**ENGINEERING TRIPOS PART IIA 2004**

Solutions to Module 3F2
Systems and Control

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# Module 3F2 - SYSTEMS AND CONTROL - 2004

## SOLUTIONS

1 (a) $\dot{x} = Ax + Bu, \quad y = Cx + Du$

Take Laplace transforms: $s\bar{x}(s) - x(0) = A\bar{x}(s) + B\bar{u}(s)$

and $\bar{y}(s) = C\bar{x}(s) + D\bar{u}(s)$

Hence $(sI - A)\bar{x}(s) = B\bar{u}(s) + x(0)$

so $\underline{\bar{x}}(s) = (sI-A)^{-1}B\bar{u}(s) + (sI-A)^{-1}x(0)$

For transfer function assume that $x(0) = 0$.

Hence $\bar{y}(s) = \underline{\underline{\left[C(sI-A)^{-1}B + D\right]}}\,\bar{u}(s)$

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(b) $A = -\frac{1}{T_a}, \quad B = \frac{1}{10}, \quad C = \frac{9K_a}{T_a}, \quad D = \frac{K_a}{10}$

Hence, from (a), $K(s) = \dfrac{\frac{9K_a}{T_a} \times \frac{1}{10}}{s + \frac{1}{T_a}} + \dfrac{K_a}{10}$

ANY OF THESE (AND OTHERS) ACCEPTABLE

$$= \frac{1}{10}\,\frac{\frac{9K_a}{T_a} + K_a\left(s + \frac{1}{T_a}\right)}{s + \frac{1}{T_a}}$$

$$= \underline{\underline{\frac{K_a}{10} \times \frac{sT_a + 10}{sT_a + 1}}}$$

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(c)

$$\frac{\bar{\psi}(s)}{\bar{\rho}(s)} = \frac{K}{s(1+sT)} \Rightarrow s(1+sT)\bar{\psi}(s) = K\bar{\rho}(s) \Rightarrow \dot{\psi} + T\ddot{\psi} = K\rho$$

Let $\underline{x} = \begin{bmatrix} \psi \\ \dot{\psi} \end{bmatrix}$. Then $\dot{x}_1 = \dot{\psi} = x_2$ and $\dot{x}_2 = \ddot{\psi}$

$$= -\frac{x_2}{T} + \frac{K\rho}{T}$$

Hence $\underline{\underline{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1/T \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ K/T \end{bmatrix}\rho, \quad \psi = [1, 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0\rho}}$

(d) Let $\underline{x}_c = \begin{bmatrix} x_1 \\ x_2 \\ x_a \end{bmatrix}$ be the state vector of the closed loop.

From (b) we have
$$\dot{x}_a = -\frac{1}{T}x_a + \frac{1}{10}(\psi_r - \psi)$$
$$= -\frac{1}{T}x_a + \frac{1}{10}\psi_r - \frac{1}{10}x_1$$

$\psi_r$ is External input

So putting this together with result from (c):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1/T & 0 \\ -1/10 & 0 & -1/T_a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_a \end{bmatrix} + \begin{bmatrix} 0 \\ K/T \\ 0 \end{bmatrix}\rho + \begin{bmatrix} 0 \\ 0 \\ 1/10 \end{bmatrix}\psi_r$$

But from (b),
$$\rho = \frac{9K_a}{T_a}x_a + \frac{K_a}{10}(\psi_r - x_1)$$
$$= \underbrace{\begin{bmatrix} -\frac{K_a}{10}, & 0, & \frac{9K_a}{T_a} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_a \end{bmatrix}}_{\text{state feedback}} + \frac{K_a}{10}\psi_r$$

so
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_a \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1/T & 0 \\ -1/10 & 0 & -1/T_a \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -\frac{KK_a}{10T} & 0 & \frac{9KK_a}{TT_a} \\ 0 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_a \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{KK_a}{10T} \\ \frac{1}{10} \end{bmatrix}\psi_r$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{KK_a}{10T} & -\frac{1}{T} & \frac{9KK_a}{TT_a} \\ -1/10 & 0 & -1/T_a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_a \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{KK_a}{10T} \\ \frac{1}{10} \end{bmatrix}\psi_r$$

$$\psi = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_a \end{bmatrix}$$ since $x_1 = \psi$ by definition.

