ENGINEERING TRIPOS PART IIA 2004

Solutions to Module 3D4
Structural Analysis and Stability

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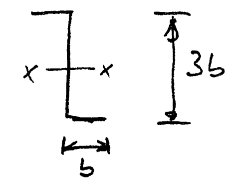
3D4. 2003-04.

1. a)

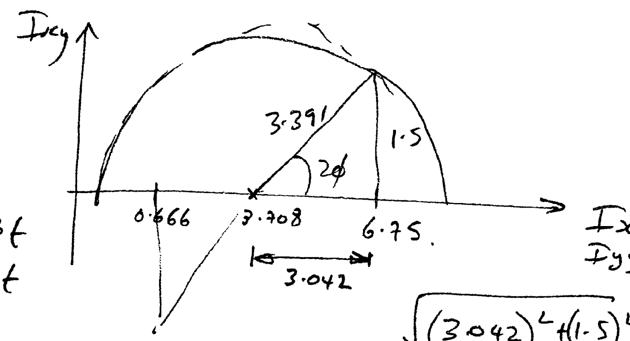


$$I_{xx} = \frac{(3b)^3 t}{12} + 2 \cdot bt(1.5b)^2 = (2.25 + 4.5)b^3 t = 6.75 b^3 t. \quad \left(= \tfrac{27}{4} b^3 t\right)$$

$$I_{yy} = \frac{(2b)^3 t}{12} = 0.667\, b^3 t \quad \left(= \tfrac{2}{3} b^3 t\right)$$

$$I_{xy} = -2(bt)\left(\tfrac{b}{2}\right)(1.5b) = -1.5 b^3 t \quad \left(= -\tfrac{3}{2} b^3 t\right)$$

b) Principal axes method ⟵
(or matrix method)

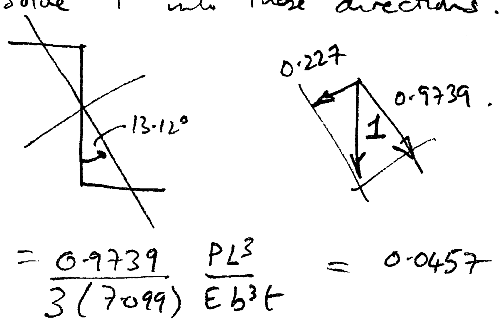


$$I_{11} = 3.708 + 3.391 = 7.099 \quad b^3 t$$
$$I_{22} = 3.708 - 3.391 = 0.318 \quad b^3 t$$

$$\sqrt{(3.042)^2 + (1.5)^2} = 3.391$$

$$\tan 2\phi = \frac{1.5}{3.042} = 0.493 \quad \therefore 2\phi = 0.458 \text{ radians} = 26.2° \Rightarrow \underline{\underline{\phi = 13.12°}}$$

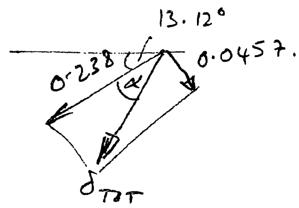
Resolve P into these directions.



$$\delta = \frac{PL^3}{3EI} \quad \text{(data bk)}$$

$$\delta_1 = \frac{0.9739}{3(7.099)} \frac{PL^3}{Eb^3 t} = 0.0457 \frac{PL^3}{Eb^3 t}$$

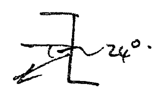
$$\delta_2 = \frac{0.227}{3(0.3176)} \frac{PL^3}{Eb^3 t} = 0.238 \frac{PL^3}{Eb^3 t}.$$



$$\delta_{TOT} = \sqrt{\delta_1^2 + \delta_2^2} = 0.242 \frac{PL^3}{Eb^3 t}.$$

Direction: $\tan\alpha = \frac{0.0457}{0.238} \Rightarrow \alpha = 10.9°.$

$\therefore$ δ_{TOT} is at $10.9 + 13.1 = 24°$ to horiz



3D4.

