ENGINEERING TRIPOS PART IIA 2004

Solutions to Module 3F1
Signals and Systems
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Module 3F1, April 2004 – SIGNALS AND SYSTEMS – Solutions

1 (a) **Nyquist plot**

A Nyquist plot can often be most easily plotted by separating the frequency response of the system into its magnitude and phase components. For the given system this is achieved as follows.

$$\text{Let } H(e^{j\theta}) = G(e^{j\theta})\, C(e^{j\theta}) = \frac{e^{-10j\theta}}{1 - e^{-j\theta}} = \frac{e^{-10j\theta}}{e^{-j\theta/2}(e^{j\theta/2} - e^{-j\theta/2})} = \frac{e^{-9.5j\theta}}{2j\sin(\theta/2)}$$

$$\text{Hence} \qquad \angle H(e^{j\theta}) = -9.5\theta - \tfrac{\pi}{2} \qquad \text{and} \qquad |H(e^{j\theta})| = \frac{1}{2\sin(\theta/2)}$$

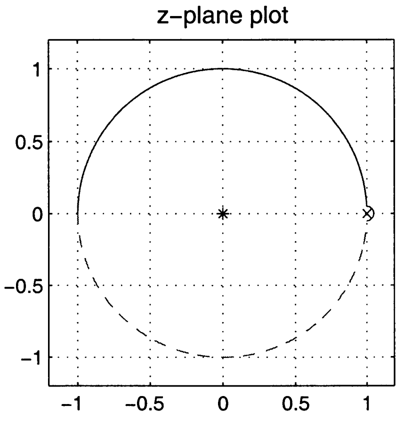
When $\theta = \pm\pi$: $\quad H(-1) = \frac{1}{2} \quad$ (end points)

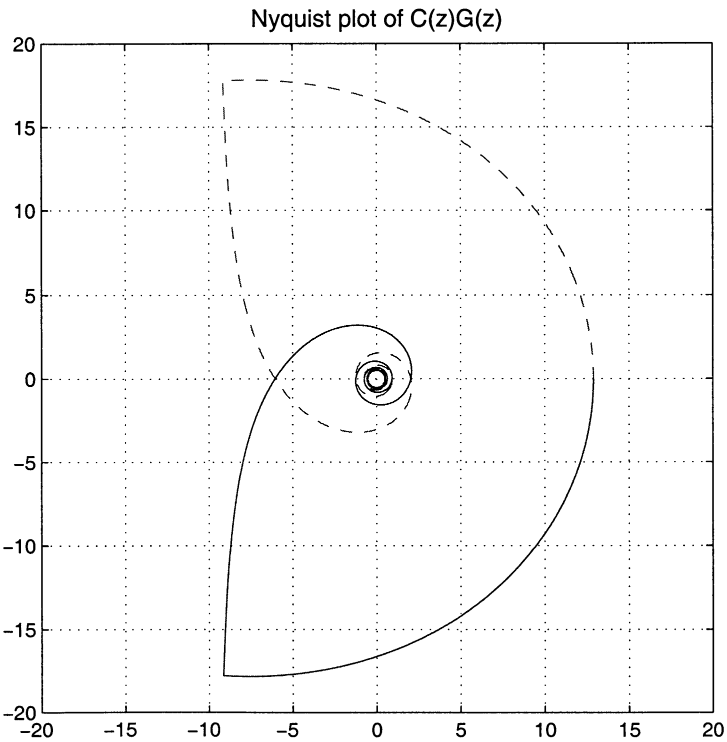
Near $\theta = 0$: $\quad H(e^{j\theta}) \simeq \dfrac{1 - 9.5j\theta + \ldots}{2j(\theta/2)} = -9.5 + \dfrac{1}{j\theta} \ldots \quad$ (asymptotes to $\pm j\infty$)

To find the poles in the z-plane, we note that $H(z) = \dfrac{1}{z^9(z-1)}$

Hence there are 9 poles at $z = 0$ and one pole at $z = 1$.

