ENGINEERING TRIPOS PART IIA 2004

Solutions to Module 3F4
Data Transmission
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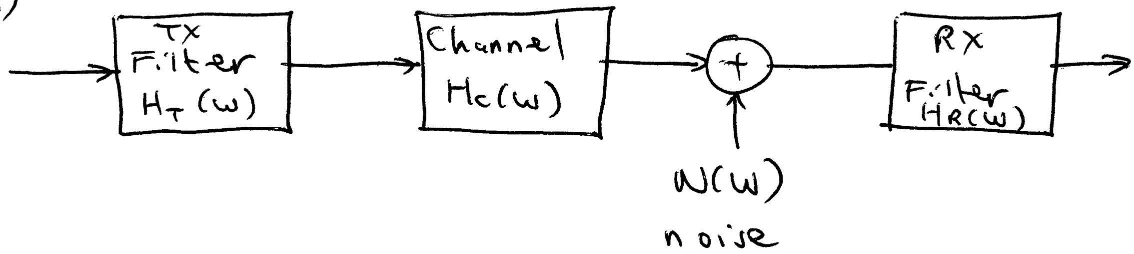
Module 3F4 - Data Transmission
Solutions Q4

1. a)



Noise assumptions – Gaussian pdf with zero mean and uniform psd (i.e, white)

b) From Parseval's Theorem

$$E_T = \int_{-\infty}^{\infty} h_T(t)^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H_T(\omega)|^2 \, d\omega$$

we are told that

$$H_T(\omega) \, H_C(\omega) \, H_R(\omega) = k \, P_R(\omega).$$

Substituting into the expression for E_T gives,

$$E_T = \frac{1}{2\pi} \int k^2 \frac{|P_R(\omega)|^2}{|H_C(\omega)|^2 \, |H_R(\omega)|^2} \, d\omega$$

c) Eye opening $= V_1 - 0 = V_1 = A_1 h(0)$
$= A_1 k \, p_R(0)$
$= k$

Noise power $= \sigma_v^2$

$$\therefore \frac{\text{eye opening}}{\sqrt{\text{noise power}}} = \frac{k}{\sigma_v^2}$$

To maximise this we should minimise $\frac{\sigma_v}{k}$ (or equivalently $\frac{\sigma_v^2}{k^2}$.

So,

$$\sigma_v^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\omega) \, |H_R(\omega)|^2 \, d\omega$$

From part (b),

$$k^2 = \frac{E_T}{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|P_R(\omega)|^2}{|H_c(\omega)|^2 |H_R(\omega)|^2} d\omega}$$

So,

$$\frac{\sigma_v^2}{k^2} = \frac{1}{(2\pi)^2 E_T} \int_{-\infty} N(\omega)|H_R(\omega)|^2 d\omega \int_{-\infty} \frac{|P_R(\omega)|}{|H_c(\omega)|^2 |H_R(\omega)|^2} d\omega$$

Schwartz's inequality states

$$\int_{-\infty} |F(\omega)|^2 d\omega \int |G(\omega)|^2 d\omega \geq \left| \int_{-\infty}^{\infty} F(\omega)\, G(\omega)\, d\omega \right|^2$$

with equality if $F(\omega) = \lambda\, G^*(\omega)$, where λ is an arbitrary constant.

Compare with previous expression for $\frac{\sigma_v^2}{k^2}$ and let
$F(\omega) = \sqrt{N(\omega)}\,|H_R(\omega)|$ and

$$G(\omega) = \frac{|P_R(\omega)|}{|H_c(\omega)|\,|H_R(\omega)|}$$

yielding,

$$\frac{\sigma_v^2}{k^2} \geq \frac{1}{(2\pi)^2 E_T} \left| \int_{-\infty}^{\infty} \sqrt{N(\omega)} \frac{|P_R(\omega)|}{|H_c(\omega)|} d\omega \right|^2$$

The given channel noise spectrum is,

$$N(\omega) = \frac{N_0}{2} |H_c(\omega)|^2, \quad \text{so}$$

$$\frac{\sigma_v^2}{k^2} \geq \frac{1}{(2\pi)^2 E_T} \left| \int_{-\infty}^{\infty} \sqrt{\frac{N_0}{2}}\, |P_R(\omega)|\, d\omega \right|^2$$

$$\geq \frac{N_0}{2E_T} \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} |P_R(\omega)|\, d\omega \right|^2$$

Now,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |P_R(\omega)|\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_R(\omega)\, d\omega \quad \text{since}$$

$P_R(\omega)$ is real and positive (as stated in question).

