ENGINEERING TRIPOS PART IIA 2004

Solutions to Module 3E4
Modelling Choice
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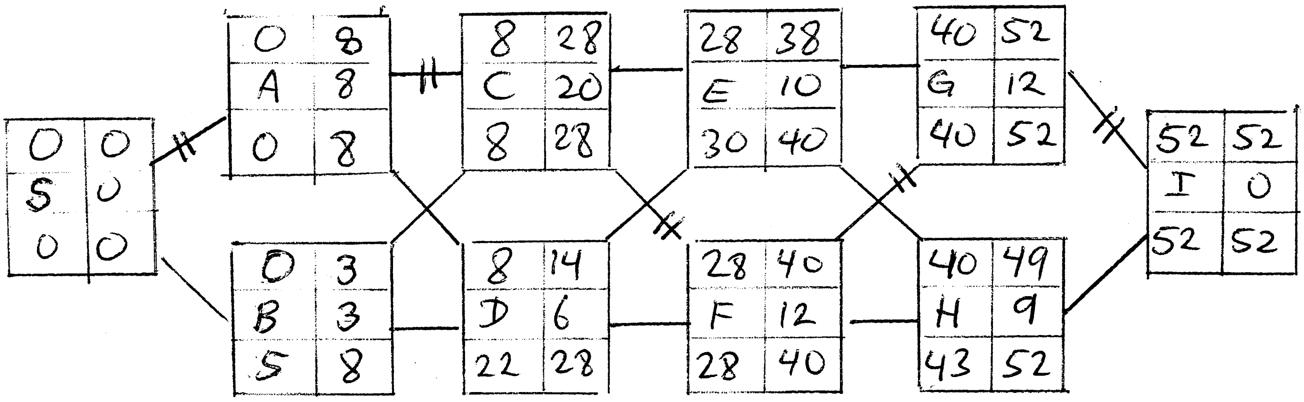
1. (a) (i)



Critical path shown with ─╫─
Critical activities: A, C, F, G. (non dummy)
(length of project: 52)

(ii) only task C has asymmetric β-distribution

→ average duration for C is $\frac{1}{6}(15+80+30)=20.83$.

→ since, first, C was critical, second, its (average) duration was increased and, finally, no other (average) durations have changed, the critical path in PERT is <u>same</u> as in part (i).

→ (average) critical path length A, C, F, G = 52.83

→ variance of CP length is sum of vars. of critical tasks A,C, F, G

= $(4^2 + 15^2 + 64 + 64)/36$

= $369/36 = 10.25$

Std dev. = $\sqrt{10.25} \approx 3.20$

Therefore random variable for project length is $T \sim N(52.83, 3.20)$, and T_{95} is such that

$$P(T \le T_{95}) = 0.95$$

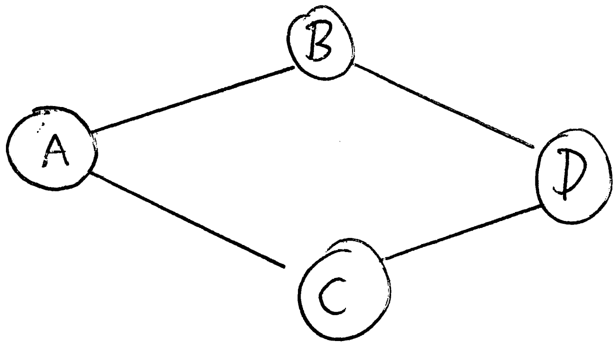
From *standard normal table*, $X \sim N(0,1)$ has

$$P(X \le 1.645) = 0.95$$
$$P(X \le 2.326) = 0.99$$

So

$$T_{95} = 52.83 + 3.20 \times 1.645$$
$$\approx 58.1$$
$$T_{99} \approx 60.3$$

1) a) **(iii)** Network similar to example in lecture



Task	Expect. Duration
A	1
B	2
C	1
D	1

C.P. is obviously A-B-D, expected length 4. PERT estimates statistical properties of random completion time by looking only at critical activities. But if variance of duration of task C is relatively large, in many realisations would have *A-C-D critical instead of A-B-D*. So risk of project overrunning is (optimistically) under-estimated by PERT.