To avoid the pole at $z = 1$ we use an *indentation* of the z-plane trajectory (below left) to produce the Nyquist plot (below right). The ten poles result in a total of 10 anticlockwise encirclements of the origin in the full Nyquist plot as θ goes from $-\pi$ to π (five encirclements for each half of the Nyquist plot).





To find the point where the Nyquist plot crosses the negative real axis first:

$$\angle H(e^{j\theta_0}) = -\pi \text{ , so from the above formula for } \angle H,\ \theta_0 = \frac{\pi - \frac{\pi}{2}}{9.5} = 0.1653 \text{ rad.}$$

Hence $|H(e^{j\theta_0})| = \dfrac{1}{2\sin(\theta_0/2)} = 6.055$, and the point is therefore $(-6.055 + 0j)$. [40%]

(b) **Range of k for stability**

Since there are no unstable open-loop poles (the pole at $z = 1$ is *just* inside the indented z-plane trajectory), we require no encirclements by the Nyquist plot of the point $-\frac{1}{k}$ to obtain closed-loop stability. Therefore $-\frac{1}{k}$ must lie to the left of the real-axis crossing point $(-6.055 + 0j)$ and also not be anywhere on the positive real axis (the right-hand semicircular arc is actually out at almost infinity if the indentation around $z = 1$ is very small!).

Therefore $-\dfrac{1}{k} < -6.055$, and so $0 < k < 0.165$ for stability. [20%]

(c) **Phase margin**

At the phase margin frequency θ_1, the loop gain is of unit magnitude.

Therefore $|k\,H(e^{j\theta_1})| = \dfrac{0.1}{2\sin(\theta_1/2)} = 1$, and so $\sin(\theta_1/2) = 0.05$ and $\theta_1 = 0.1$ rad.

Phase margin $= \pi + \angle H(e^{j\theta_1}) = \pi - \dfrac{\pi}{2} - 9.5\theta_1 = \dfrac{\pi}{2} - 0.95 = 0.621 \text{ rad} = 35.6°$ [20%]

(d) **Largest delay m**

A delay of m sample periods gives an extra factor of z^{-m} in the loop transfer function. To maintain stability, we need the associated extra phase shift at $z = e^{j\theta_1}$ to be less than the phase margin of the original system.

Hence $m\,\theta_1 < 0.621$ rad , and so $m < 6.21$.

Therefore $m = 6$ is the largest delay for which the loop remains stable. [20%]

2 (a) **Discrete-time Transformations**

Forward difference (Euler's method):

$$\dot{x} \simeq \frac{x(t+T) - x(t)}{T}$$

Backward difference:

$$\dot{x} \simeq \frac{x(t) - x(t-T)}{T}$$

In terms of transform variables s and z, these correspond to the substitutions:

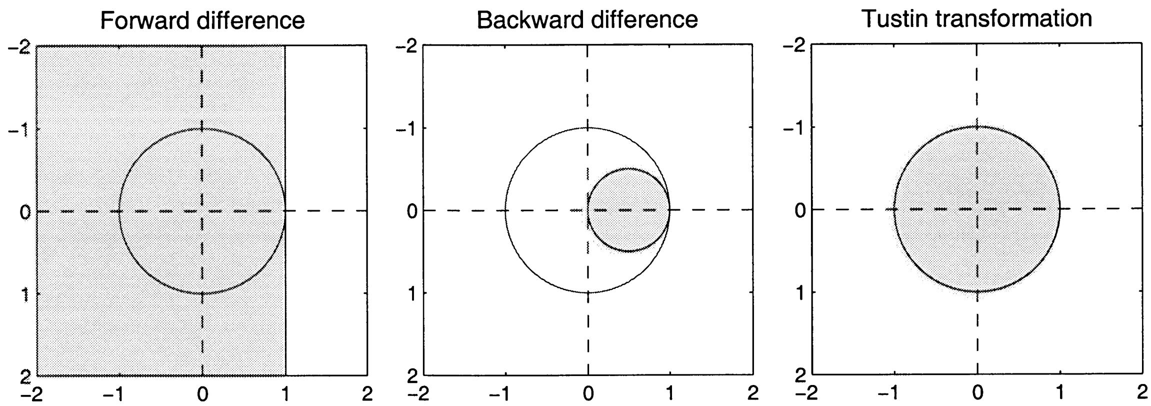
$$s = \frac{z-1}{T} \text{ (forward)} \qquad \text{and} \qquad s = \frac{z-1}{zT} \text{ (backward)}$$

