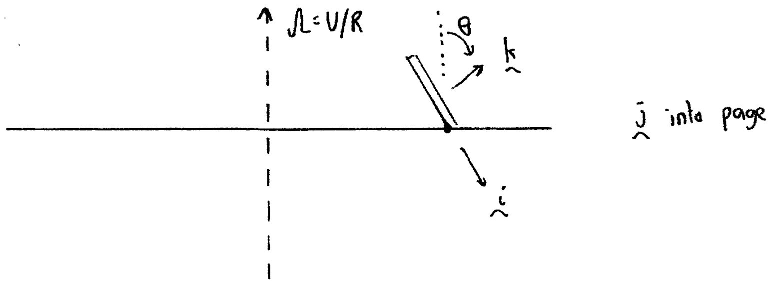
$$\text{SHM} \Rightarrow \underline{\omega_n = \frac{2Mga}{CW}}$$

Now $\dfrac{\dot{\alpha}}{\phi} = \dfrac{4Mga}{CW} \Rightarrow \omega_n\dfrac{\alpha}{\phi} = \dfrac{4Mga}{CW} \Rightarrow \dfrac{\alpha}{\phi} = \dfrac{4Mga}{CW}\,\dfrac{CW}{2Mga} = 2$

$\Rightarrow \underline{\alpha = 2\phi}$ $\qquad$ [25%]

Note $\dot{\alpha}_{max} = \omega_n\alpha_{max}$

2. a) The 'cone' is a red herring. Effectively the coin is rolling on a horizontal circular path, and it may as well be on a flat table top.

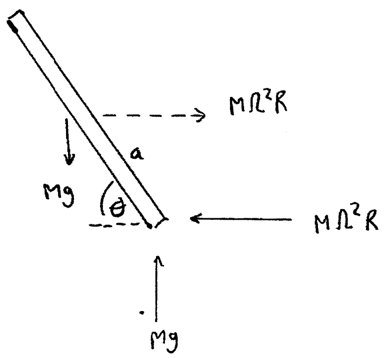



Force diagram:

Note that $\Omega_1 = -\Omega\sin\theta$

$\Omega_2 = \dot{\theta}$

$\Omega_3 = \Omega\cos\theta$




[25%]

b) $Q_2 = -Mga\cos\theta + MR\Omega^2 a\sin\theta$ [25%]

c) For no slip must have $V = -a\omega_3 \Rightarrow \omega_3 = -V/a = -R\Omega/a$ [25%]

d) Gyro equation $Q_2 = (A\Omega_3 - C\omega_3)\omega_1$ (steady state)

$\Rightarrow Q_2 = -(A\Omega\cos\theta + CR\Omega/a)\Omega\sin\theta$

For a disk, $A = \frac{1}{4}Ma^2$ and $C = \frac{1}{2}Ma^2$

small term since $a^2 << aR$

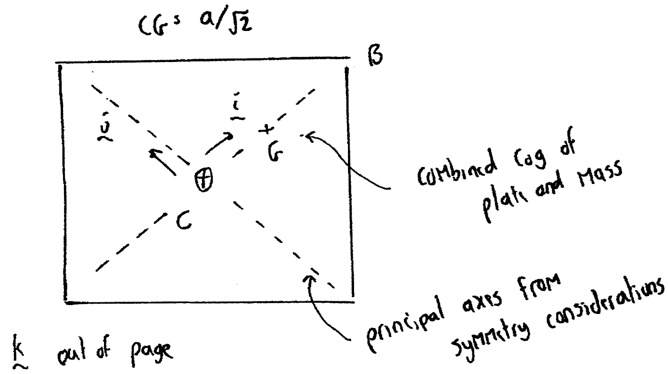
Thus $-Mga\cos\theta + MR\Omega^2 a\sin\theta = -\frac{1}{4}Ma^2\Omega^2\cos\theta\sin\theta - \frac{1}{2}MaR\Omega^2\sin\theta$

$\Rightarrow \frac{3}{2}MRa\Omega^2\sin\theta = Mga\cos\theta \Rightarrow \Omega^2 = \frac{2g}{3R\tan\theta}$

