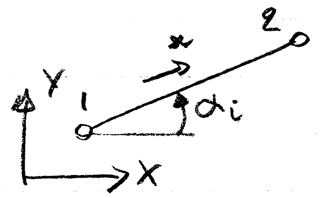
ENGINEERING TRIPOS PART IIA 2004

Solutions to Module 3D7
Finite Element Methods
Principal Assessor: Dr K Soga
Second Assessor: Dr D Symons

1

(a) Stiffness matrix for a pin-jointed element, in global coordinate system, is given in data sheet.



$$\frac{AE}{l_i}\left[\begin{array}{cc|cc} u^2 & uv & -u^2 & -uv \\ & v^2 & -uv & -v^2 \\ \hline & & u^2 & uv \\ \text{symm.} & & & v^2 \end{array}\right]$$

$\underbrace{\begin{bmatrix} u^2 & uv \\ & v^2 \end{bmatrix}}_{K'_{22}}$

where: $u = \cos\alpha_i$, $v = \sin\alpha_i$

For convenience, arrange all local member axes to go towards joint B $\therefore$ overall stiffness matrix is obtained by superposing matrices K'_{22} for the members I, .. IV

I	$\alpha_I = 0°$	$u = 1$, $v = 0$	$K'_{22} = \frac{AE}{L}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
II	$\alpha_{II} = 90°$	$u = 0$, $v = 1$	$AE/L \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$
III	$\alpha_{III} = 135°$	$u = -\sqrt{2}/2$, $v = \sqrt{2}/2$	$AE/\sqrt{2}L \begin{bmatrix}1/2 & -1/2\\-1/2 & 1/2\end{bmatrix}$
IV	$\alpha_{IV} = 180°$	$u = -1$, $v = 0$	$AE/L \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

Stiffness equations for node B:
$$\frac{AE}{L}\begin{bmatrix}1+\frac{1}{2\sqrt{2}}+1 & -\frac{1}{2\sqrt{2}}\\ -1/2\sqrt{2} & 1+1/2\sqrt{2}\end{bmatrix}\begin{bmatrix}d_{BX}\\d_{BY}\end{bmatrix} = \begin{bmatrix}p_{BX}\\p_{BY}\end{bmatrix}$$

(b) $p_{BX} = F/\sqrt{2}$, $p_{BY} = F/\sqrt{2}$ $\therefore$

$$\frac{AE}{L}\begin{bmatrix}2.3536 & -0.3536\\ -0.3536 & 1.3536\end{bmatrix}\begin{bmatrix}d_{BX}\\d_{BY}\end{bmatrix} = \begin{bmatrix}F/\sqrt{2}\\F/\sqrt{2}\end{bmatrix}$$

$$\therefore \begin{bmatrix}d_{1X}\\d_{1Y}\end{bmatrix} = \frac{FL}{AE}\begin{bmatrix}0.3943\\0.6254\end{bmatrix}$$

(c) Element compatibility matrix in global coordinates relates nodal displacements to extension:

$$\underset{\sim}{H}'^{T}_i \underset{\sim}{d}'_i = \underset{\sim}{e}_i$$

extension

displ. components of nodes 1, 2 of element i

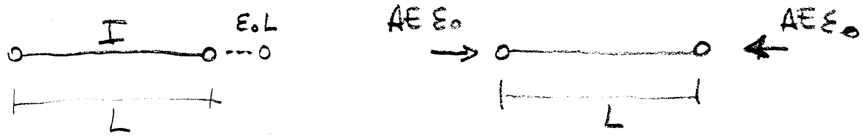
compatibility matrix

from datasheet $\quad \underset{\smile}{H}_i'^{T} = [-u \;\; -v \;\; u \;\; v]$

Node 1 of bar III is fixed; displacement comp. of node 2 are d_{BX}, d_{BY} $\therefore$

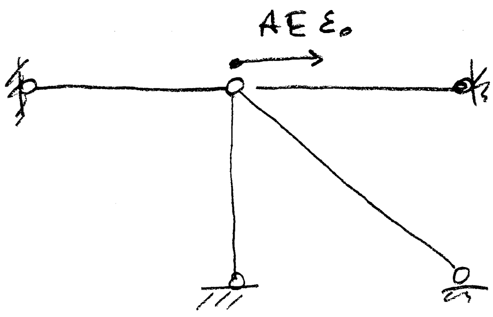
$$e_{III} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.3943 \\ 0.6254 \end{bmatrix} \frac{FL}{AE} = 0.1634 \frac{FL}{AE}$$

(d) We could begin by considering bar I in isolation, and then apply two equal and opposite forces to make its length $= L$:



Next, connect bar I to rest of structure (while keeping the opposite forces on it). Structure fits together perfectly.

