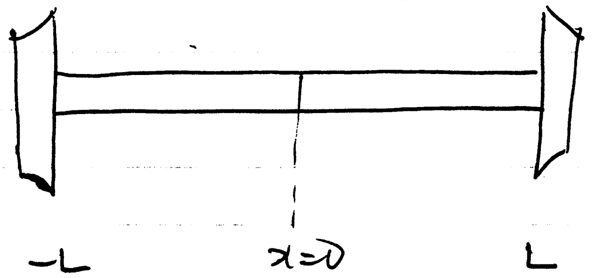
ENGINEERING TRIPOS PART IIA 2004

Solutions to Module 3C6
Vibration
Principal Assessor: Dr D Cebon
Second Assessor: Professor J Woodhouse

1. (a)



Rotational displacement of shaft $\theta(x,t)$.
Equation of motion for shaft $G\frac{\partial^2\theta}{\partial x^2} = \rho\frac{\partial^2\theta}{\partial t^2}$

Boundary condition at $x = \pm L$:
$$GJ\frac{\partial\theta}{\partial x} = -K\frac{\partial^2\theta}{\partial t^2}, \quad J = \frac{\pi a^4}{2} \text{ (data sheet)}$$

[15%]

(b) For a mode, $\theta = u(x)e^{i\omega t}$.
So $G\frac{d^2u}{dx^2} = -\rho\omega^2 u$, so $u = \left.\begin{matrix}\sin\\ \cos\end{matrix}\right) kx$, $k = \omega\sqrt{\frac{\rho}{G}}$

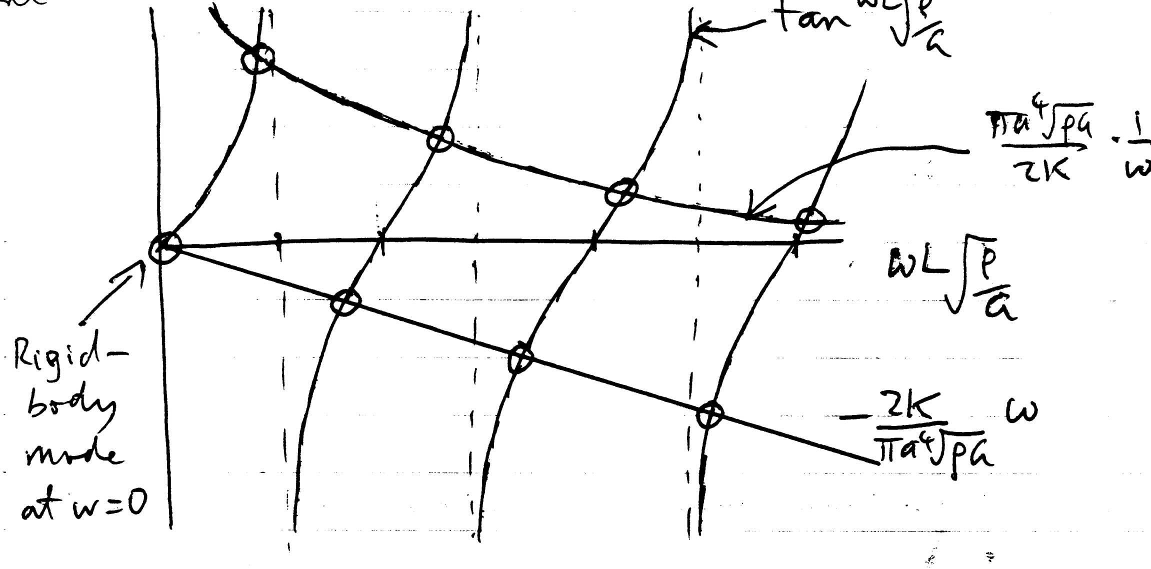
$\sin kx$ gives antisymmetric modes, $\cos kx$ symmetric.
Substitute in boundary conditions:
$\sin kx \rightarrow GJk\cos kL = K\omega^2\sin kL$ (antisym)
$\cos kx \rightarrow -GJk\sin kL = K\omega^2\cos kL$ (symm)

$$\therefore \tan kL = \tan\omega L\sqrt{\frac{\rho}{G}} = \begin{cases}\dfrac{kGJ}{K\omega^2} & \text{(anti)}\\[2ex] -\dfrac{K\omega^2}{GJk} & \text{(sym)}\end{cases}$$