1) **b) i)** x_1 = amount spent on advertising (in £k)
x_2 = amount spent on promotions (£k)

Goals:
- Sales volume 80k
- Brand index: no goal is given, so consider the max. possible value which occurs if all £30k is spent on marketing, goal = $2.5 \times 30 = 75$ units
- Budget: £25k

<u>Constraints with deviational vars:</u>

Goal constraints:

$$\text{sales}: 3x_1 + x_2 = 80 + o_1 - u_1 \quad (\text{or} \geq 80 - u_1)$$
$$\text{brand}: 1.5x_1 + 2.5x_2 = 75 - u_2 \quad (\text{overshoot } o_2 \text{ optional})$$
$$\text{budget}: x_1 + x_2 = 25 + o_3 - u_3 \quad (\text{or} \leq 25 + o_3)$$
$$\text{all } o_i, u_i \geq 0$$

hard constraints:

$$\text{budget}: x_1 + x_2 \leq 30 \qquad (\text{or } o_3 \leq 5)$$
(equality ok)
$$x_1, x_2 \geq 0$$
(could include sales ≥ 80 where instead of goals)

<u>Objective</u>: Introduce positive weights $w_1 \geq w_2, w_3$:

$$\min \frac{w_1 u_1}{80} + \frac{w_2 u_2}{75} + \frac{w_3 o_3}{25} \quad \text{subject to previous constraints}$$

1) b) **ii)** The values (outcomes to be maximised are
s = sales levels
b = brand index

Since $\begin{pmatrix} s \\ b \end{pmatrix} = \begin{bmatrix} 3 & 1 \\ 1.5 & 2.5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ get $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{6} \begin{bmatrix} 2.5 & -1 \\ -1.5 & 3 \end{bmatrix} \begin{pmatrix} s \\ b \end{pmatrix}$.