1(c). Load passes through $(b, 0)$. Torque $= Pb$ = const along length.

Need torsional rigidity GJ.

$$J = \text{"}\sum bt^3/3\text{"} = \frac{t^3}{3}\left[b + 3b + b\right] = \frac{5bt^3}{3}.$$

$$\Gamma = \frac{d^2}{4} I_{yy} = \frac{(3b)^2}{4}(0.666)b^3 t = 1.5 b^5 t.$$

$$\lambda = \sqrt{\frac{E\Gamma}{GJ}} = \sqrt{\frac{E}{G}}\left(\frac{1.5\, b^5 t}{5bt^3} \cdot 3\right)^{1/2} = 0.9486\sqrt{\frac{E}{G}} \cdot \frac{b}{t} \cdot b$$

Now $\frac{b}{t} = \frac{b}{b/10} = 10$ $\qquad G = \frac{E}{2(1+\nu)}$ so $\frac{E}{G} = 2(1+\nu)$

$\approx 2(1.3) = 2.6$ say.

(or leave as E/G).

$$\underline{\lambda = 9.486\sqrt{\frac{E}{G}} \cdot b.} \approx \cancel{15.29b}\ \underline{\underline{15.29\, b}}$$

i). $\theta = \frac{TL}{GJ} = \frac{P.b.\ 20b}{G} \cdot \frac{3}{5bt^3} = \left(\frac{P.b.}{G t^3}\right) 12 = 12 \times 10^3 \frac{P}{Gb^2}$

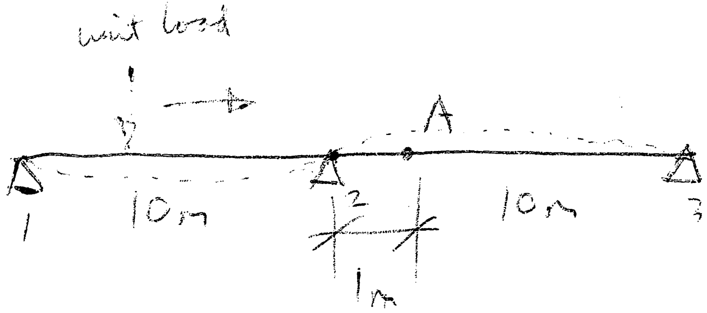
ii) Use "rule" that effective length, $\approx L - \lambda$

$= (20 - 15.29)b \approx = \underline{\underline{4.7b.}}$

$$\therefore \theta = \left(\frac{4.7}{20}\right) \times \left(\frac{12\, Pb}{Gt^3}\right) \approx \underline{\underline{2.82\frac{Pb}{Gt^3}.}} = 2.82 \times 10^3 \frac{P}{Gb^2}$$

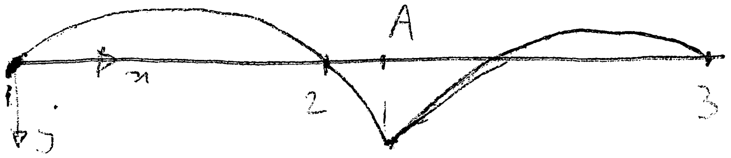
Q2 3D4

(a)



Influence line for sagging moment

Sagging +ve

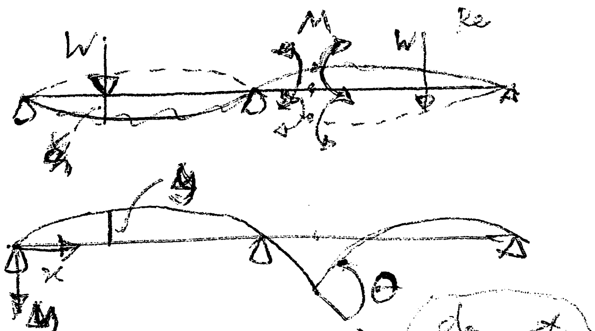


(b) Betti's Reciprocal theorem

$$M\theta = Wy$$

① Real moment & moving load

② Displacement in structure with 'kink' θ



Use Macaulay's to find deflected shape with 'kink'

