ENGINEERING TRIPOS PART IIA 2004

Solutions to Module 3C7
Mechanics of Solids
Principal Assessor: Dr V Deshpande
Second Assessor: Dr T J Lu

1(a) $\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + F_x = 0$ – automatically satisfied

$F_x = 0$

$F_y = -\rho g$

$\dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + F_y = 0 \qquad \rho g - \rho g = 0$ OK ✓

at $y = 0 \qquad -\sigma_{yy} A = \rho g h A \quad \Rightarrow \quad c = -\rho g h$

$$\sigma_{yy} = \rho g (y - h)$$

(b) $\varepsilon_{xx} = -\dfrac{\nu \sigma_{yy}}{E} = \dfrac{\partial u}{\partial x} = -\dfrac{\nu \rho g}{E}(y-h)$

$$u = \frac{\nu \rho g}{E}(h-y)x + f(y)$$

$\varepsilon_{yy} = \dfrac{\sigma_{yy}}{E} = \dfrac{\partial v}{\partial y} = \dfrac{\rho g}{E}(y-h) \Rightarrow v = \dfrac{\rho g}{E}\left(\dfrac{y^2}{2} - hy\right) + g(x)$

$\gamma_{xy} = 0 = -\dfrac{\nu \rho g x}{E} + f'(y) + g'(x) = 0$

$$f'(y) = 0, \quad f(y) = c_1$$

$$g(x) = \frac{\nu \rho g x^2}{2E} + c_2$$

$u = \dfrac{\nu \rho g}{E}(h-y)x$ as $u = 0$ @ $x = 0$

$v = \dfrac{\rho g}{E}\left(\dfrac{y^2}{2} - hy\right) + \dfrac{\nu \rho g x^2}{2E}$ ($c_2 = 0$ as $v = 0$ @ $x = 0$, $y = 0$)

[Note that B.C. (A) still holds under the small strain assumption]

(c) $$U = \int_0^h \frac{1}{2}\, \sigma_{yy}\, \varepsilon_{yy}\, dy\, A$$

$$= \frac{A}{2E} \int_0^h \rho^2 g^2 (y-h)^2\, dy = \frac{A\rho^2 g^2 h^3}{6E}$$

(d) $$\varepsilon_{xx} = -\frac{\nu\sigma_{yy}}{E} + \alpha\Delta T$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} + \alpha\Delta T$$

$$\Rightarrow \quad u = \frac{\nu\rho g}{E}(h-y)x + \alpha\Delta T x$$

$$v = \frac{\rho g}{E}\left(\frac{y^2}{2} - hy\right) + \frac{\nu\rho g x^2}{2E} + \alpha\Delta T y$$

2. (a) $T(r) = T_0(1 - r^2/b^2)$

$$\sigma_{rr} = A - \frac{B}{r^2} - \frac{E\alpha}{r^2}\int_0^r r\,T_0(1 - r^2/b^2)\,dr$$

$$\sigma_{\theta\theta} = A + \frac{B}{r^2} + \frac{E\alpha}{r^2}\int_0^r r\,T_0(1 - r^2/b^2)\,dr - E\alpha T_0(1 - r^2/b^2)$$

} For plane stress only (Data card)

$$\int_0^r r\,T_0(1 - r^2/b^2)\,dr = \frac{1}{2}T_0 r^2\left(1 - \frac{1}{2}\frac{r^2}{b^2}\right)$$

$$\Rightarrow \quad \sigma_{rr} = A - \frac{B}{r^2} - \frac{E\alpha T_0}{2}\left(1 - \frac{1}{2}\frac{r^2}{b^2}\right)$$

B.C. ① σ_{rr} finite at $r = 0$ $\Rightarrow$ $B = 0$

② $\sigma_{rr} = 0$ at $r = b$ $\Rightarrow$ $A = E\alpha T_0/4$

$$\Rightarrow \quad \sigma_{rr} = -\frac{E\alpha T_0}{4}(1 - r^2/b^2)$$

$$\sigma_{\theta\theta} = +\frac{E\alpha T_0}{4} + \frac{E\alpha T_0}{2}\left(1 - \frac{1}{2}\frac{r^2}{b^2}\right) - E\alpha T_0(1 - r^2/b^2)$$

$$= -\frac{E\alpha T_0}{4}(1 - 3r^2/b^2)$$

Convert to plane strain by replacing E by $\frac{E}{1-\nu^2}$ and α by $\alpha(1+\nu)$

$$\Rightarrow \quad \sigma_{rr} = -\frac{E\alpha T_0}{4(1-\nu)}(1 - r^2/b^2), \quad \sigma_{\theta\theta} = -\frac{E\alpha T_0}{4(1-\nu)}(1 - 3r^2/b^2)$$