Hard constraints become

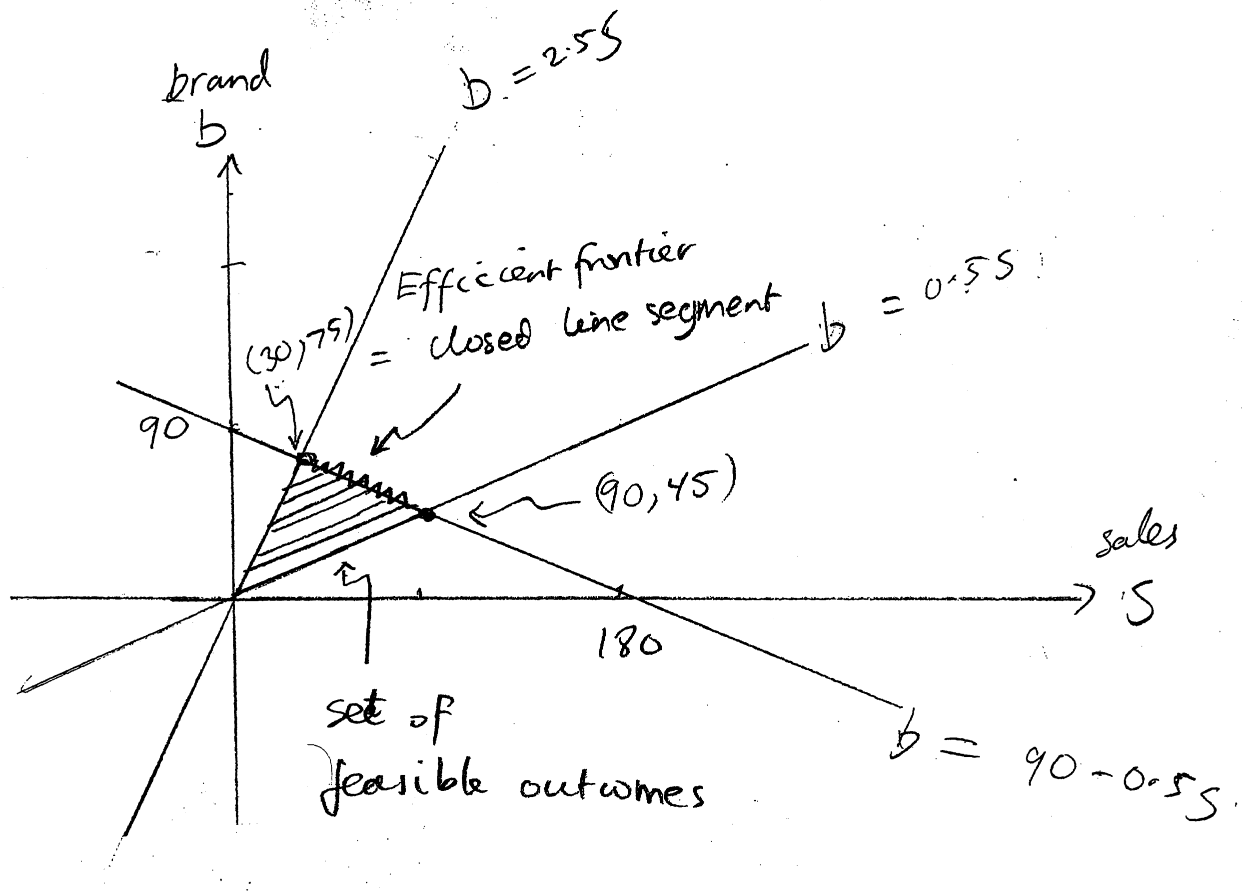
$$\frac{1}{6}(2.5s - b - 1.5s + 3b)$$

budget: $= \frac{1}{6}s + \frac{1}{3}b \leq 30$

$$\left[\Leftrightarrow b \leq 90 - \frac{1}{2}s \right]$$

non-neg: $\frac{1}{6}(2.5s - b) \geq 0 \; [\Leftrightarrow b \geq 2.5s]$

$-1.5s + 3b \geq 0 \; [\Leftrightarrow b \geq 0.5s]$



Eff. frontier is interval connecting the points (30,75) and (90,45).

2) NLP

a) Constraints are convex in x_1, x_2, x_3 for any fixed value of t. To see this:

i) $t^2 \geq x_1^2 + x_2^2 + x_3^2$

$\Leftrightarrow 0 \geq g(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - t^2$

where Hessian of g is 2 × identity matrix, which is pos. semi-def. (for all x), hence g *is convex.*

ii) $tx_2 \leq x_1$ and non-negativity of x_1, x_2 are linear constraints, hence convex.

Now consider objective.

Problem is *min*, so need Hessian of objective to be psd for convexity. Fix t, and consider $f_t(x) = tx_1^2 - 4tx_1x_2 + x_2^2 - x_3 + e^{x_3^2}$.

$$\nabla f_t(x) = \left(2tx_1 - 4tx_2, -4tx_1 + 2x_2 - 1 + 2x_3e^{x_3^2}\right)$$

$$\nabla^2 f_t(x) = \begin{bmatrix} 2t & -4t & 0 \\ -4t & 2 & 0 \\ 0 & 0 & (2+4x_3^2)e^{x_3^2} \end{bmatrix}$$

So $D^2 f_t(x)$ is psd

$$\Leftrightarrow \begin{bmatrix} 2t & -4t \\ -4t & 2t \end{bmatrix} \text{ is psd and } \left(2+4x_3^2\right)e^{x_3^2} \geq 0 \qquad \forall x$$

$$\Leftrightarrow \begin{bmatrix} 2t & -4t \\ -4t & 2t \end{bmatrix} \text{is psd}$$

because $\left(2+4x_3^2\right)e^{x_3^2} \geq 0, \forall x_3$ (since exp. ≥ 0)

$\Leftrightarrow \quad 2t \geq 0$ and $0 \leq$ determinant $= 4t - 16t^2 = 4t(1-4t)$

$\Leftrightarrow t \geq 0$ and $([4t \geq 0$ and $(1\text{-}4t) \geq 0]$
or $[4t \leq 0$ and $(1\text{-}4t) \leq 0])$

$\Leftrightarrow \quad t \geq 0$ and $t \leq \frac{1}{4}$

So convex problem for $0 \leq t \leq \frac{1}{4}$ and nonconvex problem otherwise.

2) **b)** Algorithm from lecture notes:

Problem: $\min_{x \in R^n} f(x)$ subj to $Ax = b$, $x \geq 0$

where $x \in R^n$, and we have data: $A \in R^{m \times n}, b \in R^m$

Initially: - need feasible iterate $x^\circ \left(Ax^\circ = b, x^\circ \geq 0\right)$
- set iteration counter $k = 0$.