dy at −1

$$EI\frac{d^2y}{dx^2} = -R_1x - R_2\{x-10\}$$

$$EI\frac{dy}{dx} = -\frac{R_1x^2}{2} - R_2\frac{\{x-10\}^2}{2} + A - EI\{x-11\}^0$$

$$EI\,y = -\frac{R_1x^3}{6} - R_2\frac{\{x-10\}^3}{6} + Ax + B - EI\{x-11\}^1$$

Moment equil about $x=20$ Q2 3D4 cont'd.

$$20R_1 + 10R_2 = 0 \quad \therefore \quad R_2 = -2R_1$$

@ $x = 20$, $y=0$

$$\therefore \quad 0 = -R_1\frac{8000}{6} - R_2 \cdot \frac{1000}{6} + 20A - 9EI$$

$$0 = -R_1\frac{8000}{6} + R_1\frac{2000}{6} + \frac{2000}{6}R_1 - 9EI$$

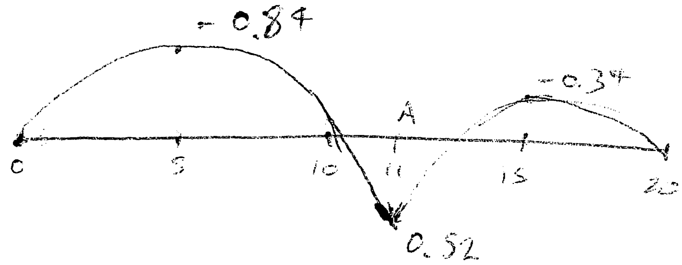
$$0 = -\frac{\cancel{4}000}{\cancel{6}3}R_1 - 9EI \quad \therefore \quad R_1 = \frac{-27}{2000}EI$$

$$R_2 = \frac{54}{2000}EI \qquad A = \frac{100}{\cancel{6}_2} \times \frac{-\cancel{27}^{9}}{\cancel{2000}}EI = \frac{-9EI}{40}$$

Thus

$$y = \frac{27}{2000}\frac{x^3}{6} - \frac{54}{2000}\frac{\{x-10\}^3}{6} - \frac{9x}{40} - \{x-11\}$$

x	y
0	0
5	−0.84375
10	0 ✓
11	0.51525
15	−0.34375
20	0 ✓

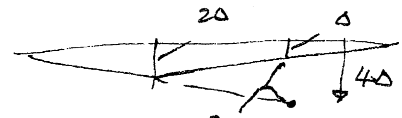


Distributed load, either on all −ve areas for most hogging, or on all +ve regions for worst

3D4. Q3

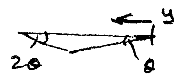
a) Classical – e.v.s ~ loads (due to decomposition $K_{mat} + PK_{Geom}$).
Non-classical – e.v.s ~ stiffnesses. (e.v.s of total tangent stiffness matrix.)
etc.

b). Let deflection of E be Δ.



i)
$\therefore$ Strain Energy $=$ "$\frac{1}{2} k x^2$" $= \frac{1}{2} k (3\Delta)^2 = \underline{\frac{9}{2} k \Delta^2}$

Ext. ~~Work~~ Work Done: need y:



$$y = L(1-\cos 2\theta) + 2L(1-\cos\theta)$$
$$= L\left(1-\left(1-\frac{(2\theta)^2}{2}+\dots\right)\right) + 2L\left(1-\left(1-\frac{\theta^2}{2}+\dots\right)\right)$$
$$= L\,2\theta^2 + 2L\frac{\theta^2}{2} = 3L\theta^2.$$

and $\theta = \Delta/L \Rightarrow y = 3L\left(\frac{\Delta}{L}\right)^2 = \frac{3\Delta^2}{L}$