So,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} P_R(\omega)\, e^{j\omega t}\, d\omega = p_R(t)$$ (this is the IFFT)

and at $t=0$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} P_R(\omega)\, d\omega = p_R(0) = 1 .$$

So, the maximum value of

$$\frac{\text{eye opening}}{\sqrt{\text{noise power}}} = \frac{k}{\sigma_v} = \sqrt{\frac{2E_T}{N_0}} .$$

2.) a) The meaning of systematic linear binary blockcode:

Block code: Data symbols are grouped into blocks, and each block of data is separately coded into a single codeword.

Binary: Each data and codeword symbol is either '1' or '0'.

Linear: Any two valid codewords when added together using modulo-2 arithmetic produce a valid codeword.

Systematic: Any valid codeword contains the corresponding data word as part of the code word (usually the first part).

b) $$c = dG = d \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

ie, the codewords are linear combinations of the a_i vectors.

Now,

$s = c_r H^T$ where $c_r = c + e$.

$s = (c+e) H^T$

$s = cH^T + eH^T$

For valid code words, $cH^T = 0$ so

$s = eH^T$

c) $$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \therefore \begin{array}{l} a_1 = 101110 \\ a_2 = 010111 \end{array}$$

There are 4 possible data words, i.e, 00, 01, 10 and 11.

codeword c_i is the linear combination of the g_i vectors, so

$$\begin{aligned}
d_1 &= 00 \quad \therefore \quad c_1 = 000000 \\
d_2 &= 01 \quad \therefore \quad c_2 = 010111 \\
d_3 &= 10 \quad \therefore \quad c_3 = 101110 \\
d_4 &= 11 \quad \therefore \quad c_4 = 111001
\end{aligned}$$

All non-zero codewords have distance 4 from the all-zero codeword (and also each other) so,

$$d_{min} = 4.$$

$\therefore$ max number of detectable errors $= d_{min} - 1 = 3$

$\therefore$ max number of correctable errors $= \left\lfloor \frac{d_{min}-1}{2} \right\rfloor = 1$

d) From (b), $s = eH^T$

Now for a systematic code

$G = [I \mid P]$ and $H = [-P^T \mid I]$

For the given (6,2) code then

I is a 2×2 identity matrix in G and

I is a $(6-2=4)$ 4×4 identity matrix in H, so

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \therefore \quad H = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad H^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For all possible single-bit error patterns

e						$s = eH^T$			
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	1
0	0	0	0	1	0	0	0	1	0
0	0	0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0	0	0
0	1	0	0	0	0	0	1	1	1
1	0	0	0	0	0	1	1	1	0

For the (6,2) code there are $2^{6-2} = 2^4 = 16$ possible syndromes.

1 syndrome for zero errors

6 syndromes for all 1-bit errors

$\frac{6(6-1)}{2} = 15$ syndromes for all 2-bit errors.

So for all upto 1-bit errors need 7 syndromes

" " " " 2-bit errors need 22 syndromes.

See code has 16 available syndromes, ∴ code is <u>not</u> perfect.

3. (a) **Phasors**

Let the modulated wave be:

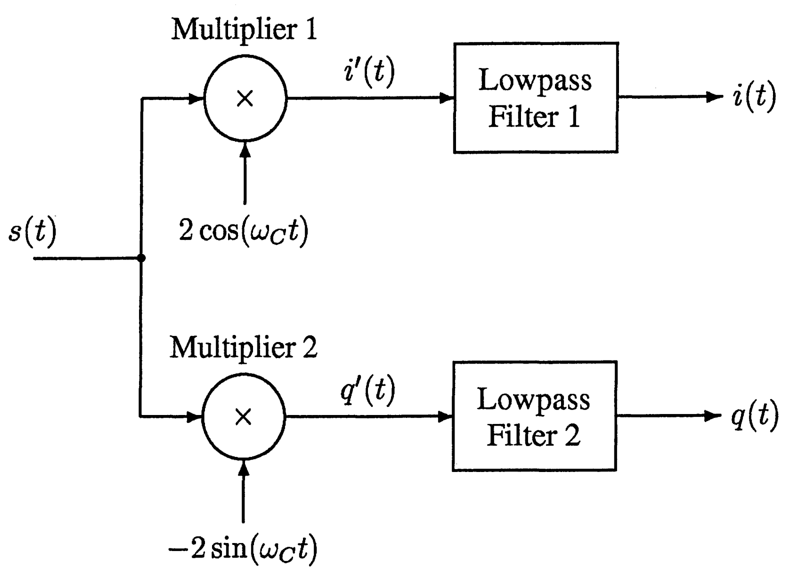
$$s(t) = a(t)\cos(\omega_C t + \phi(t))$$