Iteration k:

- Given feasible x^k and scalar $t > 0$, let $y^k(t)$ = nearest point in feasible set to x^k-$t\nabla f(x^k)$
- Use linesearch to find suitable step size $t > 0$, i.e. should at least satisfy

$$f(y^k(t)) < f(x^k) \qquad (*)$$

Denote this value of t by t_k.

- Define $x^{k+1} = y^k(t_k)$ and go to iteration k+1.

Notes on algorithm

1. All iterates remain feasible (each $y^k(t)$ is feasible)

2. If (*) cannot be accomplished for any $t > 0$, then x^k is already stationary for the NLP (i.e. $y^k(t) = x^k$ for all $t > 0$).

3. Convergence: Algorithm is likely (unless some x^k is stationary) to take an infinite no. of iterations. In this case, if $\{x_k\}_{k=0}^{\infty}$ has a convergent subsequence with limit x^*, then x^* is stationary.

c) $\max\ -x_1^2 - x_2^2 - x_3^2 =: f(x)$
subj to $x_1, x_2, x_3 \geq 0$
$- x_1 - 2x_2 + 3x_3 = 1$

$x^* = (0,0,\frac{1}{3})$.

L.P. at x^* obtained by linearising nonlinear functions.
Have $\nabla f(x) = -(2x_1, 2x_2, 2x_3)$, so $\nabla f(x^*) = (0,0,-2/3)$.

LP: $\max - 2x_3$
subj to $- x_1 - 2x_2 + 3x_3 = 1$
$x_1, x_2, x_3 \geq 0$
($x_3 \geq 0$ optional since inactive at x^*).

Optimality of x^*: x_3 is only basic variable.

$A = [-1 \ \ -2 \ \ 3] \in R^{1\times 3}$
$B = 3$ basic matrix corresp. to x_3
$N = [1 \ \ 2]$ non basic columns of A
$P^T = [0 \ \ 0 \ \ -2/3]$ objective vector
$P_B^T = -2$, $P_N^T = [0 \ \ 0]$

Reduced cost:

$$= P_N - N^T B^{-T} P_B$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{bmatrix} -1 \\ -2 \end{bmatrix} \times \frac{1}{3} \times \frac{-2}{3} = \begin{bmatrix} -2/9 \\ -4/9 \end{bmatrix} \leq 0$$

Since x^* is BFS of LP and reduced cost ≤ 0, then x^* is optimal for LP.
Shadow cost for LP gives Lagrange multiplier for NLP:
shadow cost $\lambda = B^{-T} Q_B = \frac{1}{3} \times -\frac{2}{3} = -\frac{2}{9}$

$\therefore$ Lagrange mult $= -\frac{2}{9} \approx -0.2222$

$\therefore$ changing RHS from 1 to 1.15 gives approx. change in NLP optimal value of

$$(1.15 - 1) \times -0.2222 \approx -0.0333$$

$\therefore$ new opt value $\approx f(x^*) - 0.0333$

$$= -\left(\frac{1}{3}\right)^2 - 0.0333$$

$$= -0.1444$$

3(a) Define decision variables

$B_i, i = 1, 2, 3$: The number of Basic Computers to be made in plant i

$D_i, i = 1, 2, 3$: The number of Deluxe Computers to be made in plant i

$L_i, i = 1, 2, 3$: The number of Laptop Computers to be made in plant i

Then the objective is to maximise the total profit:

$$400(B_1 + B_2 + B_3) + 750(D_1 + D_2 + D_3) + 980(L_1 + L_2 + L_3)$$

The assembly hours at each plant is limited. Hence

$$\begin{aligned} 3B_1 + 4D_1 + 8L_1 &\leq 20,000, \quad \text{plant 1} \\ 3B_2 + 4D_2 + 8L_2 &\leq 30,000, \quad \text{plant 2} \\ 3B_3 + 4D_3 + 8L_3 &\leq 10,000, \quad \text{plant 3} \end{aligned}$$

The fabrication hours at each plant is limited. Hence

$$\begin{aligned} 8B_1 + 12D_1 + 16L_1 &\leq 100,000, \quad \text{plant 1} \\ 8B_2 + 12D_2 + 16L_2 &\leq 100,000, \quad \text{plant 2} \\ 8B_3 + 12D_3 + 16L_3 &\leq 70,000, \quad \text{plant 3} \end{aligned}$$