$\varepsilon_{zz} = 0$ $\Rightarrow$ $\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) = -\frac{\nu E\alpha T_0}{2(1-\nu)}(1 - 2r^2/b^2)$

plane strain

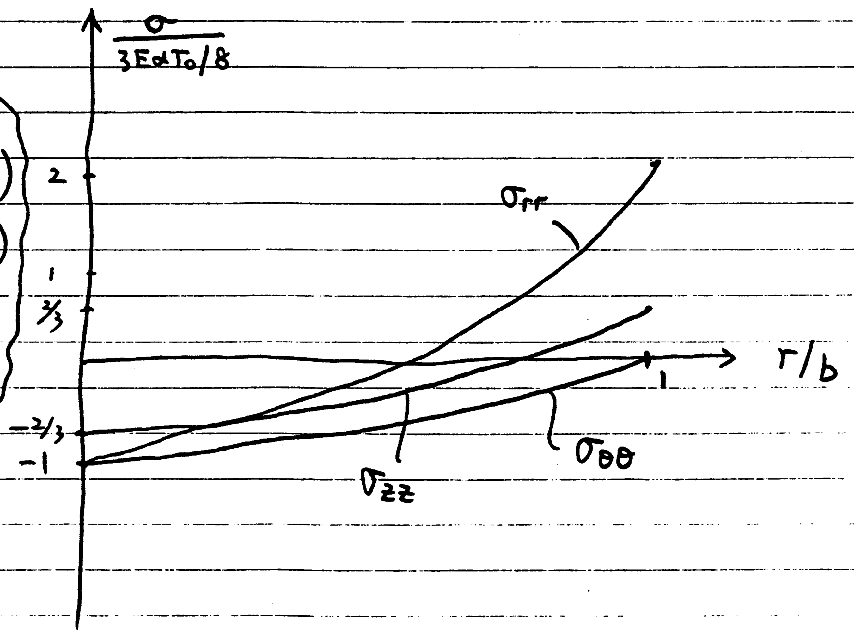
(b)

$\nu = \frac{1}{3}$

$$\sigma_{rr} = -\frac{3E\alpha T_0}{8}\left(1 - \frac{r^2}{b^2}\right)$$

$$\sigma_{\theta\theta} = -\frac{3E\alpha T_0}{8}\left(1 - 3\frac{r^2}{b^2}\right)$$

$$\sigma_{zz} = -\frac{E\alpha T_0}{4}\left(1 - 2\frac{r^2}{b^2}\right)$$



2. (c) $\frac{u}{r} = \varepsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr} - \nu\sigma_{zz})$

$$\Rightarrow \quad u = \frac{r}{E}\left(-\frac{E\alpha T_0}{4(1-\nu)}\right)\left\{1 - \frac{3r^2}{b^2} - \nu\left(1-\frac{r^2}{b^2}\right) - 2\nu^2\left(1-2\frac{r^2}{b^2}\right)\right\}$$

At $r=b$, $\quad u = -\frac{\alpha b T_0}{4(1-\nu)}(-2+2\nu^2) = \frac{\alpha b T_0(1+\nu)}{2}$

(d) $$\tau_{max} = \frac{\max\left(|\sigma_{rr}-\sigma_{\theta\theta}|,\ |\sigma_{\theta\theta}-\sigma_{zz}|,\ |\sigma_{zz}-\sigma_{rr}|\right)}{2}$$

Tresca criterion $\quad \tau_{max} = \frac{Y}{2}$ at yielding

Inspection shows that τ_{max} occurs at $r=b$, with

$$\tau_{max} = \left.\frac{|\sigma_{rr}-\sigma_{\theta\theta}|}{2}\right|_{r=b} = \left.\frac{E\alpha T_0}{4(1-\nu)}\frac{r^2}{b^2}\right|_{r=b} = \frac{E\alpha T_0}{4(1-\nu)}$$

If no yielding allowed, then the maximum T_0 allowable is

$$T_0 = \frac{2(1-\nu)Y}{E\alpha}$$

Q3

(a) $\phi = A r \theta \sin\theta$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \frac{2A}{r}\cos\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = 0$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right] = 0$$