Finally, remove forces, hence need to analyse effect of removing $AE\,\varepsilon_0$. Therefore, the load case to be analysed is:

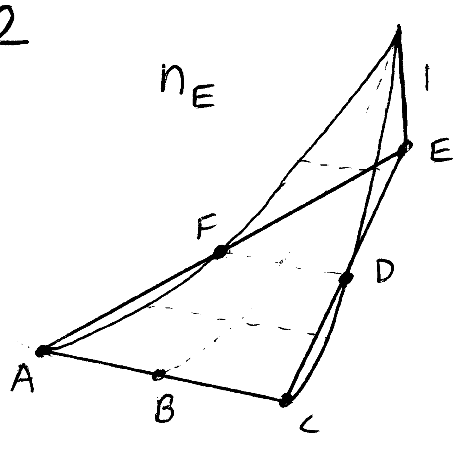


3D7

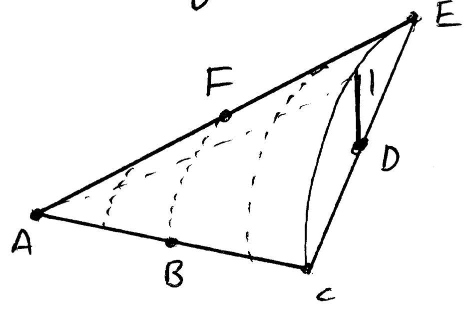
Qu. 2

(a)

n_E



n_D



Convert data sheet shape functions $(x = 2\xi,\ y = 4\eta)$

$$n_E = \eta(2\eta - 1) = \frac{y}{4}\left(\frac{y}{2} - 1\right)$$

$$n_D = 4\xi\eta = 4\frac{x}{2}\frac{y}{4} = \frac{1}{2}xy$$

(b)

Displacement field:

$$u_x = n_D u_{Dx} + n_E u_{Ex}$$
$$= \left[\tfrac{1}{2}xy \times 3 + \tfrac{y}{4}\left(\tfrac{y}{2} - 1\right) \times -2\right] \times 10^{-4}$$
$$= \left[\tfrac{3}{2}xy + \tfrac{y}{2} - \tfrac{y^2}{4}\right] \times 10^{-4}$$

$$u_y = n_D u_{Dy} + n_E u_{Ey}$$
$$= \left[\tfrac{1}{2}xy \times 1 + \tfrac{y}{4}\left(\tfrac{y}{2} - 1\right) \times -2\right] \times 10^{-4}$$
$$= \left[\tfrac{1}{2}xy + \tfrac{y}{2} - \tfrac{y^2}{4}\right] \times 10^{-4}$$

3D7 Qu. 2 (cont.)

ε

Strain field: @ P $x=1$ $y=0.5$

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = 10^{-4}\left[\frac{3}{2}y\right] \quad \left(= \tfrac{3}{4}\times 10^{-4} \text{ @ } P\right)$$

$$\varepsilon_y = \frac{\partial u_y}{\partial y} = 10^{-4}\left[\frac{1}{2}x + \frac{1}{2} - \frac{y}{2}\right] \quad \left(= \tfrac{3}{4}\times 10^{-4} \text{ @ } P\right.$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 10^{-4}\left[\frac{3}{2}x + \frac{1}{2} - \frac{y}{2} + \frac{1}{2}y\right]$$
$$= 10^{-4}\left[\frac{3}{2}x + \frac{1}{2}\right] \quad \left(= 2\times 10^{-4} \text{ @ } P\right)$$