(e) Integral action can be incorporated by adding another state variable, say $x_4$, such that

$$\dot{x}_4 = \psi_r - \psi = e \text{ (the heading error)},$$

and making $\rho$ depend on $x_4$ as well as $x_1$ and $x_2$.

The ship transfer function already has an integrator — see (6). So an integrator is not needed to ensure zero steady-state heading error *if* there are no input disturbances (or if these are zero on average).

But there *are* likely to be steady input disturbances, eg currents, wind, which act in a similar way to the rudder. So integral action *is* likely to be needed in practice.

[One candidate pointed out that such steady disturbances are more likely to be corrected by adjusting the value of $\psi_r$. (That is the traditional solution adopted in navigation.)]

[Credit was awarded for any sensible discussion along the lines outlined above.]

2. (a) Poles are at 0, −2, −20.

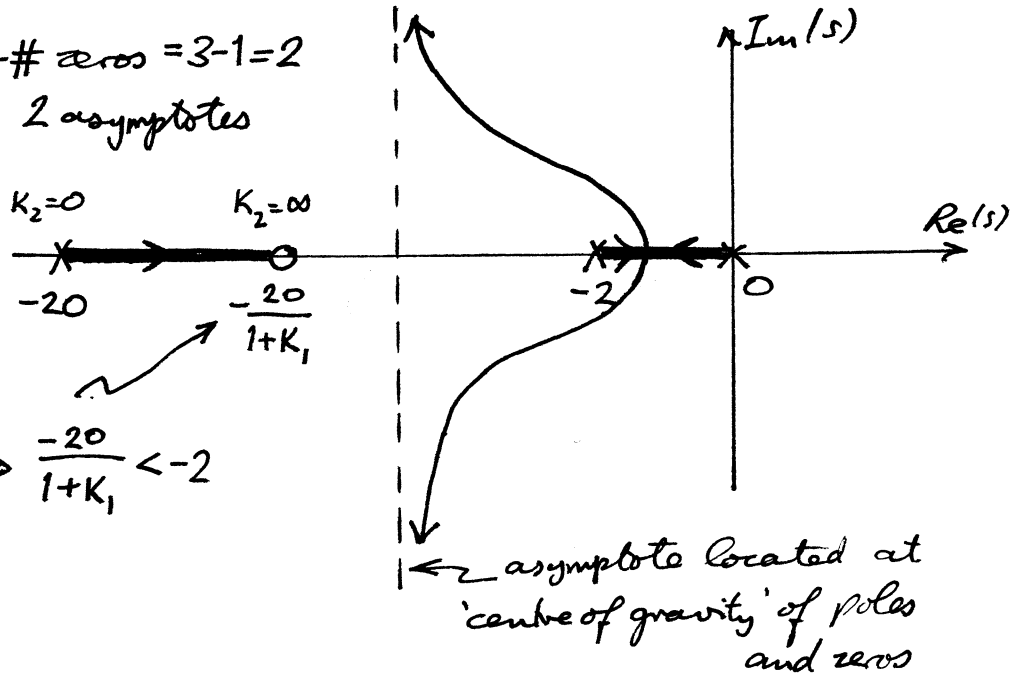
Zeros: Compensator is $\dfrac{(1+K_1)s+20}{s+20}$

so there is a zero at $\dfrac{-20}{1+K_1}$. $K_1>0 \Rightarrow \dfrac{-20}{1+K} > -20$.

So the root-locus diagram for $K_2$ variations is:

#poles − # zeros = 3−1 = 2

⇒ 2 asymptotes




$K_1<9 \Rightarrow \dfrac{-20}{1+K_1} < -2$

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(b) The point at which the asymptotes intersect the real axis is

$$\frac{\Sigma \text{ poles} - \Sigma \text{ zeros}}{\#\text{poles} - \#\text{zeros}} = \frac{(0-2-20)-\left(\frac{-20}{1+K_1}\right)}{3-1}$$

$$= -11 + \frac{10}{1+K_1}.$$

So increasing $K_1$ moves the asymptotes to the left, hence 'pulls' the complex branches of the diagram leftwards. That is, it tends to improve closed-loop stability.