Note $a(t)$ and $\phi(t)$ are difficult to combine, so we consider the cosine term as the real part of a complex exponential:

$$\begin{aligned} s(t) &= Re[a(t)\, e^{j(\omega_C t + \phi(t))}] \\ &= Re[a(t)\, e^{j\phi(t)}\, e^{j\omega_C t}] \\ &= Re[\underbrace{p(t)}_{\substack{\text{modulation}\\ \text{phasor}}} \quad \underbrace{e^{j\omega_C t}}_{\substack{\text{carrier}\\ \text{wave}}}] \end{aligned}$$

$$\text{where } p(t) = \underbrace{a(t)}_{\substack{\text{ampl.}\\ \text{of } p(t)}} \quad \underbrace{e^{j\phi(t)}}_{\substack{\text{phase}\\ \text{of } p(t)}}$$

(b) **Quadrature Demodulator:**



A Quadrature Demodulator, shown above, obtains $i(t)$ and $q(t)$, the real and imaginary parts of $p(t)$, from the modulated signal $s(t)$.

From multiplier 1:

$$\begin{aligned} i'(t) &= s(t) \times 2\cos(\omega_C t) \\ &= [i(t)\cos(\omega_C t) - q(t)\sin(\omega_C t)] \times 2\cos(\omega_C t) \\ &= 2i(t)\cos^2(\omega_C t) - 2q(t)\sin(\omega_C t)\cos(\omega_C t) \\ &= i(t) + i(t)\cos(2\omega_C t) - q(t)\sin(2\omega_C t) \end{aligned}$$

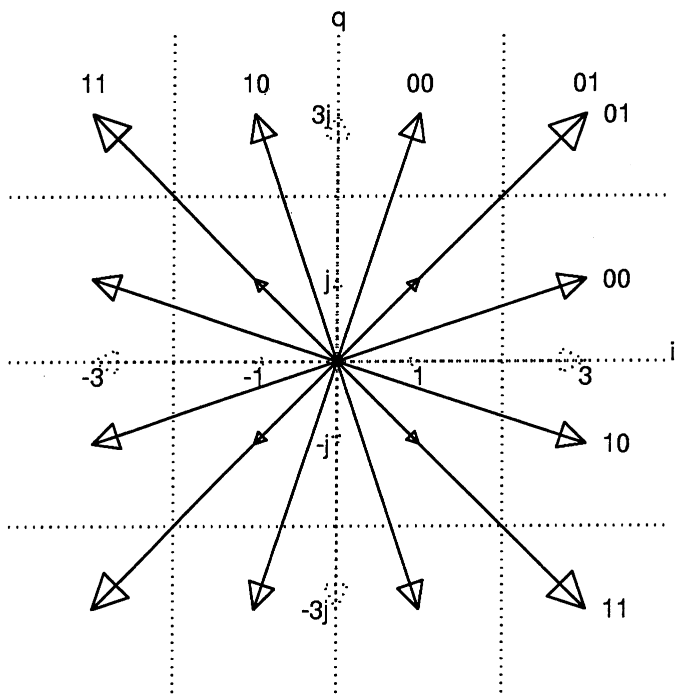
Hence the output of lowpass filter 1 is $i(t)$, since the two terms modulated onto carriers at $2\omega_C$ are rejected by the filter.