But $\frac{GJk}{K\omega^2} = \frac{G}{K\omega^2}\frac{\pi a^4}{2}\omega\sqrt{\frac{\rho}{G}} = \frac{\pi a^4}{2}\frac{\sqrt{G\rho}}{K\omega}$ [30%]

(c) To solve, plot $\tan\omega L\sqrt{\frac{\rho}{G}}$ against ω, and overlay the two right-hand side functions.

1 cont



[30%]

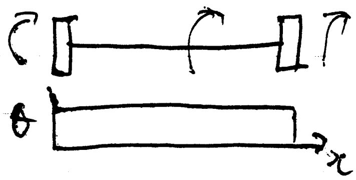
(d) (i) As $K \to 0$ the intersections on this graph move towards
$$\begin{cases} \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ \dots & \text{(anti)} \\ 0,\ \pi,\ 2\pi,\ 3\pi,\ \dots & \text{(sym)} \end{cases}$$

These are the solutions for a free-free shaft, with $\partial\theta/\partial x = 0$ at both ends

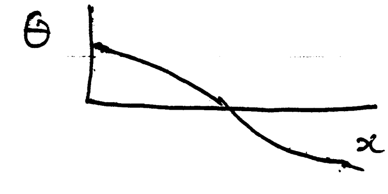
(ii) As $K \to \infty$, the intersection become
$$\begin{cases} 0,\ \pi,\ 2\pi,\ 3\pi & \text{(anti)} \\ 0,\ \pi/2,\ 3\pi/2,\ \dots & \text{(sym)} \end{cases}$$

These are fixed-fixed resonant frequencies of the shaft, plus a low-frequency antisymmetric mode in which the shaft twists quasi-statically and the two wheels rotate in opposite directions.

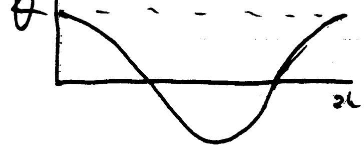
Mode shapes : $w=0$ Rigid rotation



Second mode is antisymmetric :



3rd mode symmetric



[25%]

2. (a) At a pinned end of a beam, $w = 0$ and $\frac{\partial^2 w}{\partial x^2} = 0$.

For a mode, $w = u(x) e^{i\omega t}$.
So need $\frac{d^4 u}{dx^4} = \alpha^4 u$, $\alpha^4 = \frac{12 \rho\, \omega^2}{E h^2}$

So $u = K_1 \cos\alpha x + K_2 \sin\alpha x + K_3 \cosh\alpha x + K_4 \sinh\alpha x$

$$\begin{cases} u(0) = 0 \rightarrow K_1 + K_3 = 0 \\ u''(0) = 0 \rightarrow K_1 - K_3 = 0 \end{cases}$$

$\therefore K_1 = K_3 = 0$

$$\begin{cases} u(L) = 0 \rightarrow K_2 \sin\alpha L + K_4 \sinh\alpha L = 0 \\ u''(L) = 0 \rightarrow -K_2 \sin\alpha L + K_4 \sinh\alpha L = 0 \end{cases}$$

So either (i) $K_2 = 0$ and $\sinh\alpha L = 0$

or (ii) $K_4 = 0$ and $\sin\alpha L = 0$

(i) is not possible except at $\omega = 0$.