Any production must be non-negative. Hence

$$\begin{aligned} B_1, B_2, B_3 &\geq 0 \\ D_1, D_2, D_3 &\geq 0 \\ L_1, L_2, L_3 &\geq 0 \end{aligned}$$

The overall optimisation problem is:

$$\begin{array}{ll} \text{max} & 400(B_1 + B_2 + B_3) + 750(D_1 + D_2 + D_3) + 980(L_1 + L_2 + L_3) \\ \text{subjectto} & 3B_1 + 4D_1 + 8L_1 \leq 20,000, \text{ plant 1} \\ & 3B_2 + 4D_2 + 8L_2 \leq 30,000, \text{ plant 2} \\ & 3B_3 + 4D_3 + 8L_3 \leq 10,000, \text{ plant 3} \\ & 8B_1 + 12D_1 + 16L_1 \leq 100,000, \text{ plant 1} \\ & 8B_2 + 12D_2 + 16L_2 \leq 100,000, \text{ plant 2} \\ & 8B_3 + 12D_3 + 16L_3 \leq 70,000, \text{ plant 3} \\ & B_1, B_2, B_3 \geq 0 \\ & D_1, D_2, D_3 \geq 0 \\ & L_1, L_2, L_3 \geq 0 \end{array}$$

3(b)(i) Convert the LP into the following standard format:

$$\begin{array}{ll} \max & 2x+3y=z \\ \text{subjectto} & x+2y+s_1 \quad\ \ =30 \\ & x+y+ \quad\ +s_2=20 \\ & x,y,s_1,s_2\geq 0 \end{array}$$

Choose s_1, s_2 as basic variables. Then we have

$$s_1 = 30 - x - 2y \tag{1}$$
$$s_2 = 20 - x - y \tag{2}$$
$$z = \quad 2x + 3y \tag{3}$$

The basic solution is $(0,0,30,20)$ with $z=0$. Since the coefficients of x and y in (3) are both positive, we may choose either x or y as an entering variable. We select y as the entering variable. The maximum y-values are 15 and 20 in (1) and (2) respectively. Hence we choose s_1 as an exiting variable. Execute one pivot. We obtain the following

$$\begin{aligned} y &= 15 - \tfrac{1}{2}x - \tfrac{1}{2}s_1 \\ s_2 &= 20 - x - 15 + \tfrac{1}{2}x + \tfrac{1}{2}s_1 \\ z &= \quad 2x + 45 - \tfrac{3}{2}x - \tfrac{3}{2}s_1 \end{aligned}$$

which is equivalent to

$$y = 15 - \tfrac{1}{2}x - \tfrac{1}{2}s_1 \tag{4}$$
$$s_2 = 5 - \tfrac{1}{2}x + \tfrac{1}{2}s_1 \tag{5}$$
$$z = 45 + \tfrac{1}{2}x - \tfrac{3}{2}s_1 \tag{6}$$

The basic solution we obtained is $(0,15,0,5)$ with $z=45$.

Looking at (6), we may choose x as an entering variable. Checking (4) and (5), the maximum x-values for (4) and (5) are 30 and 10 respectively. Therefore, s_2 is selected as an exiting variable. Execute another pivot. We obtain the following:

$$\begin{aligned} y &= 15 - \tfrac{1}{2}(10 + s_1 - 2s_2) - \tfrac{1}{2}s_1 \\ x &= 10 + s_1 - 2s_2 \\ z &= 45 + \tfrac{1}{2}(10 + s_1 - 2s_2) - \tfrac{3}{2}s_1 \end{aligned}$$

which is equivalent to

$$y = 10 - s_1 + s_2 \tag{7}$$
$$x = 10 + s_1 - 2s_2. \tag{8}$$
$$z = 50 - s_1 - s_2 \tag{9}$$

The basic solution is $(10,10,0,0)$ with $z=50$.

Since the reduced costs in (9) are all non-positive, this basic solution must be optimal. Hence the optimal solution is $(x,y)=(10,10)$ and the optimal objective function value is 50.