$\Rightarrow \underline{V = \sqrt{R^2\Omega^2} = \left(\frac{2gR}{3\tan\theta}\right)^{1/2}}$ [25%]

3 a)

Plate alone




Mass alone

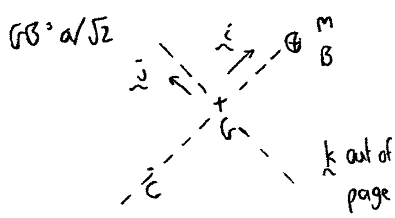



Plate is an AAC body. From the data book:

$A = \frac{1}{3}Ma^2 \quad C = \frac{2}{3}Ma^2$ about point C

Use // axis theorem to move to G:

$A_G = \frac{1}{3}Ma^2$

$B_G = \frac{1}{3}Ma^2 + M\left(\frac{a}{\sqrt{2}}\right)^2$

$C_G = \frac{2}{3}Ma^2 + M\left(\frac{a}{\sqrt{2}}\right)^2$

For the mass about G:

$A_M = 0$

$B_M = M\left(\frac{a}{\sqrt{2}}\right)^2$

$C_M = M\left(\frac{a}{\sqrt{2}}\right)^2$

add to give total:

$A = \frac{1}{3}Ma^2$

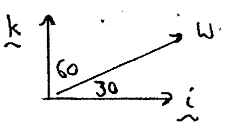
$B = \frac{1}{3}Ma^2 + \frac{1}{2}Ma^2 + \frac{1}{2}Ma^2 = \frac{4}{3}Ma^2$

$C = \frac{2}{3}Ma^2 + \frac{1}{2}Ma^2 + \frac{1}{2}Ma^2 = \frac{5}{3}Ma^2$

$$\Rightarrow \quad \underline{I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} \frac{Ma^2}{3}}$$

[35%]

b)




$\underset{\sim}{\omega} = (\cos 30\, \hat{i} + \sin 30\, \underset{\sim}{k})\,\omega$

$$\underset{\sim}{h} = I\underset{\sim}{\omega} = \frac{Ma^2}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \end{pmatrix}\omega/2 = \frac{Ma^2\omega}{6}\begin{pmatrix} \sqrt{3} \\ 0 \\ 5 \end{pmatrix}$$

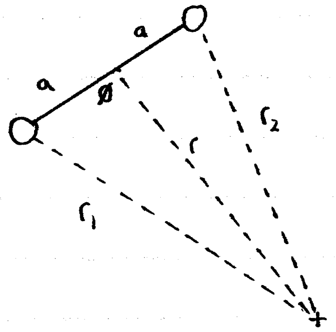
direction: $\underline{\tan^{-1}\frac{5}{\sqrt{3}}}$ to plane of plate [30%]

c) $\underset{\sim}{Q} = \underset{\sim}{h}' = \underset{\sim}{\Omega} \times \underset{\sim}{h} = \frac{\omega}{2}(\sqrt{3} \quad 0 \quad 1) \times \frac{\omega M a^2}{6}(\sqrt{3} \quad 0 \quad 5)$

$$= \left(0 \quad -\frac{\omega^2 M a^2}{\sqrt{3}} \quad 0\right)$$

Now $Q = Fb \Rightarrow \underline{|F| = \frac{\omega^2 M a^2}{\sqrt{3}\, b}}$ [35%]

4 a)




From the cosine rule:

$$r_1^2 = a^2 + r^2 - 2ar\cos\phi$$

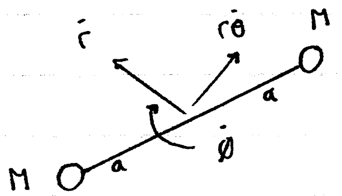
$$r_2^2 = a^2 + r^2 - 2ar\cos(\pi - \phi) = a^2 + r^2 + 2ar\cos\phi$$

$$\Rightarrow \underline{U = -GMM_e \left\{ \frac{1}{(a^2 + r^2 - 2ar\cos\phi)^{1/2}} + \frac{1}{(a^2 + r^2 + 2ar\cos\phi)^{1/2}} \right\}}$$