Stress: $\underset{\sim}{\sigma} = \underset{\sim}{D}\,\underset{\sim}{\varepsilon}$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) = \frac{200\times 10^3}{1-0.3^2}\left(\frac{3}{4} + 0.3\times\frac{3}{4}\right)\times 10^{-4}$$
$$= 21.43 \quad \text{N/mm}^2$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right) = 21.43 \quad \text{N/mm}^2$$

$$\tau_{xy} = \frac{E}{(1-\nu^2)}\frac{(1-\nu)}{2}\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = \frac{200\times 10^3}{2(1+0.3)}\times 2\times 10^{-4}$$
$$= 15.38 \quad \text{N/mm}^2$$

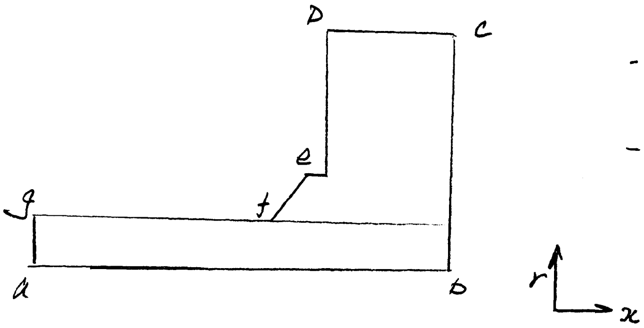
(c) Consistent mass matrix $\underset{\sim}{M}$ (where $\underset{\sim}{M}\,\ddot{\underset{\sim}{d}} + \underset{\sim}{K}\,\underset{\sim}{d} = \underset{\sim}{R}$) is obtained by calculating the kinetic energy of the element using the same shape functions as used for the displacement field (and stiffness matrix $\underset{\sim}{K}$)

Hence $$\underset{\sim}{M} = \int_V \underset{\sim}{N}^T \rho\, \underset{\sim}{N}\, dV = t\rho \int_A \underset{\sim}{N}^T \underset{\sim}{N}\, dA$$ for a 2D element

3

(a)

(i)



- Two dimensional
- axi symmetric

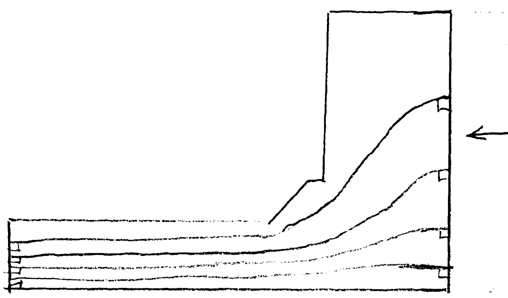
Boundary ab - $T = 50\,°C$

Boundary bc - $\frac{\partial T}{\partial x} = 0$

Boundary c-d-e-f-g - $T = 0\,°C$

Boundary ga - $\frac{\partial T}{\partial x} = 0$

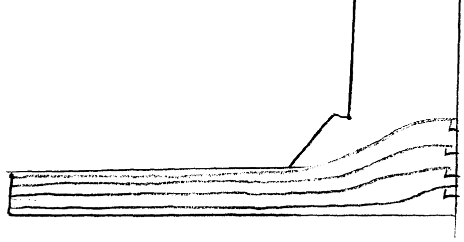
(ii)



contours normal to this plane

not equall spaced because of radial direction

(iii)



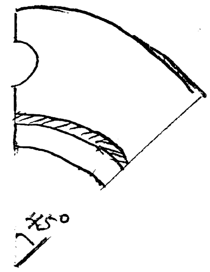
(iv) Need to apply convection boundary conditions and heat transfer coefficients are needed.

Flux boundary

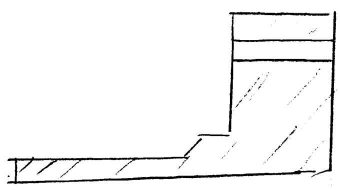
$$f = k\frac{\partial T}{\partial r} = h(T_{fl} - T_s)$$

where T_{fl} is the temperature of the fluid and T_s is the temperature of the pipe surface.

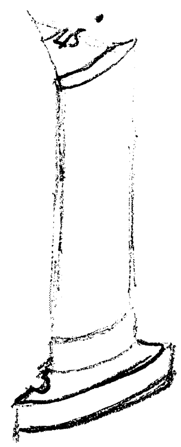
(b) (i) Need to model the section in three dimension, but only need to model 1/16 of the section by taking into account of the symmetry.