As $K_1 \to 9$, the point of intersection of the asymptotes approaches $-11 + \frac{10}{10} = -10$.

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(c) <u>Either</u> consider the closed-loop characteristic equation:

$$1 + \frac{[(1+K_1)s + 20]\,0{\cdot}1\,K_2}{s(s+2)(s+20)} = 0 \quad \text{and } K_1 = 9$$

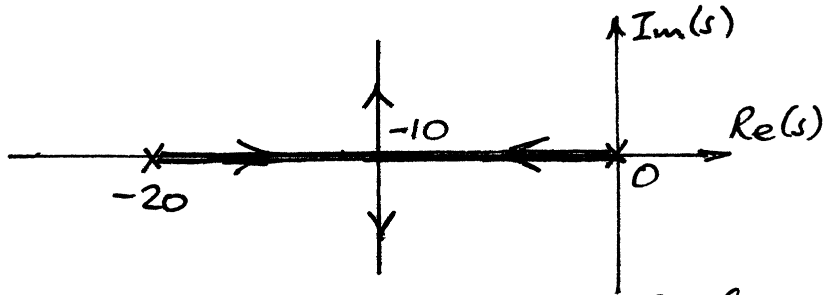
gives $$1 + \frac{K_2\cancel{(s+2)}}{s\cancel{(s+2)}(s+20)} = 0$$

hence $s(s+20) + K_2 = 0$

which is $(s+10)^2 = 0$ if $\underline{\underline{K_2 = 100}}$

<u>Or</u> use the root-locus diagram:

Note that the zero is now at $\frac{-20}{1+9} = -2$
so it cancels the pole at $-2$.
So the R-L diagram simplifies to:




so for critical damping both closed-loop poles must be at $-10$.

The open-loop transfer function is

$$\frac{K_2}{s(s+20)}$$ call this $K_2\,G(s)$.

Using the rule for calibrating R-L diagrams, the gain required to put both poles at $-10$ is

$$K_2 = \frac{1}{|G(-10)|} = 10 \times 10 = \underline{\underline{100}}$$

[Surprisingly many candidates lost marks by not knowing what 'critical damping' means. Most commonly it was interpreted as $\xi = 1/\sqrt{2}$ rather than $\xi = 1$.]

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(d) The closed-loop characteristic equation can be written as [see (c)]:

$$s(s+2)(s+20) + 0{\cdot}1K_2\left[(1+K_1)s + 20\right] = 0$$

or

$$\underbrace{\left[s(s+2)(s+20) + 0{\cdot}1\,K_2(s+20)\right]}_{K_1 \text{ not in this term}} + 0{\cdot}1\,K_2K_1 s = 0$$

or $$1 + \frac{0{\cdot}1\,K_2K_1\,s}{(s+20)\left[s(s+2)+0{\cdot}1K_2\right]} = 0$$

But $K_2 = 100$

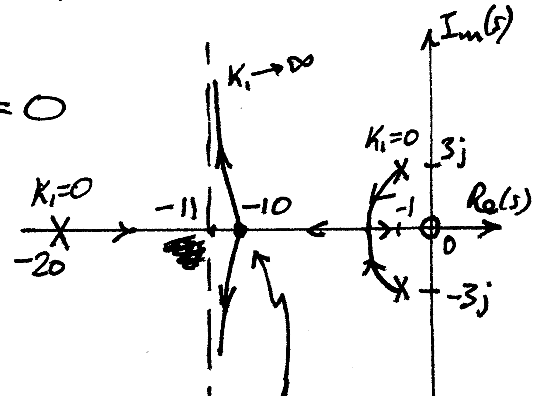
hence $$1 + \frac{10K_1\,s}{(s+20)(s^2+2s+10)} = 0$$

Poles at $-20$, $-1 \pm 3j$

Zero at $0$.

Hence R-L diagram

[2 asymptotes, crossing real axis at $\dfrac{(-20-2)-(0)}{3-1} = -11$]




NB we know from (c) that $K_1 = 9$ gives both poles at $-10$.

3. (a) Stability of $\underline{\dot{x}} = A\underline{x} + B\underline{u}$ depends on the eigenvalues of $A$ in the following way:

<u>All</u> eigenvalues in open LHP (ie excluding imaginary axis): asymptotically stable.