The Tustin transformation is derived from the trapezoidal approximation for integration and is:

$$s = \frac{2}{T}\frac{z-1}{z+1}$$

In each case the discrete-time filter in z is obtained by making the appropriate substitution for s in the continuous-time filter. [25%]

Each transformation corresponds to a mapping between the s-plane and the z-plane. The shaded areas in the diagrams below show the regions of the z-plane which correspond to the left half of the s-plane.



Stable poles must lie in the left half of the s-plane, and within the unit circle of the z-plane. Hence backward difference or Tustin transformations, applied to stable continuous-time systems, result in stable discrete-time ones. This is not necessarily true for the forward difference transformation. [25%]

2 (b) **Transformation of a Random Variable**

If $g(X)$ is monotonic increasing, the cdf of Y is given by

$$F_Y(y) = \Pr\{Y \leq y\} = \Pr\{g(X) \leq g(x)\} = \Pr\{X \leq x\} = F_X(x)$$

since $Y = g(X)$ and $y = g(x)$.

The pdf of Y may then be found by differentiating wrt y:

$$f_Y(y) = \tfrac{d}{dy}\, F_Y(y) = \tfrac{d}{dy}\, F_X(x) = \left(\tfrac{d}{dx}\, F_X(x)\right) \tfrac{dx}{dy} = f_X(x)\, \tfrac{dx}{dy}$$

Defining $\frac{dy}{dx} = g'(x)$, $\qquad f_Y(y) = \dfrac{f_X(x)}{g'(x)}$ [20%]

For the given pdfs:

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \begin{cases} y\, e^{-y^2/2} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$\therefore \frac{dx}{dy} = \frac{f_Y(y)}{f_X(x)} = 2y\, e^{-y^2/2}$$

We can solve this differential equation by multiplying by dy and integrating both sides, provided that the constant of integration for $g(X)$ is chosen such that X and Y cover the correct ranges. Hence integrating with x and y both starting at the lower limits of their respective pdfs, gives:

$$\int_{-1}^{x} dx = \int_{0}^{y} 2y\, e^{-y^2/2}\, dy = \left[-2\, e^{-y^2/2}\right]_0^y$$

$$\therefore\ x + 1 = 2 - 2\, e^{-y^2/2} \quad \text{for } y \geq 0$$

We need y in terms of x, so: $e^{-y^2/2} = \dfrac{1-x}{2}$

$$\therefore \frac{-y^2}{2} = \ln\left(\frac{1-x}{2}\right)$$

$$\therefore\ y = \sqrt{-2\ln\left(\frac{1-x}{2}\right)}$$

Hence, in terms of the random variables X and Y:

$$Y = g(X) = \sqrt{-2\ln\left(\frac{1-X}{2}\right)}$$

[30%]

3 (a) **Characteristic Function**

It is a standard property of expectations that

$$E[g(X)] = E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx \quad \text{where } Y = g(X)$$

Hence for the given definition of the Characteristic Function:

$$\Phi_X(u) = E[e^{juX}] = \int_{-\infty}^{\infty} e^{jux}\, f_X(x)\, dx$$

Now (from the Data Book) the Fourier Transform of the pdf is given by

$$\mathcal{F}(u) = \int_{-\infty}^{\infty} f_X(x)\, e^{-jux}\, dx$$

Hence

$$\Phi_X(u) = \mathcal{F}(-u)$$

[25%]