So must be (ii), so mode shapes $u = K_2 \sin\alpha x$

where $\alpha L = n\pi$, i.e. $\omega_n^2 = \left(\frac{n\pi}{L}\right)^4 \frac{E h^2}{12\rho}$. [35%]

(b) To use Rayleigh's principle, need a reasonable estimate of mode shapes of the modified system. So use mode shapes of the original system, and evaluate the Rayleigh quotient using these old mode shapes in the new expressions for potential and kinetic energy. [15%]

(c) Use approximate mode shape from (a): $u_n = \sin\frac{n\pi x}{L}$.

$$\text{So } \omega_n^2 \simeq \frac{\frac{E}{24}\int_0^L (h_0 + \Delta h)^3 \left(\frac{n\pi}{L}\right)^4 \sin^2\frac{n\pi x}{L}\, dx}{\frac{\rho}{2}\int_0^L (h_0 + \Delta h) \sin^2\frac{n\pi x}{L}\, dx}$$

$$\simeq \frac{E}{12\rho}\left(\frac{n\pi}{L}\right)^4 \frac{\int (h_0^3 + 3h_0^2 \Delta h) \sin^2\frac{n\pi x}{L}\, dx}{\int (h_0 + \Delta h) \sin^2\frac{n\pi x}{L}\, dx}$$

2 cont

$$= \frac{E}{12\rho}\left(\frac{n\pi}{L}\right)^4 \frac{h_0^3 \, L/2 + 3h_0^2 \int \Delta h \sin^2\frac{n\pi x}{L}\,dx}{h_0 \, L/2 + \int \Delta h \sin^2 \frac{n\pi x}{L}\,dx}$$

$$\simeq \frac{Eh_0^2}{12\rho}\left(\frac{n\pi}{L}\right)^4\left[1 + \frac{6}{h_0 L}\int \Delta h \sin^2\frac{n\pi x}{L}\,dx\right]\left[1 - \frac{2}{h_0 L}\int \Delta h \sin^2 \frac{n\pi x}{L}\,dx\right]$$

(binomial)

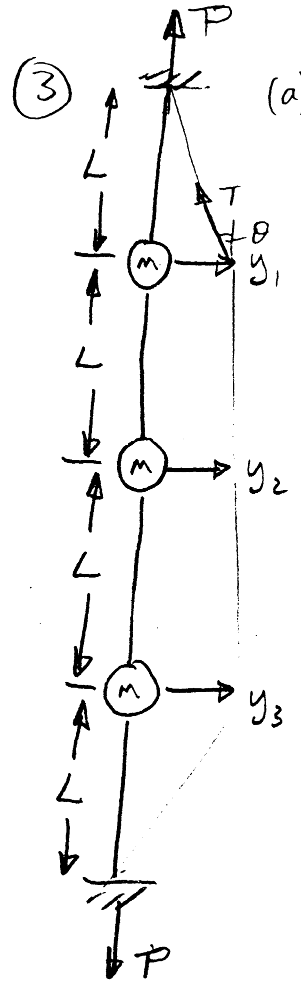
$$\simeq \frac{Eh_0^2}{12\rho}\left(\frac{n\pi}{L}\right)^4\left[1 + \frac{4}{h_0 L}\int_0^L \Delta h \sin^2\frac{n\pi x}{L}\,dx\right]$$ [35%]

(d) Relative to the centre of the beam, $\sin^2\frac{n\pi x}{L}$ is always an even function while $\Delta h = \varepsilon x$ is an odd function. So provided h_0 is replaced by the centre value $h_0 + \frac{\varepsilon L}{2}$, then the term

$\int \Delta h \sin^2 \frac{n\pi x}{L}\,dx$ gives zero contribution.