3(b)(ii) The shadow price formula is

$$\lambda = (B^T)^{-1} p_B \qquad \text{or} \qquad B^T \lambda = p_B$$

where $A = (B \quad N)$ and $p = (p_B \quad p_N)$ so that B is the optimal basic matrix. From 3(b)(i), we have

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$p = (2 \quad 3 \quad 0 \quad 0)^T, p_B = (2 \quad 3)^T.$$

Therefore we have

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^T \lambda = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

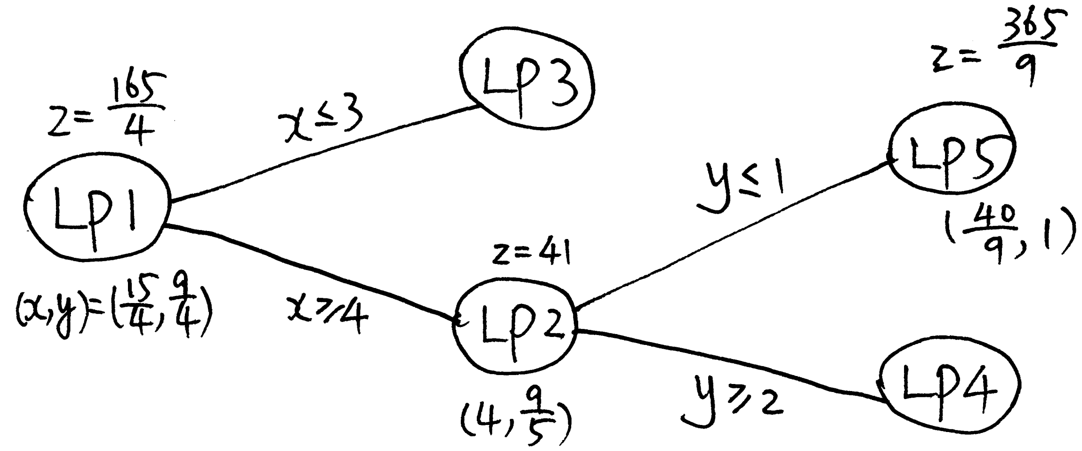
The solution is $\lambda_1 = 1$ and $\lambda_2 = 1$.

The shadow prices of both constraints are 1. The shadow price of a constraint means the profit that can be made by increasing one unit of the resource (RHS) corresponding to this constraint.

3(b)(iii) The reduced cost of x and y in the final and optimal solution are zeros (This is always true while the reduced costs for non-basic variables can be negative in a maximisation problem).

4(a)(i)

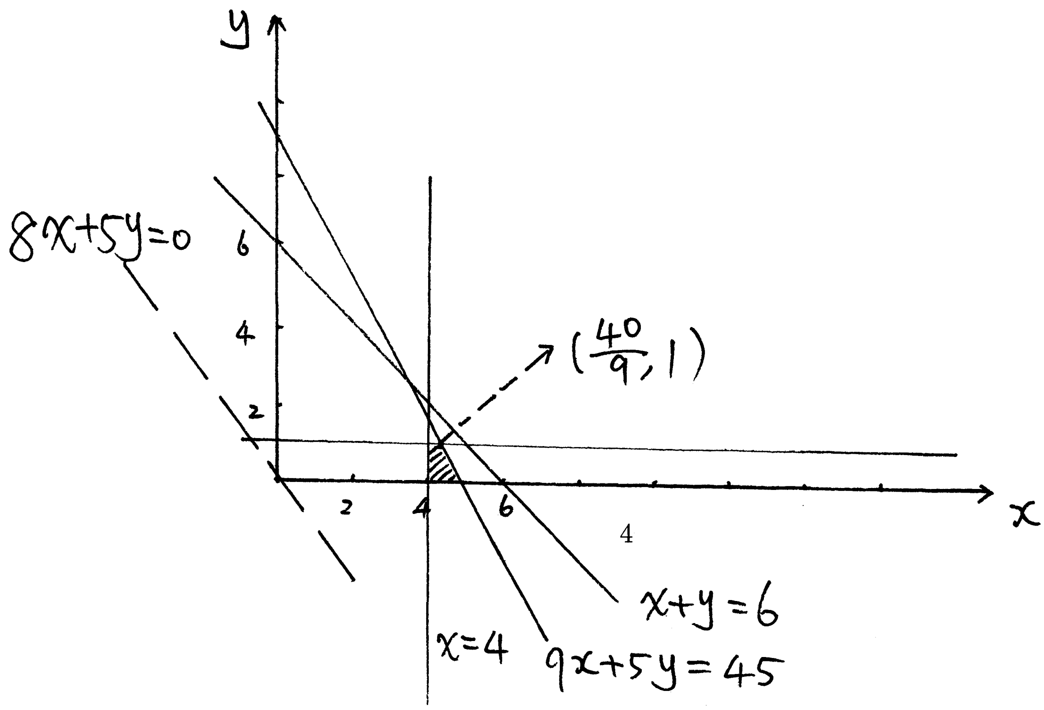
$$
\begin{aligned}
\max \quad & z = 8x + 5y \\
\text{subjectto} \quad & x + y \le 6 \\
& 9x + 5y \le 45 \\
& x, y \ge 0, \text{integer}
\end{aligned}
$$