(b) **Moments of X**

n^{th} derivative of Φ_X:

$$\begin{aligned} \frac{d^n\, \Phi_X(u)}{du^n} &= \int_{-\infty}^{\infty} (jx)^n\, e^{jux}\, f_X(x)\, dx \\ \text{Hence, at } u=0\text{:} \quad E[X^n] &= \int_{-\infty}^{\infty} x^n\, f_X(x)\, dx = \frac{1}{j^n} \left.\frac{d^n\, \Phi_X(u)}{du^n}\right|_{u=0} \end{aligned}$$

[20%]

(c) **Triangular pdf**

From the E&I Data Book, the Fourier Transform of the given triangular pdf is:

$$\mathcal{F}(u) = ab\, \text{sinc}^2\left(\frac{ub}{2}\right)$$

Using the result from part (a):

$$\begin{aligned} \Phi_X(u) &= \mathcal{F}(-u) = ab\, \text{sinc}^2\left(\frac{-ub}{2}\right) \\ &= \frac{4ab}{u^2b^2} \sin^2\left(\frac{ub}{2}\right) = \frac{2a}{u^2b}\, [1 - \cos(ub)] \\ &= \frac{2a}{u^2b}\left[1 - 1 + \frac{u^2b^2}{2!} - \frac{u^4b^4}{4!} + \frac{u^6b^6}{6!} - \ldots\right] \\ &= ab\left[1 - \frac{u^2b^2}{12} + \frac{u^4b^4}{360} - \ldots\right] \end{aligned}$$

[25%]

(d) **Moments**

Using the result from part (b):

$$E[X^0] = \frac{1}{j^0}\,\Phi_X(u)\big|_{u=0} = \Phi_X(0) = ab$$

But $E[X^0] = 1$ for a valid pdf, and so $a = \dfrac{1}{b}$.

Now differentiate Φ_X to get higher order moments:

$$E[X^1] = \frac{1}{j}\,\frac{d\,\Phi_X(u)}{du}\bigg|_{u=0} = \frac{1}{j}\left[-\frac{ub^2}{6} + \frac{u^3b^4}{90} - \ldots\right]_{u=0} = 0$$

$$E[X^2] = \frac{1}{j^2}\,\frac{d^2\,\Phi_X(u)}{du^2}\bigg|_{u=0} = -\left[-\frac{b^2}{6} + \frac{u^2b^4}{30} - \ldots\right]_{u=0} = \frac{b^2}{6}$$

$$E[X^3] = \frac{1}{j^3}\,\frac{d^3\,\Phi_X(u)}{du^3}\bigg|_{u=0} = j\left[\frac{ub^4}{15} - \ldots\right]_{u=0} = 0$$

$$E[X^4] = \frac{1}{j^4}\,\frac{d^4\,\Phi_X(u)}{du^4}\bigg|_{u=0} = \left[\frac{b^4}{15} - \ldots\right]_{u=0} = \frac{b^4}{15}$$

The kurtosis κ is given by:

$$\kappa = \frac{E[(X-\overline{X})^4]}{\sigma^4} - 3$$

Now

$$\overline{X} = E[X^1] = 0$$

and

$$\sigma = \sqrt{E[(X-\overline{X})^2]} = \sqrt{E[X^2]} = \frac{b}{\sqrt{6}}$$

$$\therefore\ \kappa = \frac{E[X^4]}{\sigma^4} - 3 = \frac{b^4}{15}\,\frac{36}{b^4} - 3 = \frac{12-15}{5} = -\frac{3}{5}$$

Kurtosis provides information about how ‘heavy’ the tails of the pdf are. The kurtosis of a Gaussian pdf is zero, so the negative kurtosis of the given triangular pdf means that it has less heavy tails than a Gaussian. [30%]

4 (a) **A memoryless source**

A memoryless source is one in which each symbol is generated independently of those that came before it. [10%]

(b) **Simple Decoder**

The probability of receiving an A given a 0 was sent is:

$$P(A|0) = \int_0^{0.5} (2 - 2V)\, dV = 0.75$$

(This probability can be easily deduced from looking at the triangular graph of the pdf.)