$\therefore$ Total Potential Energy
$$\Pi = \frac{9k\Delta^2}{2} - P\left(\frac{3\Delta^2}{L}\right)$$
$$= \left(\frac{9}{2}k - \frac{3P}{L}\right)\Delta^2$$
$$= \frac{1}{2}\left(9k - 6\frac{P}{L}\right)\Delta^2 \quad = \text{"}\tfrac{1}{2}k x^2\text{"}$$

So eigenvalue is positive ~~until~~ until $\underline{P > \frac{3}{2}kL}$
(stiffness)

ii) Total potential energy is now

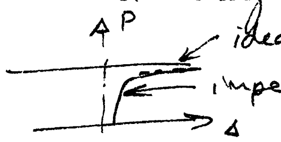
$$\Pi = \underbrace{-Q.\Delta}_{\text{conjugate.}} + \frac{1}{2}\left(9k - \frac{6P}{L}\right)\Delta^2.$$

Equilib when $\frac{\partial \Pi}{\partial \Delta} = 0 \quad \therefore \quad Q = \left(9k - \frac{6P}{L}\right)\Delta$

when $P = kL$ (not buckled yet)
$$Q = (9k - 6k)\Delta = 3k\Delta$$
$$\therefore \underline{\Delta = \frac{Q}{3k}} = \text{deflection of E.}$$

Deflection of D $= 4 \times$ deflection of E $= \underline{\frac{4Q}{3k}}$

iii) It'd be stress-free when not straight, so imperfection, so



(all that can be said without doing full, large deflection analysis).

iv). $k = k_0 + k_1 e$

$$\Pi = \text{"}\tfrac{1}{2}kx^2\text{"} - \text{"}P.y\text{"}$$

$$= \cancel{\tfrac{1}{2}(k_0 + k_1(3\Delta)^2) - Py}$$

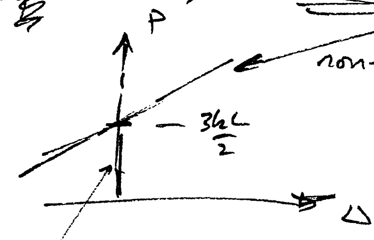
$$= \tfrac{1}{2}(k_0 + k_1 3\Delta)(3\Delta)^2 - P.y.$$

$$\Pi = \tfrac{1}{2}k_0(9\Delta^2) + \tfrac{1}{2}k_1 27\Delta^3 - P.\left(\frac{3\Delta^2}{L} + O(\Delta^4)\right)..$$

Equilib: $\frac{\partial \Pi}{\partial \Delta} = 0 \Rightarrow 9k_0\Delta + \frac{3}{2}.27.k_1\Delta^2 - 2.3\frac{P\Delta}{L}$

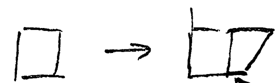
$$0 = 3\Delta\left(3k_0 - \frac{2P}{L} + 40.5k_1\Delta\right)$$

so $\Delta = 0$ or $P = \frac{3k_0L}{2} + 40.5k_1\Delta\frac{L}{2}$.



Asymmetric Bifurcation.

c) Shanley recognised that there were an infinite no. of solutions, because structure could buckle as it was loaded so that it could avoid strain reversal



bending without strain reversal.

3D4

4.

a) Column $\begin{pmatrix} M_B \\ M_C \end{pmatrix} = k \begin{bmatrix} s & sc \\ sc & s \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}$

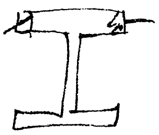
Add pin-end → beam $\begin{bmatrix} +3 & \\ & \end{bmatrix}$

Add fixed end-beam $\begin{bmatrix} & \\ & +4 \end{bmatrix}$

$$\rightarrow \begin{pmatrix} M_B \\ M_C \end{pmatrix} = k \begin{bmatrix} s+3 & sc \\ sc & s+4 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}$$

→ stable until $(s+3)(s+4) - s^2c^2 = 0$.

$\therefore$ $\underline{s^2(1-c^2) + 7s + 12 = 0}$

b).  As per notes.

c). Perry-Robertson. As per notes.