<u>Any</u> eigenvalue in open RHP: unstable.

All eigenvalues in closed LHP, with only simple $\underbrace{\text{(not repeated)}}_{\text{not quite precise, but OK.}}$ eigenvalues on imaginary axis: marginally stable.

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(b) We need eigenvalues of $A$:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 0 \\ -S/M & \lambda & G/M \\ 0 & -S/G & \lambda + \frac{R}{L} \end{bmatrix}$$

$$= \lambda \begin{vmatrix} \lambda & G/M \\ -S/G & \lambda + R/L \end{vmatrix} + 1 \begin{vmatrix} -S/M & G/M \\ 0 & \lambda + R/L \end{vmatrix}$$

$$= \lambda\left[\lambda(\lambda + R/L) + S/M\right] - S/M\,(\lambda + R/L)$$

$$= \left[\lambda^3 + \frac{R}{L}\lambda^2 + \cancel{\lambda \frac{S}{M}}\right] - \left[\cancel{\lambda\frac{S}{M}} + \frac{SR}{ML}\right]$$

$$= \lambda^3 + \frac{R}{L}\lambda^2 - \frac{SR}{ML}$$

But, from Routh-Hurwitz criteria (Data Book) a polynomial $s^3 + a_1 s^2 + a_2 s + a_3$ has all roots in open LHP only if $a_i > 0$ for each $i$. In this case $a_2 = 0$ and $a_3 < 0$ so at least one eigenvalue is <u>not</u> in open LHP, hence system <u>not</u> asymptotically stable.

A <u>much</u> smarter solution was spotted by a few candidates, using the fact that 'product of eigenvalues = determinant': Let the eigenvalues be $\lambda_1, \lambda_2, \lambda_3$. Then

$$\lambda_1 \lambda_2 \lambda_3 = \det(A) = -1 \times \begin{vmatrix} S/M & -G/M \\ 0 & -R/L \end{vmatrix}$$

$$= \frac{SR}{ML} > 0$$

So at least one of $\lambda_1, \lambda_2, \lambda_3$ must be real and positive. So system is unstable.

[An unintended difficulty for several candidates was that, if they used 's' instead of '$\lambda$' and evaluated $\det(sI-A)$ then lower-case and upper-case 's' and 'S' got confused with each other in the algebra.]

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(c) Controllability test: $\text{rank}[B, AB, A^2B] = 3$?

We have:

$$[B, AB, A^2B] = \left[\begin{array}{c|c|c} 0 & 0 & -G/ML \\ 0 & -G/ML & GR/ML^2 \\ 1/L & -R/L^2 & -H/ML + R^2/L^3 \end{array}\right]$$

It is clear by inspection (because of the 'triangular' structure) that the rows — or columns — are linearly independent, so the rank is 3 (if $G \neq 0$). $\therefore$ <u>controllable</u>

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(d) With the given values we have

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 250 & 0 & -1/4 \\ 0 & 1000 & -30 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u$$

Let $u = -[k_1, k_2, k_3]\underline{x}$ (state feedback)

Then $\dot{\underline{x}} = A\underline{x} + Bu = A\underline{x} - B\underline{k}^T\underline{x} = (A - B\underline{k}^T)\underline{x}$

or:

$$\dot{\underline{x}} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 250 & 0 & -1/4 \\ 0 & 1000 & -30 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2k_1 & 2k_2 & 2k_3 \end{bmatrix} \right\} \underline{x}$$

$$= \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 250 & 0 & -1/4 \\ -2k_1 & 1000-2k_2 & -30-2k_3 \end{bmatrix}}_{\text{call this } A_c.} \underline{x}$$

Find closed-loop poles:

$$\det(sI - A_c) = \det \begin{bmatrix} s & -1 & 0 \\ -250 & s & \frac{1}{4} \\ 2k_1 & 2k_2-1000 & s+30+2k_3 \end{bmatrix}$$

$$= s \begin{vmatrix} s & 1/4 \\ 2k_2-1000, & s+30+2k_3 \end{vmatrix} + 1 \begin{vmatrix} -250 & 1/4 \\ 2k_1 & s+30+2k_3 \end{vmatrix}$$

$$= s\left(s^2 + [30+2k_3]s - \left[\frac{k_2}{2} - 250\right]\right) + (-250)(s+30) - 500k_3 - \frac{k_1}{2}$$

$$= s^3 + [30+2k_3]s^2 + \left[\cancel{250} - \frac{k_2}{2} - \cancel{250}\right]s - \left[7500 + 500k_3 + \frac{k_1}{2}\right]$$

A.3 cont'd.