So a linearly tapered beam has the same natural frequencies (in this approximation) to a uniform beam of depth $h_0 + \frac{\varepsilon L}{2}$. [15%]

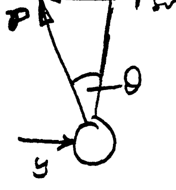
(3)



(a) KE $\quad T = \frac{1}{2} m \dot{y}_1^2 + \frac{1}{2} m \dot{y}_2^2 + \frac{1}{2} m \dot{y}_3^2$

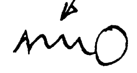
$$\Rightarrow [M] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

Consider a string



$P\sin\theta \simeq \frac{Py}{L} = \frac{P}{L} y$

$\Rightarrow$



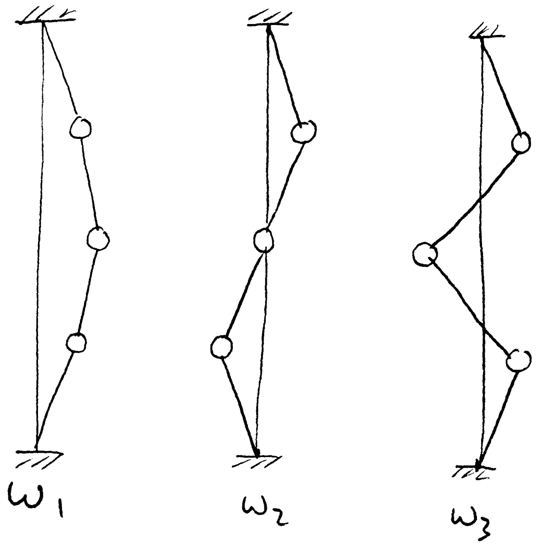
PE $\quad \therefore V = \frac{1}{2}\left(\frac{P}{L}\right) y_1^2 + \frac{1}{2}\left(\frac{P}{L}\right)(y_2 - y_1)^2 + \frac{1}{2}\left(\frac{P}{L}\right)(y_3 - y_2)^2 + \frac{1}{2}\left(\frac{P}{L}\right) y_3^2$

$$= \frac{1}{2}\left(\frac{P}{L}\right)\left[2y_1^2 + 2y_2^2 + 2y_3^2 - 2y_1 y_2 - 2y_2 y_3\right]$$

$$\therefore [K] = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{where } k = P/L$$

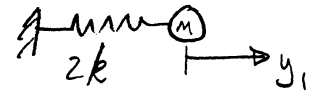
[25%]

(b) Mode shapes



Mode 2 — y_2 is zero

$\therefore$ equivalent system is



$$\Rightarrow \omega_2^2 = 2k/m = 2P/Lm$$

Eigenvalue problem is $\left([K] - \omega^2 [M]\right) \underset{\sim}{u} = 0$

$$\Rightarrow \begin{bmatrix} (2-\lambda^2) & -1 & 0 \\ -1 & (2-\lambda^2) & -1 \\ 0 & -1 & (2-\lambda^2) \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = 0 \quad \text{with } \lambda^2 = \omega^2 \frac{m}{k}$$

$$\Rightarrow (2-\lambda^2)\left[(2-\lambda^2)^2 - 1\right] + \left[-(2-\lambda^2)\right] = 0$$

③ Cont $\Rightarrow \quad (2-\lambda^2)(\lambda^4-4\lambda^2+2)=0$

$\therefore \quad \lambda_2^2=2 \quad \& \quad \lambda_{1,3}^2 = 2 \pm \sqrt{2}$

$\therefore$ Lowest natural freq is $\lambda^2 = 2-\sqrt{2}$ ie $\omega_1^2 = (2-\sqrt{2})\frac{P}{Lm}$

ie $\omega_1 = 0.765\sqrt{P/Lm}$

[40%]

(c) Rayleigh $\quad \omega_1^2 \leq \frac{V_{max}}{T^*} = \frac{\frac{1}{2}\left(\frac{P}{L}\right)\left[2y_1^2+2y_2^2+2y_3^2-2y_1y_2-2y_2y_3\right]}{\frac{1}{2}m\left[y_1^2+1.2y_2^2+y_3^2\right]}$