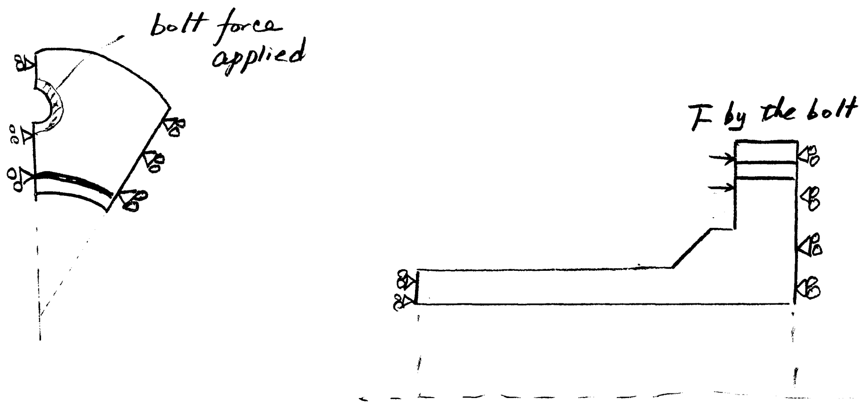
cross-sectional view



cross-section in longtitudinal direction



3D view



(ii) The reduced element uses smaller number of integration points to calculate the stiffness matrix.

Advantages

- improve the accuracy of the results by offsetting the overstiffness associated with compatible elements based on assumed displacement field

- computationally faster because of less calculation to determine the stiffness matrix

Limitations

- A spurious mode which has no resistance to nodal loads may activate.

307 Qu. 4

(a) t = thickness

A = area of whole mesh

p_X, p_Y = components of distributed load

n_j = shape functions

($n_j = 1$ @ node j, $n_j = 0$ at all other nodes)

(b) $p_X = 0 \quad \therefore \quad P_{AX} = 0 \qquad p_Y = -\rho g$

$$P_{AY} = t\int_A -\rho g \, n_A \, dA \qquad n_A = 1 - 10x - 10y$$

(x, y in metres)

$$= -t\rho g \int_{y=0}^{y=1/10}\int_{x=0}^{x=1/10-y} 1 - 10x - 10y \; dx\,dy$$

$$= -t\rho g \int_{y=0}^{y=1/10} \left[x - 5x^2 - 10xy\right]_{x=0}^{x=1/10-y} dy$$

$$= -t\rho g \int_0^{y=1/10} \tfrac{1}{10} - y - 5\left(\tfrac{1}{10} - y\right)^2 - 10\left(\tfrac{1}{10} - y\right)y \; dy$$

$$= -t\rho g \int_0^{y=1/10} \tfrac{1}{10} - y - \tfrac{5}{100} + 5x\tfrac{2}{10}y - 5y^2 - y + 10y^2 \; d$$

$$= -t\rho g \int_0^{1/10} \tfrac{1}{20} - y + 5y^2 \; dy$$

$$= -t\rho g \left[y/20 - y^2/2 + \tfrac{5}{3}y^3\right]_0^{1/10}$$

3D7 Qu. 4 (cont.)

$$P_{AY} = -t\rho g \left\{ \frac{1}{200} - \frac{1}{200} + \frac{5}{3000} \right\}$$

$$= \frac{-1}{1000} \times 2000 \times 10 \times \frac{5}{3000} = \frac{-1}{30} \text{ N}$$

$$P_{JY} = 2t \int_A -\rho g \, n_J \, dA \qquad n_J = 10y$$

Elements ABJ and JBI have equal contributions

$$P_{JY} = -2t\rho g \int_0^{1/10} \int_0^{1/10 - y} 10y \, dx \, dy$$

$$= -2t\rho g \int_0^{1/10} 10 \left[xy \right]_0^{1/10 - y} dy$$

$$= -2t\rho g \int_0^{1/10} 10 \left(\tfrac{1}{10} - y \right) y \, dy$$

$$= -2t\rho g \int_0^{1/10} y - 10y^2 \, dy$$

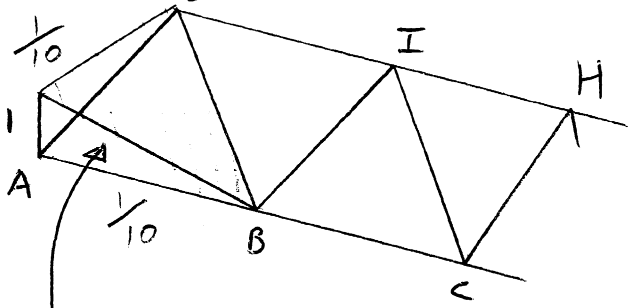
$$= -2t\rho g \left[\frac{y^2}{2} - \frac{10y^3}{3} \right]_0^{1/10}$$