Similarly from multiplier 2:

$$\begin{aligned} q'(t) &= s(t) \times [-2\sin(\omega_C t)] \\ &= [i(t)\cos(\omega_C t) - q(t)\sin(\omega_C t)] \times [-2\sin(\omega_C t)] \\ &= -2i(t)\cos(\omega_C t)\sin(\omega_C t) + 2q(t)\sin^2(\omega_C t) \\ &= q(t) - q(t)\cos(2\omega_C t) - i(t)\sin(2\omega_C t) \end{aligned}$$

and the output of lowpass filter 2 is $q(t)$.

(c) **16-QAM phasor constellation**

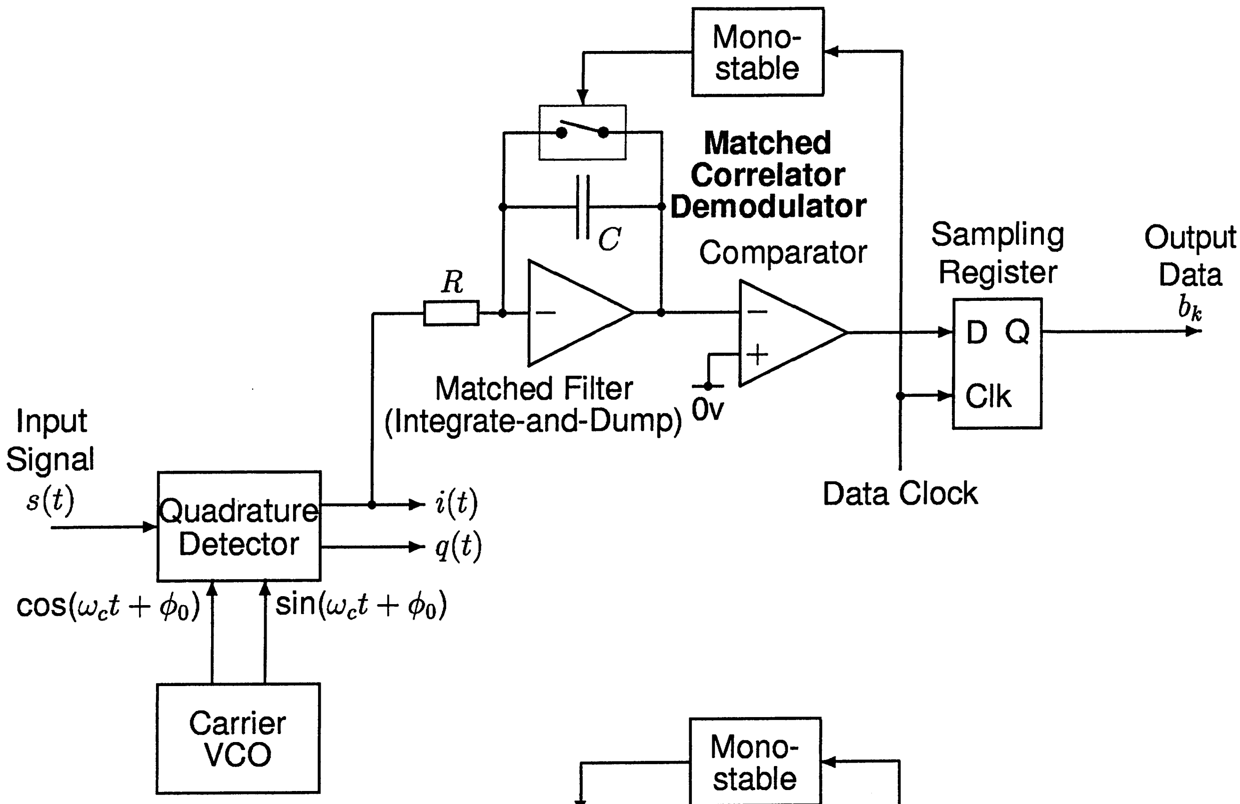


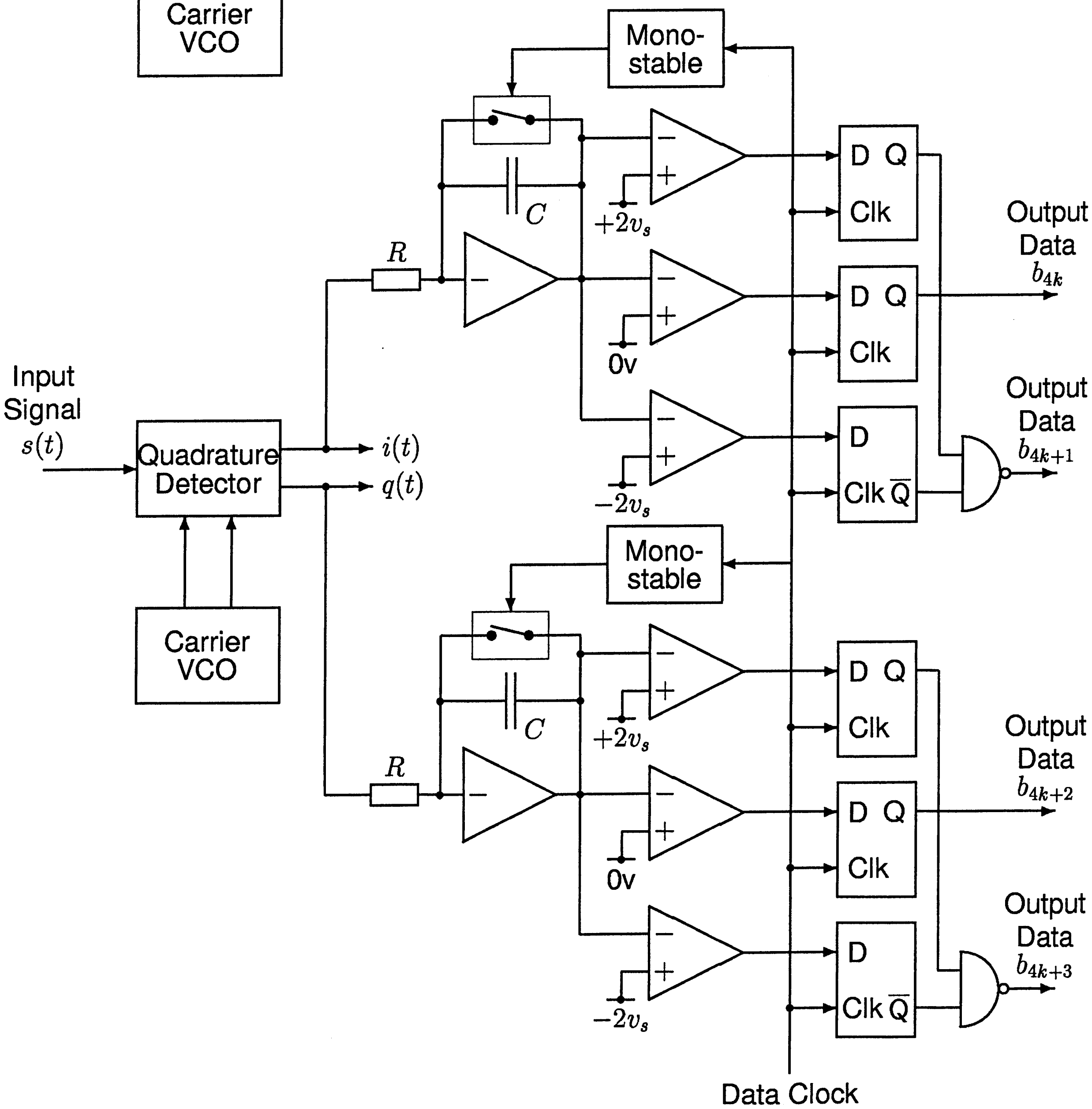
The constellation points are labelled with a Gray unit-distance code, so that detection errors to an adjacent state will only ever cause single bit errors.

For equiprobable symbols, the decision boundaries for both the i and q components should be mid-way between the constellation points, as shown by the dotted lines above.

(d) **16-QAM detector**

The lecture notes show a BPSK detector of the form shown in the upper diagram on the next page. To create a QAM detector, first a quadrature 'integrate-and-dump' matched filter must be added, connected to the $q(t)$ output of the quadrature detector. Then three comparators and sampling registers must be connected to each matched filter output in order to detect which of the four possible amplitude states each component is in. These set the detection thresholds at 0 and $\pm 2v_s$, as shown in the constellation diagram for part(c). Finally, the less significant output bit of each bit-pair is low only if the signal lies between the $\pm 2v_s$ thresholds, so it can be formed by a NAND function on the outer two comparator states. Hence the block diagram should be as shown in the lower figure on the next page.