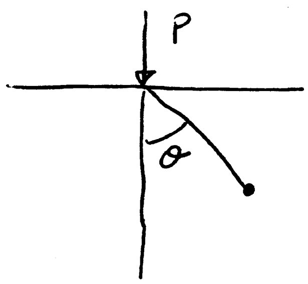
Boundary conditions are $\sigma_{\theta\theta} = \sigma_{r\theta} = 0$ on $y = 0 \Rightarrow$ satisfied

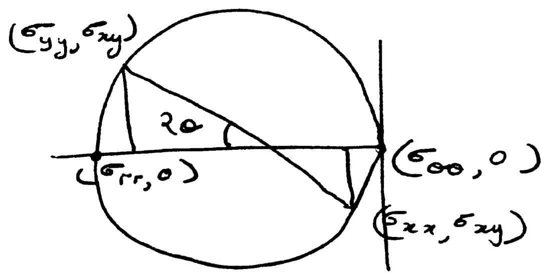
Equilibrium in vertical direction $P = -\int_{-\pi/2}^{\pi/2} \sigma_{rr} \cos\theta \, r \, d\theta$

$$P = -2A\frac{\pi}{2}$$

$$\Rightarrow A = -\frac{P}{\pi} \qquad (10\%)$$

(b)





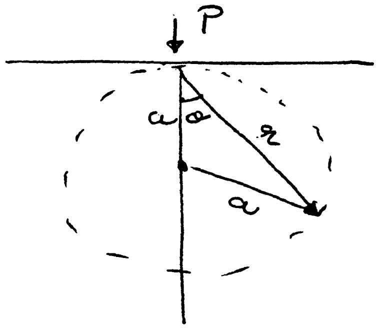
$$\sigma_{xx} = \frac{\sigma_{rr}}{2}(1 - \cos 2\theta) = \sigma_{rr}\sin^2\theta = -\frac{2F}{\pi}\frac{y x^2}{(x^2+y^2)^2}$$

$$\sigma_{yy} = \frac{\sigma_{rr}}{2}(1 + \cos 2\theta) = \sigma_{rr}\cos^2\theta = -\frac{2F}{\pi}\frac{y^3}{(x^2+y^2)^2}$$

$$\sigma_{xy} = \frac{\sigma_{rr}}{2}\sin 2\theta = \frac{2F}{\pi}\frac{xy}{(x^2+y^2)^2}$$

$$\therefore \quad \sigma_{yy}(x, a) = -\frac{2F}{\pi a} \frac{1}{\left[1 + \frac{x^2}{a^2}\right]^2} \qquad (30\%)$$

(c)



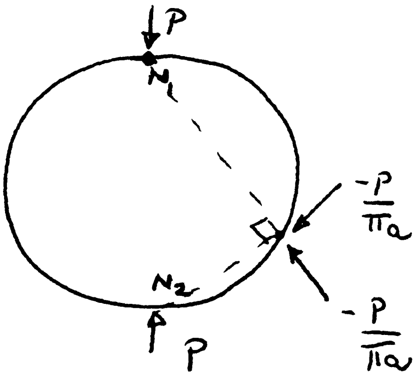
On a circle of radius a

$$2a\cos\theta = r \Rightarrow \frac{\cos\theta}{r} = \frac{1}{2a}$$

$$\sigma_{rr} = -\frac{2P}{\pi} \cdot \frac{1}{2a} = -\frac{P}{\pi a}$$

(20%)

(d)



Superposition of forces at N_1 & N_2 gives hydrostatic pressure on a circle of radius a. Remove it by applying a hydrostatic tension

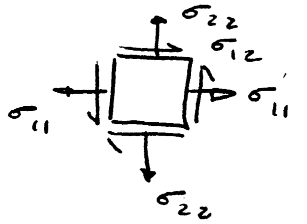
$$\sigma_{xx}'(x, y) = \sigma_{yy}'(x, y) = \frac{F}{\pi a}$$

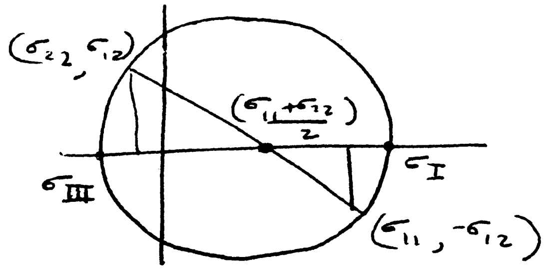
$$\therefore \quad \sigma_{yy}(x, a) = 2\sigma_{yy}(x, a) + \sigma_{yy}'(x, a)$$

$$= \frac{F}{\pi a}\left[1 - \frac{4}{\left[1 + \frac{x^2}{a^2}\right]^2}\right] \qquad (40\%)$$