The branch-and-bound tree has been constructed. We now continue to finish the branch-and-bound procedure for this example. Select LP5 as the next node. Form LP at node LP5:

$$
\begin{aligned}
\max \quad & z = 8x + 5y \\
\text{subjectto} \quad & x + y \le 6 \\
& 9x + 5y \le 45 \\
& x \ge 4 \\
& y \le 1 \\
& x, y \ge 0
\end{aligned}
$$

The optimal solution of LP5 is $x = \frac{40}{9}, y = 1$ and $z = \frac{365}{9}$. This optimal solution can be obtained using the graphically approach as shown in the following graph.

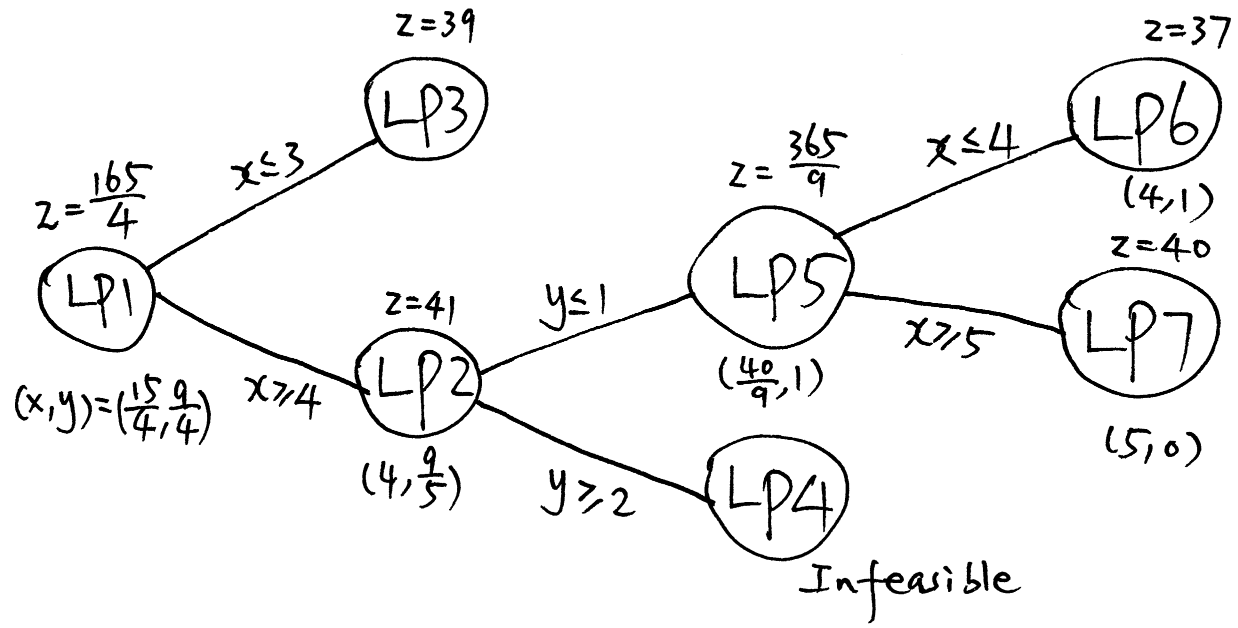


The optimal solution $(40/9, 1)$ for LP5 is still a fractional solution. Therefore we perform a branching step at node LP5 by adding $x \leq 4$ in one branch and $x \geq 5$ in another. We then obtain two more child nodes LP6 and LP7 of LP5 as shown in the second branch-and-bound tree.