Mono-
stable
Matched
Correlator
Demodulator
Comparator
Sampling
Register
Output
Data
b_k
R
C
D Q
Clk
Matched Filter
(Integrate-and-Dump)
0v
Input
Signal
$s(t)$
Quadrature
Detector
$i(t)$
$q(t)$
Data Clock
$\cos(\omega_c t + \phi_0)$
$\sin(\omega_c t + \phi_0)$
Carrier
VCO


Mono-
stable
C
R
D Q
Clk
$+2v_s$
0v
$-2v_s$
Output
Data
b_{4k}
Output
Data
b_{4k+1}
Input
Signal
$s(t)$
Quadrature
Detector
$i(t)$
$q(t)$
Carrier
VCO
D
Clk $\overline{Q}$
Output
Data
b_{4k+2}
Output
Data
b_{4k+3}
Data Clock

4. (a) **Principles of OFDM:**

The aim of OFDM is to demultiplex the high-speed bit stream into N streams, each at $1/N$ of the original rate, which are then modulated onto N separate carrier waves. Typically $N \approx 1000$ to 2000.

The inverse FFT may be used to put QPSK or QAM data on each of N carriers, spaced by $1/T$ Hz, where T is the IFFT block period. Each carrier is an IFFT basis function which is multiplied by the modulation phasor ($\pm 1 \pm j$ in the case of QPSK). In this way the carriers are orthogonal to each other and may be demodulated by an equivalent FFT process without mutual interference at the receiver. The mutual orthogonality of the IFFT basis functions, means that there should be no interference between each modulated carrier and its neighbours. Orthogonality is not affected by the modulation process, because the modulation rate is no faster than once per FFT block period, so each modulated carrier is a pure tone for the duration of the block period T.

The OFDM signal has much improved resilience to typical multipath delays because of the much lower modulation rate on each carrier, compared with a single carrier system operating at the same overall bit rate. The use of a Guard Band between each IFFT block allows for variations in path delay up to the duration of the Guard Band, before any signal degradation occurs. By using error correction coding, the loss of some of the carriers due to frequency-selective fading (another characteristic of multipath propagation) can also be tolerated.

(b) **Spectral Efficiency**

Let $m = \log_2 M$, so the number of bits per M^2-QAM symbol $= 2m$.

Symbol rate for each carrier $= \dfrac{1}{T + \Delta T}$ sym/s.

No. of carriers for the user $= N - n$

$\therefore$ User bit rate (after allowing for ECC) $= \dfrac{N-n}{T+\Delta T} \, . \, 2m \, . \, R$ bit/s.

Carrier spacing = tone spacing of FFT $= \dfrac{1}{T}$ Hz

$\therefore$ Bandwidth of OFDM signal $\approx \dfrac{N}{T}$ Hz

$$\text{Hence spectral efficiency} \approx \frac{(N-n) \, . \, 2mR}{T+\Delta T} \, . \, \frac{T}{N} = \boxed{\frac{(N-n)T \, . \, 2R\log_2 M}{N(T+\Delta T)}}$$