$M_e$ = mass of earth

[15%]

b) For the kinetic energy:




$$T = 2\tfrac{1}{2}M[\dot{r}^2 + (r\dot{\theta})^2] + \tfrac{1}{2}2Ma^2\dot{\phi}^2$$

velocity of centre of gravity  Note $I_G = 2Ma^2$

<u>Lagrange for $\theta$</u>

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}}\right] - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = 0$$

$$\underline{\frac{d}{dt}[2Mr^2\dot{\theta}] = 0}$$ conservation of angular momentum about A — (1)

<u>Lagrange for $r$</u>

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{r}}\right] - \frac{\partial T}{\partial r} + \frac{\partial U}{\partial r} = 0$$

$$\underline{\frac{d}{dt}[2M\dot{r}] - 2Mr\dot{\theta}^2 + GMM_e\left\{\frac{r - a\cos\phi}{r_1^3} + \frac{r + a\cos\phi}{r_2^3}\right\} = 0} \quad \text{— (2)}$$

<u>Lagrange for $\phi$</u>

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\phi}}\right] - \frac{\partial T}{\partial \phi} + \frac{\partial U}{\partial \phi} = 0$$

$$\underline{\frac{d}{dt}[2Ma^2\dot{\phi}] + GMM_e\left\{\frac{ar\sin\phi}{r_1^3} - \frac{ar\sin\phi}{r_2^3}\right\} = 0} \quad \text{— (3)}$$

[45%]

c) For $a \ll r$, in equation (2) we can replace $r_1$ and $r_2$ with $r$, and neglect the term dependent on a

Thus (2) $\Rightarrow$ $2M\ddot{r} - 2Mr\dot{\theta}^2 + 2GMM_e\left(\frac{1}{r^2}\right) = 0$

and for $\dot{r} = \ddot{r} = 0$ this gives $\dot{\theta}^2 = GM_e/r^3 \Rightarrow \underline{\dot{\theta} = \sqrt{GM_e/r^3}}$

Note also that equation (1) $\Rightarrow \dot{\theta}$ = constant, which is consistent with the above result. [20%]

d) For $\phi$ small, $\cos\phi \simeq 1$ $\Rightarrow$ $r_1^2 \simeq a^2+r^2-2ar = (r-a)^2$

$$r_2^2 \simeq a^2+r^2+2ar = (r+a)^2$$

Thus $\frac{1}{r_1^3} = \frac{1}{(r-a)^3} = \frac{1}{r^3}\left(1-\frac{a}{r}\right)^{-3} \simeq \frac{1}{r^3}\left(1+\frac{3a}{r}\right)$

Also $\frac{1}{r_2^3} = \frac{1}{(r+a)^3} = \frac{1}{r^3}\left(1+\frac{a}{r}\right)^{-3} \simeq \frac{1}{r^3}\left(1-\frac{3a}{r}\right)$

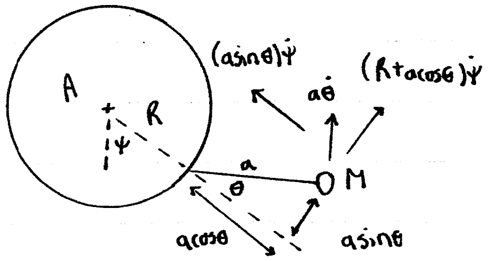
$$\frac{1}{r_1^3}-\frac{1}{r_2^3} \simeq \frac{6a}{r^4}$$

So equation (3) becomes : $2Ma^2\ddot{\phi} + \frac{6GMM_e}{r^3}a^2\phi = 0$

$$\Rightarrow \ddot{\phi} + \frac{3GM_e}{r^3}\phi = 0 \Rightarrow \omega_n = \sqrt{3}\left(\frac{GM_e}{r^3}\right)^{\frac{1}{2}} = \underline{\sqrt{3}\,\dot{\theta}}$$