Q4

(a) (i)





As $\varepsilon_{33} = 0$ & $\nu = \frac{1}{2}$

$$\sigma_{33} = \frac{1}{2}(\sigma_{11} + \sigma_{22}) = \sigma_{II}$$

Hence, the principal stresses are

$$\sigma_{I} = \frac{\sigma_{11}+\sigma_{22}}{2} + \sqrt{\frac{(\sigma_{11}-\sigma_{22})^2}{4} + \sigma_{12}^2}$$

$$\sigma_{II} = \frac{\sigma_{11}+\sigma_{22}}{2}$$

$$\sigma_{III} = \frac{\sigma_{11}+\sigma_{22}}{2} - \sqrt{\frac{(\sigma_{11}-\sigma_{22})^2}{4} + \sigma_{12}^2}$$

Von Mises yield condition is

$$\sqrt{\frac{1}{2}\left[(\sigma_{I}-\sigma_{II})^2 + (\sigma_{II}-\sigma_{III})^2 + (\sigma_{III}-\sigma_{I})^2\right.} = \sigma_{Y}$$

$$\Rightarrow \frac{1}{2}\left\{\left[\frac{(\sigma_{11}-\sigma_{22})^2}{4} + \sigma_{12}^2\right](1+1+4)\right\} = \sigma_{Y}^2$$

(20%)

$$\Rightarrow (\sigma_{11}-\sigma_{22})^2 + 4\sigma_{12}^2 = \frac{4}{3}\sigma_{Y}^2$$

(ii) For $\sigma_{11} = S$ & $\sigma_{22} = \sigma_{12} = 0$

$$S = \frac{2}{\sqrt{3}}\sigma_Y \quad \Rightarrow \quad \frac{S}{\sigma_Y} = \frac{2}{\sqrt{3}}$$

(10%)

(iii) For Tresca criterion

$$\sigma_I = S, \quad \sigma_{II} = \frac{S}{2}, \quad \sigma_{III} = 0$$

(10%)

$$\therefore \quad \tau_{max} = \frac{\sigma_I - \sigma_{III}}{2} = \frac{S}{2}$$

applying Tresca $\tau_{max} = \frac{S}{2} = \tau_Y = \frac{\sigma_Y}{2}$

$$\Rightarrow \quad \frac{S}{\sigma_Y} = 1$$

(b)

(i) The general expression for the yield criterion based on max. principal deviatoric stress is

$$|\sigma_I - \sigma_m| = C, \quad |\sigma_{II} - \sigma_m| = C, \quad |\sigma_{III} - \sigma_m| = C$$

where $\sigma_m = \dfrac{\sigma_I + \sigma_{II} + \sigma_{III}}{3}$

$\therefore$ we get

$$|2\sigma_I - \sigma_{II} - \sigma_{III}| = 3C; \quad |2\sigma_{II} - \sigma_{III} - \sigma_I| = 3C$$

$$|2\sigma_{III} - \sigma_{II} - \sigma_I| = 3C$$

We determine constant c from a uniaxial test $\sigma_I = \sigma_Y$ & $\sigma_{II} = \sigma_{III} = 0$

$\Rightarrow$ $3c = 2\sigma_Y$ or $c = \frac{2}{3}\sigma_Y$ & the yield criteria are

$$|2\sigma_I - \sigma_{II} - \sigma_{III}| = 2\sigma_Y$$

$$|2\sigma_{II} - \sigma_{III} - \sigma_I| = 2\sigma_Y$$

$$|2\sigma_{III} - \sigma_{II} - \sigma_I| = 2\sigma_Y$$

(30%)