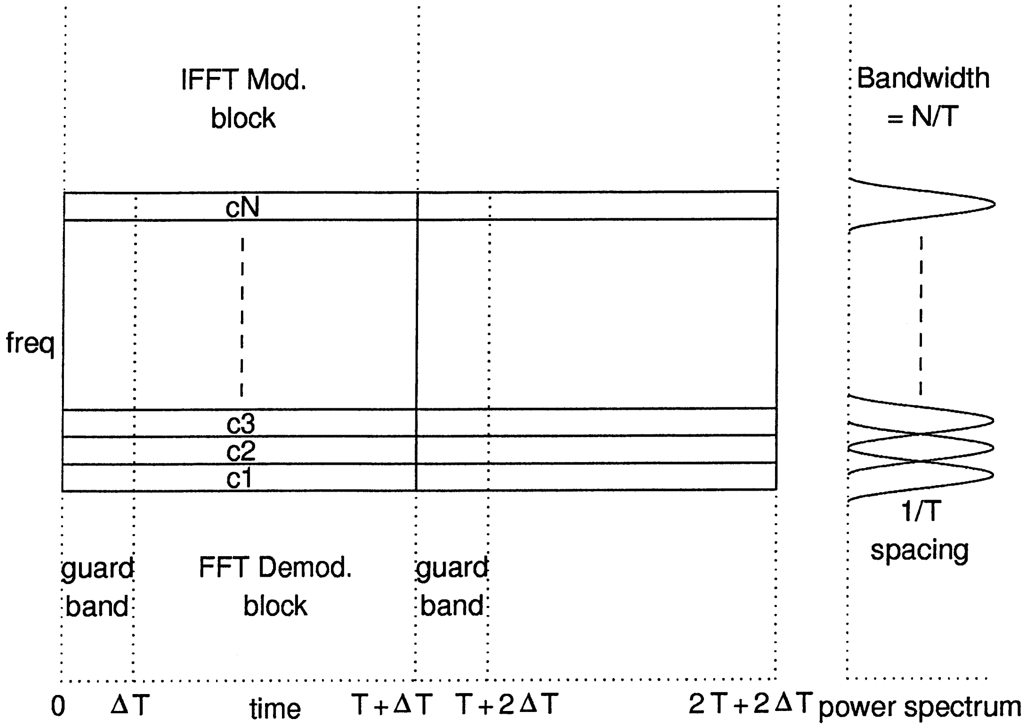


Fig 6.2: Orthogonal Frequency Division Multiplexing (OFDM) with N carriers.

(c) **DVB system**

Multipath delays mean that the phase transitions (due to the modulation) on the various carriers of the OFDM signal can occur at different times, and this could upset the orthogonality property because some modulation transitions would need to occur during the demodulator FFT analysis block if each block followed its predecessor immediately. To avoid this problem, a *guard period* ΔT is inserted between consecutive blocks in the modulator and demodulator (see fig 6.2, above, from the 3F4 lecture notes). For optimum demodulator performance, the modulator extends (by periodic extension) the inverse FFT output waveform into the guard period before each block, so that the transmitted waveform is continuous from the point where the modulation transitions occur at the start of each guard period. The FFT demodulator analyses the interval from ΔT to $T + \Delta T$. In this way multipath delays varying from 0 to ΔT can be tolerated without any modulation transitions intruding into the FFT analysis interval and spoiling the orthogonality of the carrier waves. Unfortunately the guard periods either reduce the throughput of the system or increase its bandwidth in the ratio $T : (T + \Delta T)$.

With a guard band $\Delta T = 7\mu$s, the maximum path delay variation before phase transitions intrude into the FFT analysis interval is 7μs. Assuming that the radio waves travel at the velocity of light, c:

Maximum path length difference $= c\Delta T = 3 \,.\, 10^8 \,.\, 7 \,.\, 10^{-6} = \boxed{2100 \text{ m.}}$

For a single carrier system, operating at 24 Mbit/s, the symbol rate with 64-QAM ($M = 8$) would be at least $24/6 = 4$ Msym/s. If $R = 0.6$ ECC were used, this would increase to $4/0.6 = 6.67$ Msym/s. Hence the path delays would need to be much less than the symbol period of 0.25μs (or 0.15μs with ECC), say less than 0.02μs. Hence the path length difference would need to be less than about $\boxed{6 \text{ m}}$, which would not be practical in a typical urban environment where reflections off buildings are commonplace.

Engineering Triops Part 2A
Module 3F4. Data Transmission, May 2004 - Comments

1. Parts (a) and (b) were well answered. In part (c) a common mistake was to neglect the effect of the receiver filter when evaluating the received noise power.

2. This question was in general answered very well. The only notable issues concerned less than comprehensive descriptions in part (a) and some misunderstandings about what constitutes a perfect code in part (d).

3. The second most popular question. Tended to be answered either well or quite poorly. Given the bookwork nature of parts (a) and (b) it was surprising that they were not answered better than they were. No candidate produced a complete answer for part (c).

4. The least popular question but was answered reasonably well. Some difficulties were experienced with part (b) by some candidates. Some candidates neglected (or were unable) to estimate the maximum path length difference for a single carrier system which was required for the second part of (c).