[20%]

5 a) Consider the velocity of the mass:




Velocity components are : $(R+a\cos\theta)\dot\psi + a\dot\theta\cos\theta$

and : $(a\sin\theta)\dot\psi + a\dot\theta\sin\theta$

$$T = \tfrac{1}{2}M\left[\{R\dot\psi + (\dot\psi+\dot\theta)\,a\cos\theta\}^2 + \{(\dot\psi+\dot\theta)\,a\sin\theta\}^2\right] + \tfrac{1}{2}I\dot\psi^2$$

$$\underline{T = \tfrac{1}{2}M\left[R^2\dot\psi^2 + 2R\dot\psi(\dot\psi+\dot\theta)\,a\cos\theta + a^2(\dot\psi+\dot\theta)^2\right] + \tfrac{1}{2}I\dot\psi^2}$$ [20%]

b) For $\psi$: $\quad \dfrac{d}{dt}\left[\dfrac{\partial T}{\partial\dot\psi}\right] - \dfrac{\partial T}{\partial\psi} = G$

$$\Rightarrow \frac{d}{dt}\left[MR^2\dot\psi + MR(\dot\psi+\dot\theta)a\cos\theta + MR\dot\psi\, a\cos\theta + Ma^2(\dot\psi+\dot\theta) + I\dot\psi\right] = G$$

$$\Rightarrow \underline{\frac{d}{dt}\left[(I + MR^2 + Ma^2 + 2MRa\cos\theta)\dot\psi + (Ma^2 + MRa\cos\theta)\dot\theta\right] = G} \quad \text{—— } (*)$$

For $\theta$: $\quad \dfrac{d}{dt}\left[\dfrac{\partial T}{\partial\dot\theta}\right] - \dfrac{\partial T}{\partial\theta} = 0$

$$\Rightarrow \frac{d}{dt}\left[MRa\dot\psi\cos\theta + Ma^2(\dot\psi+\dot\theta)\right] + MRa\,\dot\psi(\dot\theta+\dot\psi)\sin\theta = 0$$

$$\Rightarrow MRa\ddot\psi\cos\theta - MRa\dot\psi\dot\theta\sin\theta + Ma^2(\ddot\psi+\ddot\theta) + MRa\dot\psi\dot\theta\sin\theta + MRa\dot\psi^2\sin\theta = 0$$

$$\underline{Ma^2(\ddot\psi+\ddot\theta) + MRa\ddot\psi\cos\theta + MRa\dot\psi^2\sin\theta = 0}$$ [35%]

c) From the diagram above, angular momentum about A is given by:

$$\text{Mom} = I\dot\psi + \left[(R+a\cos\theta)\dot\psi + a\dot\theta\cos\theta\right]\left[R+a\cos\theta\right]M + \left[(a\sin\theta)\dot\psi + a\dot\theta\sin\theta\right]Ma\sin\theta$$

$$\underline{\text{Mom} = I\dot\psi + MR^2\dot\psi + 2MRa\dot\psi\cos\theta + MRa\dot\theta\cos\theta + Ma^2\dot\psi + Ma^2\dot\theta}$$ [20%]

$(*) \Rightarrow \dfrac{d}{dt}[\text{Mom}] = 0$, conservation of angular momentum

d) From the $\theta$ equation of motion:

$$Ma^2(\underset{\downarrow}{\underset{0}{\ddot{\psi}}} + \ddot{\theta}) + MRa\underset{\downarrow}{\underset{0}{\ddot{\psi}}}\cos\theta + MRa\underset{\downarrow}{\underset{\Omega^2}{\dot{\psi}^2}}\underset{\downarrow}{\underset{\theta}{\sin\theta}} = 0$$

$$Ma^2\ddot{\theta} + MRa\Omega^2\theta = 0$$

$$\Rightarrow \quad \omega_n^2 = \frac{MRa\Omega^2}{Ma^2} \quad \Rightarrow \quad \underline{\omega_n = \Omega\sqrt{\frac{R}{a}}}$$ [25%]