But we want

$$\det(sI - A_c) = (s+3)(s+1+2j)(s+1-2j)$$
$$= (s+3)(s^2+2s+5)$$
$$= s^3 + 5s^2 + 11s + 15$$

Hence we need

$$30 + 2k_3 = 5 \quad \Rightarrow \quad \underline{\underline{k_3 = -\frac{25}{2}}}$$

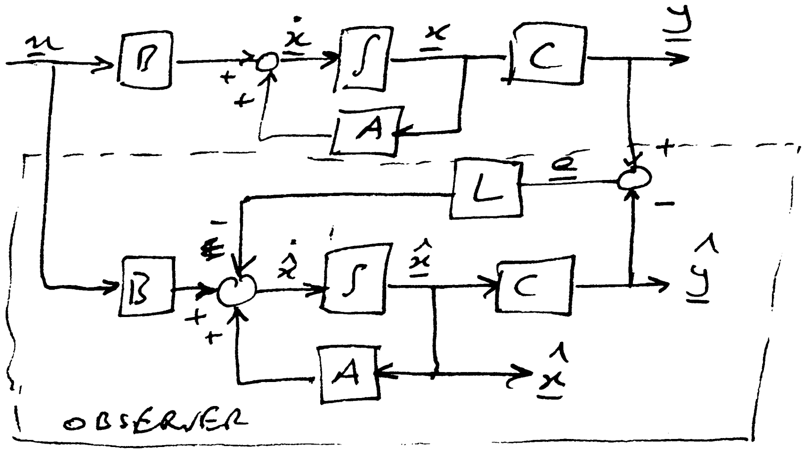
$$-\frac{k_2}{2} = 11 \quad \Rightarrow \quad \underline{\underline{k_2 = -22}}$$

$$-7500 - 500k_3 - \frac{k_1}{2} = 15 \quad \Rightarrow \quad -7500 + 6250 - \frac{k_1}{2} = 15$$

$$\Rightarrow k_1 = 2(-15 - 1250)$$

$$\underline{\underline{= -2530}}$$

A.4 (a) A state observer is used to estimate the state of a system from its input and output measurements. It consists of a copy of the system, as shown in the diagram, with feedback from the output error to correct the state estimate.




(b)

Let $\underline{e} = \underline{x} - \hat{\underline{x}}$ be the state estimation error.

Since $\dot{\underline{x}} = A\underline{x} + B\underline{u}$

and $\dot{\hat{\underline{x}}} = A\hat{\underline{x}} + B\underline{u} + L(\underline{y} - \hat{\underline{y}})$

$= A\hat{\underline{x}} + B\underline{u} + L(C\underline{x} - C\hat{\underline{x}})$

we have $\dot{\underline{e}} = \dot{\underline{x}} - \dot{\hat{\underline{x}}} = A(\underline{x} - \hat{\underline{x}}) - LC(\underline{x} - \hat{\underline{x}})$

$= (A - LC)(\underline{x} - \hat{\underline{x}})$

$= (A - LC)\,\dot{\underline{e}}$

Hence $\underline{e}(t) = \exp([A-LC]t)\,\underline{e}(0)$

so the observer is asymptotically stable if and only if each eigenvalue of $(A-LC)$ has negative real part.

A.4 (c) $\delta$, $\theta$ and $z$ are measured.

$\theta$ and $z$ are outputs, so the 'C' matrix is

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The state vector can be estimated if the system is observable, i.e. if

$$\text{rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = 4 \qquad (\text{since state dimension} = 4).$$

Now
$$CA = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -u_0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -u_0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & -u_0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

It can be seen by inspection that the first 4 rows (or the columns) are linearly independent; hence the matrix has rank 4.
(No need to evaluate $CA^2$ or $CA^3$.)

Hence the entire state vector can be estimated from measurements of $\delta$, $z$ and $\theta$.

[Pleasingly well answered question. Almost everyone knew what they were doing.]