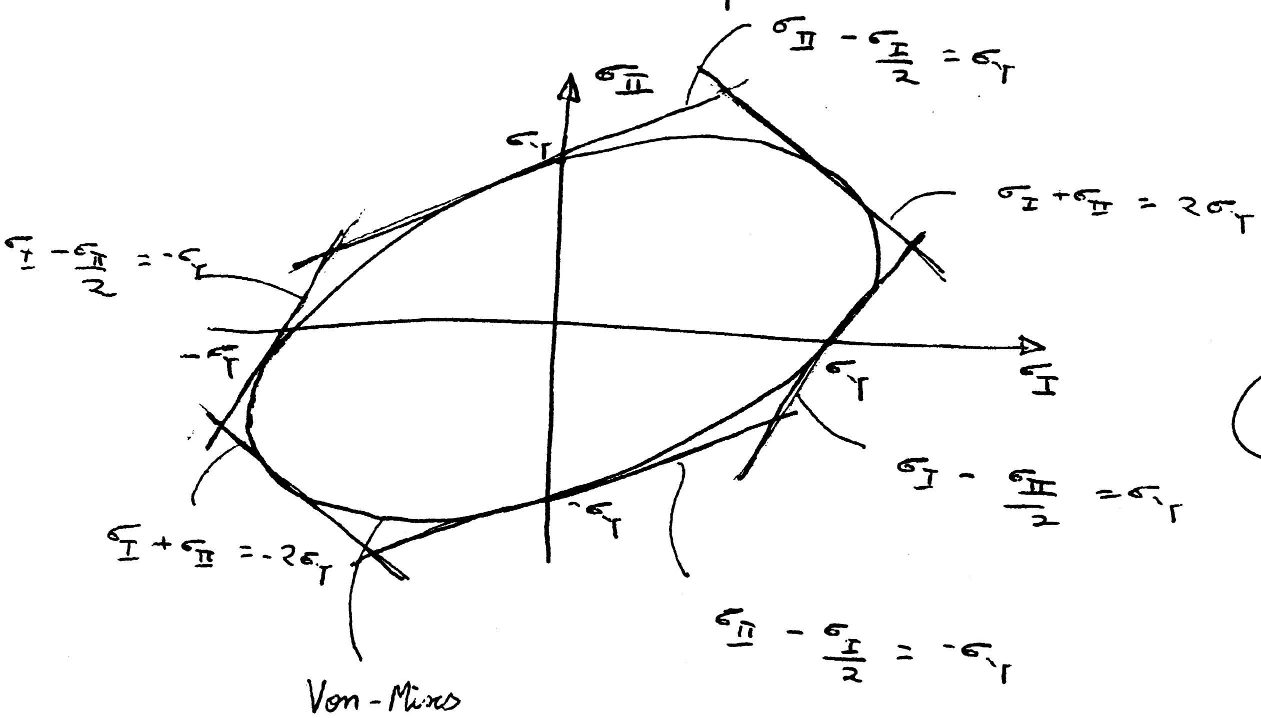
(ii)

In plane stress ($\sigma_{III} = 0$) the new yield criterion reduces to

$$\sigma_I - \frac{\sigma_{II}}{2} = \pm\sigma_Y, \quad \sigma_{II} - \frac{\sigma_I}{2} = \pm\sigma_Y, \quad \sigma_{II} + \sigma_I = \pm 2\sigma_Y$$

while the von-Mises yield criterion is

$$\sigma_I^2 + \sigma_{II}^2 - \sigma_I\sigma_{II} = \sigma_Y^2$$



(20%)

Answers to 3C7: Mechanics of Solids: 2004

1. (b) $u = \frac{\nu\rho g}{E}(h-y)x, \ v = \frac{\rho g}{E}(\frac{y^2}{2} - hy) + \frac{\nu\rho g x^2}{2E}$

(c) $U = \frac{A\rho^2 g^2 h^3}{6E}$

(d) $u = \frac{\nu\rho g}{E}(h-y)x + \alpha\Delta T x, \ v = \frac{\rho g}{E}(\frac{y^2}{2} - hy) + \frac{\nu\rho g x^2}{2E} + \alpha\Delta T y$

2 (a)

$$\sigma_{rr} = -\frac{E\alpha T_o}{4(1-\nu)}\left(1 - \frac{r^2}{b^2}\right),$$

$$\sigma_{\theta\theta} = -\frac{E\alpha T_o}{4(1-\nu)}\left(1 - \frac{3r^2}{b^2}\right)$$

$$\sigma_{zz} = -\frac{\nu E\alpha T_o}{2(1-\nu)}\left(1 - \frac{2r^2}{b^2}\right)$$

(b) $u = \frac{\alpha b T_o(1+\nu)}{2}$

(c) $T_o = \frac{2(1-\nu)Y}{E\alpha}$

3 (a) $A = -\frac{P}{\pi}$

(d) $\sigma_{yy}(x,a) = \frac{F}{\pi a}\left(1 - \frac{4}{\left[1 + x^2/a^2\right]^2}\right)$

4(a) (ii) $S = \frac{2}{\sqrt{3}}\sigma_Y$

(iii) $S = \sigma_Y$