Assume mode shape is $[1 \;\alpha\; 1]^T$

$$\Rightarrow \quad R = \frac{2P}{Lm}\left(\frac{\alpha^2-2\alpha+2}{2+1.2\alpha^2}\right)$$

To find exact ω_1 set $\frac{dR}{d\alpha}=0$

$$\Rightarrow \quad (2+1.2\alpha^2)(2\alpha-2)-(\alpha^2-2\alpha+2)(2.4\alpha)=0$$

$$\Rightarrow \quad 2.4\alpha^2-0.8\alpha-4=0$$

$$\Rightarrow \quad \alpha = \frac{0.8 \pm 6.248}{4.8} = \underset{\text{lowest mode}}{1.468}, \; -1.135$$

$$\therefore \quad \omega_1^2 = \frac{2P}{Lm}\left(\frac{1.468^2-2(1.468)+2}{2+1.2(1.468)^2}\right) = 0.532\frac{P}{Lm}$$

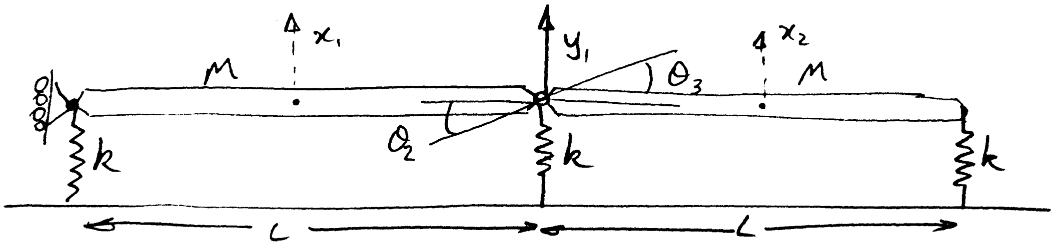
$$\therefore \quad \omega_1 = 0.729\sqrt{P/Lm}$$

ie 4.7% decrease in 1st natural frequency

[35%]

[Alternatively determine (or guess) first mode of original system $\underline{u}^{(1)} = \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix}$ and use Rayleigh with this mode shape & modified KE to account for $1.2M_2$]

(4)



(a) KE $\quad T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} I \dot{\theta}_2^2 + \frac{1}{2} m \dot{x}_2^2 + \frac{1}{2} I \dot{\theta}_3^2$

$I = \frac{mL^2}{12}$; $\quad x_1 = y_1 - L/2\,\theta_2$; $\quad x_2 = y_1 + L/2\,\theta_3$

$\therefore\ T = \frac{1}{2} m (\dot{y}_1 - L/2\,\dot{\theta}_2)^2 + \frac{1}{24} mL^2 \dot{\theta}_2^2 + \frac{1}{2} m (\dot{y}_1 + \frac{L}{2}\dot{\theta}_3)^2 + \frac{1}{24} mL^2 \dot{\theta}_3^2$

$= \frac{1}{2} m \left(2\dot{y}_1^2 + \frac{L^2}{3}\dot{\theta}_2^2 + \frac{L^2}{3}\dot{\theta}_3^2 - L\dot{y}_1\dot{\theta}_2 + L\dot{y}_1\dot{\theta}_3 \right)$

$$\therefore [M] = m \begin{array}{c} \begin{bmatrix} 2 & -L/2 & L/2 \\ -L/2 & L^2/3 & 0 \\ L/2 & 0 & L^2/3 \end{bmatrix} \end{array} \begin{array}{c} y_1 \\ \theta_2 \\ \theta_3 \end{array}$$

PE $\quad V = \frac{1}{2} k y_1^2 + \frac{1}{2} k (y_1 - L\theta_2)^2 + \frac{1}{2} k (y_1 + L\theta_3)^2$