We now select node LP6 for further explorations. Form the LP at LP6:

$$
\begin{array}{ll}
\max & z = 8x + 5y \\
\text{subjectto} & x + y \leq 6 \\
& 9x + 5y \leq 45 \\
& x \geq 4 \\
& y \leq 1 \\
& x \leq 4 \\
& x, y \geq 0
\end{array}
$$

It is easy to see that the optimal solution at LP6 is $x = 4$ and $y = 1$ with $z = 37$. We do not need to further explore at node LP6 since we have obtained an integer optimal solution.



We next select node LP7. Form the LP at node LP7:

$$
\begin{array}{ll}
\max & z = 8x + 5y \\
\text{subjectto} & x + y \leq 6 \\
& 9x + 5y \leq 45 \\
& x \geq 4 \\
& y \leq 1 \\
& x \geq 5 \\
& x, y \geq 0
\end{array}
$$

The optimal solution of the LP at node LP7 is $x = 5$ and $y = 0$, which is an integer solution. Hence there is no need to explore further at node LP7.

We next choose node LP4 and solve the LP at this node. It turns out that the LP at node LP4 is infeasible, which implies that no further exploration is needed at node LP4.

Then we choose node LP3 and solve the LP at this node. By the graphical approach, it is easy to find out the optimal solution at node LP3, which is $x = 3, y = 3$ with $z = 39$. No further exploration is required at node LP3.

From the branch-and-bound tree, we can see that all nodes have been explored. Therefore, the optimal solution to the ILP is $x = 5, y = 0$ with $z = 40$.

The order of the nodes that have been visited in the whole branch-and-bound tree is

$$LP1 \longrightarrow LP2 \longrightarrow LP5 \longrightarrow LP6 \longrightarrow LP7 \longrightarrow LP4 \longrightarrow LP3$$

Note that different orders may exist if students explore the branch-and-bound tree differently.

4(a)(ii) The LP at node LP5 is

$$\begin{array}{ll} \text{max} & z = 8x + 5y \\ \text{subjectto} & x + y \leq 6 \\ & 9x + 5y \leq 45 \\ & x \geq 4 \\ & y \leq 1 \\ & x, y \geq 0 \end{array}$$

4(a)(iii) There are several possibilities using this heuristic approach for solving ILP.

- The optimal solution of ILP is obtained
- An sub-optimal solution of ILP is obtained
- An infeasible solution to ILP is obtained

For the ILP considered in this question, we could apply this approach. The optimal solution of the LP at the root node is $(15/4, 9/4)$. If we round it down to the nearest integer point, we get (3, 2). $(x, y) = (3, 2)$ is feasible to the ILP, but it is not an optimal solution. Hence the heuristic approach fails to solve this ILP.

4(b) Define the decision variables

$x_i, i = 1, 2, 3$: The number of items to be made on machine i

$y_i, i = 1, 2, 3$: The binary variable so that $y_i = 1$ if and only if machine i is used to produce the product.

Then the objective is to minimise the total cost:

$$20x_1 + 24x_2 + 22x_3 + 1000y_1 + 950y_2 + 850y_3$$

The production constraint is

$$x_1 + x_2 + x_3 \geq 1800$$

The production capacity constraints are

$$\begin{aligned} x_1 &\leq 1000 \\ x_2 &\leq 1200 \\ x_3 &\leq 1400 \end{aligned}$$

The linking constraints are

$$\begin{aligned} x_1 &\leq 1000y_1 \\ x_2 &\leq 1200y_2 \\ x_3 &\leq 1400y_3 \end{aligned}$$

The linking constraints ensure that the fixed cost on machine i is taken into account if some items are produced on machine i.

The non-negative constraints and integrality constraints are

$$\begin{aligned} &x_1, x_2, x_3 \geq 0, \text{integer} \\ &y_1, y_2, y_3 \in \{0, 1\} \end{aligned}$$

The overall optimisation problem is

$$\begin{array}{ll} \text{min} & 20x_1 + 24x_2 + 22x_3 + 1000y_1 + 950y_2 + 850y_3 \\ \text{subjectto} & x_1 + x_2 + x_3 \geq 1800 \\ & x_1 \leq 1000 \\ & x_2 \leq 1200 \\ & x_3 \leq 1400 \\ & x_1 \leq 1000y_1 \\ & x_2 \leq 1200y_2 \\ & x_3 \leq 1400y_3 \\ & x_1, x_2, x_3 \geq 0, \text{integer} \\ & y_1, y_2, y_3 \in \{0, 1\} \end{array}$$