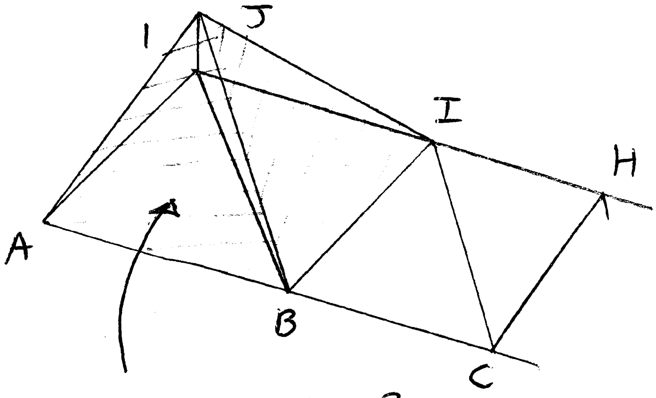
$$= -2 \times \frac{1}{1000} \times 2000 \times 10 \left\{ \frac{1}{200} - \frac{1}{300} \right\}$$

$$= \frac{-2}{30} \text{ N}$$

3D7 Qu. 4 (cont.)

Alternatively find P_{AY} and P_{JY} from the volume of the pyramidal shape functions:





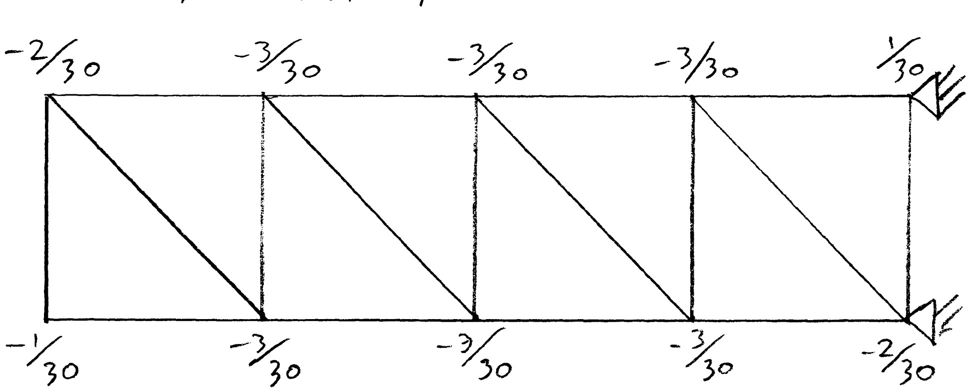
$$\text{volume} = \frac{1}{3}\left(\frac{1}{10}\right)^2 \frac{1}{2} = \frac{1}{600}$$

$$\text{volume} = \frac{1}{3}\left(\frac{1}{10}\right)^2 = \frac{1}{300}$$

$$\therefore\ P_{AY} = -\rho g t \times \frac{1}{600} = -2000 \times 10 \times \frac{1}{1000} \times \frac{1}{600} = -\frac{1}{30}\ \text{N}$$

$$\therefore\ P_{JY} = -\rho g t \times \frac{1}{300} = -\frac{2}{30}\ \text{N}$$

(c) Hence find P_{jY} for all nodes:



(all in N)

$$\text{Check total} = -(2\times1 + 2\times2 + 6\times3)/30 = -\frac{4}{5}\ \text{N}$$

$$= -\rho g t A = -2000\times10\times\frac{1}{1000}\times\frac{1}{10}\times\frac{4}{10} = -\frac{4}{5}\ \text{N}$$

(d) Constant strain triangles – not good for linear stress distribution due to bending. Could improve mesh density but better to use higher order elements e.g. 6 node LST.