$= \frac{1}{2} k \left[3y_1^2 + L^2\theta_2^2 + L^2\theta_3^2 - 2Ly_1\theta_2 + 2Ly_1\theta_3 \right]$

$$\therefore [K] = k \begin{bmatrix} 3 & -L & L \\ -L & L^2 & 0 \\ L & 0 & L^2 \end{bmatrix} \begin{array}{c} y_1 \\ \theta_2 \\ \theta_3 \end{array}$$

[30%]

(b) Eigenvalue Problem

$([K] - \omega^2 [M])\, \underline{u} = 0$

$$\begin{bmatrix} (3 - 2\lambda^2) & L(-1 + \lambda^2/2) & L(1 - \lambda^2/2) \\ L(-1 + \lambda^2/2) & L^2(1 - \lambda^2/3) & 0 \\ L(1 - \lambda^2/2) & 0 & L^2(1 - \lambda^2/3) \end{bmatrix} \begin{Bmatrix} y_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = 0 \qquad (\lambda^2 = \omega^2 m/k) \quad \text{——— (1)}$$

$\Rightarrow L^4(3-2\lambda^2)(1-\lambda^2/3)^{\cancel{2}} - L^4(-1+\lambda^2/2)(-1+\lambda^2/2)\cancel{(1-\lambda^2/3)} - L^4(1-\lambda^2/2)(1-\lambda^2/2)\cancel{(1-\lambda^2/3)}$

$\Rightarrow \left.\begin{array}{l} \lambda^2 = 3 \\ \omega^2 = 3k/m \end{array}\right\}$ $\quad$ & $\quad (3-2\lambda^2)(1-\lambda^2/3) - 2(1-\lambda^2/2)^2 = 0$

ie $\quad \lambda^4 - 6\lambda^2 + 6 = 0 \;\Rightarrow\; \underline{\underline{\lambda^2 = 3 \pm \sqrt{3}}}$ $\quad$ [20%]

ie $\omega^2 = (3 \pm \sqrt{3})\, k/m$

4 Cont

Eigenvectors

Row 2 of ①: $\cancel{\angle}(-1+\lambda^2/2)y_1 \quad + \angle^{\cancel{2}}(1-\lambda^2/3)\,\theta_2 = 0$

ie $\dfrac{y_1}{\theta_2} = \dfrac{\angle(1-\lambda^2/3)}{(1-\lambda^2/2)}$ ——— ②

Row 3 of ①: $\cancel{\angle}\,(1-\lambda^2/2)\,y_1 \quad + \angle^2(1-\lambda^2/3)\,\theta_3 = 0$

ie $\dfrac{y_1}{\theta_3} = \dfrac{\angle(1-\lambda^2/3)}{(\lambda^2/2-1)} \equiv -\dfrac{y_1}{\theta_2}$ ie $\theta_2 \equiv -\theta_3$ — ③

except when $y_1=0$

for $\lambda^2=3$ $\quad \dfrac{y_1}{\theta_2} = 0 \Rightarrow y_1 = 0$

So Row 1 of ①: $\cancel{\angle}(-1+3/2)\theta_2 \quad + \angle(1-3/2)\,\theta_3 = 0$

$$\theta_2/\theta_3 = 1 \longrightarrow \theta_2=\theta_3 \Rightarrow \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

for $\lambda^2 = 3-\sqrt{3}$ $\quad \dfrac{y_1}{\theta_2} = \dfrac{\angle\,(1-(3-\sqrt{3})/3)}{(1-(3-\sqrt{3})/2)} = \angle\left(1+\dfrac{1}{\sqrt{3}}\right)$

& $\dfrac{y_1}{\theta_3} = -\angle(1+1/\sqrt{3}) \Rightarrow \begin{Bmatrix} 1 \\ 0{\cdot}634 \\ -0{\cdot}634 \end{Bmatrix}$

for $\lambda^2 = 3+\sqrt{3}$ $\quad \dfrac{y_1}{\theta_2} = \dfrac{\angle\,(1-(3+\sqrt{3})/3)}{(1-(3+\sqrt{3})/2)} = \angle(1-1/\sqrt{3})$