Hence $P(B|0) = 0.25$.

Similarly $P(A|1) = 0.25$ and $P(B|1) = 0.75$.

Hence $P(A) = P(A|0)P(0) + P(A|1)P(1) = 0.75 * 0.5 + 0.25 * 0.5 = 0.5$
and $P(B) = 1 - P(A) = 0.5$.

Mutual information, I(output;input) = H(output) $-$ H(output|input).

H(output) = $-(0.5\log_2(0.5) + 0.5\log_2(0.5)) = 1$

H(output|input) = H(output|0) P(0) + H(output|1) P(1)

H(output|0) = $-(0.75\log_2(0.75) + 0.25\log_2(0.25))$ = 0.3113 + 0.5 = 0.8113

and H(output|1) = $-(0.25\log_2(0.25) + 0.75\log_2(0.75))$ = 0.8113

so H(output|input) = 0.8113

and hence I(output;input) = H(output) $-$ H(output|input) $= 1 - 0.8113 = 0.1887$ [30%]

(c) **Soft Decoder**

Repeat the same process - except there are now four output symbols with probabilities given by:

$$P(A|0) = 7/16 = 0.4375$$
$$P(B|0) = 5/16 = 0.3125$$
$$P(C|0) = 3/16 = 0.1875$$
$$P(D|0) = 1/16 = 0.0625$$

$$P(A|1) = 0.0625$$
$$P(B|1) = 0.1875$$
$$P(C|1) = 0.3125$$
$$P(D|1) = 0.4375$$

Hence H(output|0) = H(output|1) = $-(0.0625\log_2(0.0625) + 0.1875\log_2(0.1875) + 0.3125\log_2(0.3125) + 0.4375\log_2(0.4375)) = 1.7490$

So H(output|input) = 1.7490

Now $P(A) = P(B) = P(C) = P(D) = 0.25$

Hence H(output) $= -4\,(0.25\log_2(0.25)) = 2$

So I(output;input) $= 2 - 1.7490 = 0.2510$

This value is higher than for the simple decoder. By decoding into more symbols, the soft decoder is able to capture more information from the received electrical signal. [35%]

(d) **Continuous (unquantised) Decoder**

If the voltage range is split into a very large number of bins, we need to calculate the mutual information between the continuous variable V and the transmitted symbols. This requires calculating two entropies.

The probability density of receiving a voltage V, in the range $0 \leq V < 1$, is given by

$$P(V) = P(0)P(V|0) + P(1)P(V|1) = 0.5 * (2 - 2V) + 0.5 * 2V = 1$$

Thus $H(\text{output}) = H(V) = -\int_0^1 1 \ \log_2(1) \ dV = 0$

For $H(\text{output}|\text{input})$:

$$\begin{aligned}
H(V|0) &= -\int_0^1 2V \log_2(2V) \ dV \\
&= -\int_0^1 2V \left(1 + \log_2(V)\right) dV \\
&= -\int_0^1 2V \ dV - \int_0^1 \frac{2V}{\ln(2)} \ln(V) \ dV \\
&= -1 - \frac{2}{\ln(2)} \int_0^1 V \ln(V) \ dV \\
&= -1 - \frac{2}{\ln(2)} \left[\frac{V^2 \ln(V)}{2} - \frac{V^2}{4}\right]_0^1 \\
&= -1 + \frac{1}{2\ln(2)} = -0.2789
\end{aligned}$$

Similarly $H(V|1) = -0.2789$

Hence $I(V;\text{input}) = H(V) - H(V|\text{input}) = 0 - (-0.2789) = 0.2789$

Note that this limiting result for many bins is only a little larger than when there were only 4 bins in part (c)! [25%]