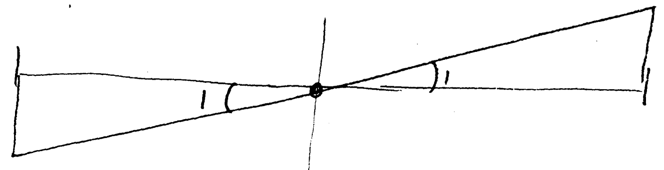
& $\dfrac{y_1}{\theta_3} = -\angle(1-1/\sqrt{3}) \Longrightarrow \begin{Bmatrix} 1 \\ 2{\cdot}366 \\ -2{\cdot}366 \end{Bmatrix}$

Not needed in solution

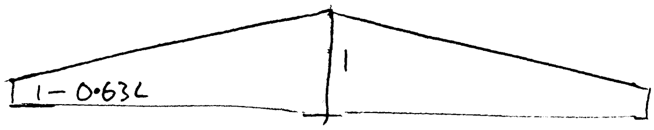
~~At~~ System is symmetric ∴ All modes must be either symm or anti-symm. Only anti-symm mode is $\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$
Only possible symm mode with $y_1=0$ is $\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ which is trivial
∴ all other modes must have $y_1 \neq 0$.

[20%]

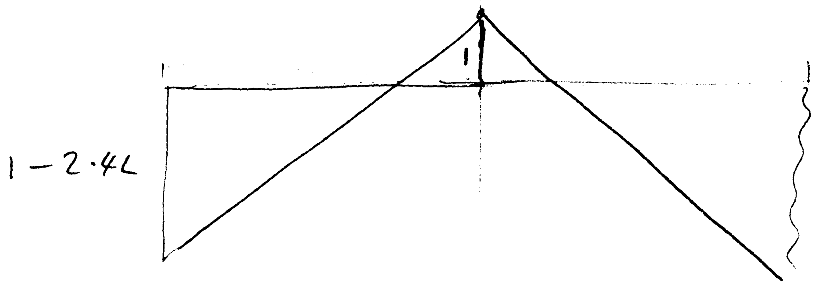
4 cont



$\lambda^2 = 3 \quad \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$



$\lambda^2 = 3 - \sqrt{3} \quad \begin{Bmatrix} 1 \\ 0.63 \\ -0.63 \end{Bmatrix}$

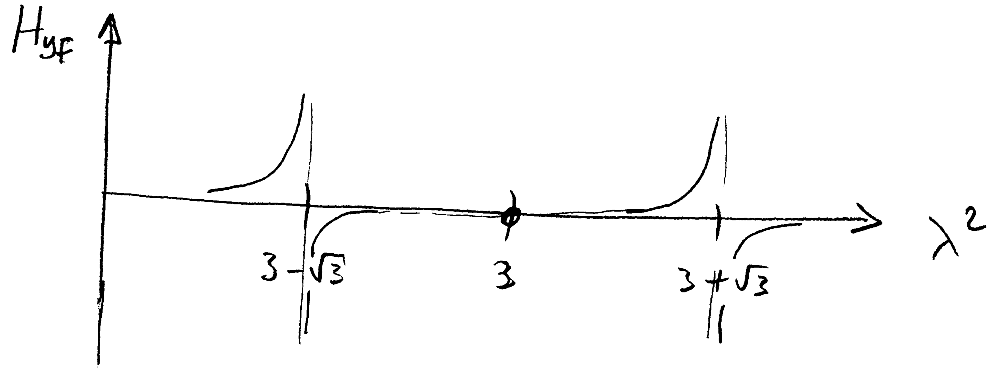


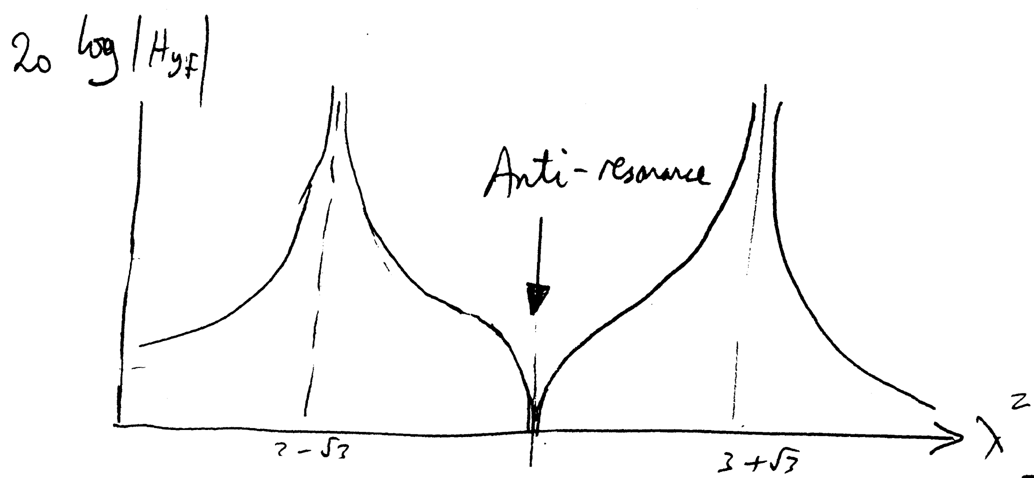
$\lambda^2 = 3 + \sqrt{3} \quad \begin{Bmatrix} 1 \\ 2.4 \\ -2.4 \end{Bmatrix}$

Not needed in solution

4 d Driving point Impedance: $u_J u_K$ is always +ve.

$$y/F = \sum \frac{y_1^{(n)} y_1^{(n)}}{\omega_n^2 - \omega^2} = \underset{3-\sqrt{3}}{\frac{y_1^{(1)} y_1^{(1)}}{\omega_1^2 - \omega^2}} + \frac{(0)(0)}{\omega_2^2 - \omega^2} + \underset{3+\sqrt{3}}{\frac{y_1^{(3)} y_1^{(3)}}{\omega_3^2 - \omega^2}}$$





[80%]

ENGINEERING TRIPOS PART IIB

Module 3C6 Examination, 2004

Answers

1. See crib

2. (a) $w = 0 \quad and \quad \dfrac{\partial^2 w}{\partial x^2} = 0$ at $x = 0$ and $x = L$;

   Modes: $u = \sin\left(\dfrac{n\pi x}{L}\right); \quad \omega_n = \left(\dfrac{n\pi}{L}\right)^4 \dfrac{Eh^2}{12\rho}$

   (b), (c) See crib

   (d) With this approximation, a tapered beam has same natural frequencies as a uniform beam of thickness $h_0 + \varepsilon L/2$.

3. (a) $[M] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \quad [K] = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad with \quad k = P/L$

   (b) $\omega_1^2 = \left(2-\sqrt{2}\right)\dfrac{k}{m}; \quad \omega_2^2 = 2\dfrac{k}{m}; \quad \omega_1^2 = \left(2+\sqrt{2}\right)\dfrac{k}{m}$

   (c) 4.7% decrease in lowest natural frequency

4. (a) $[M] = m\begin{bmatrix} 2 & -L/2 & L/2 \\ -L/2 & L^2/3 & 0 \\ L/2 & 0 & L^2/3 \end{bmatrix} \quad [K] = k\begin{bmatrix} 3 & -L & L \\ -L & L^2 & 0 \\ L & 0 & L^2 \end{bmatrix}$

   (b) $\omega_1^2 = \left(3-\sqrt{3}\right)\dfrac{k}{m}; \quad \omega_2^2 = 3\dfrac{k}{m}; \quad \omega_1^2 = \left(3+\sqrt{3}\right)\dfrac{k}{m}$

   (c), (d) See crib

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