UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIA

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Tuesday 4 May 2004 9 to 10.30

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Module 3C5

DYNAMICS

*Answer not more than **three** questions.*

*All questions carry the same number of marks.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

*Attachment:*
*3C5 Datasheet (5 pages).*

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

(TURN OVER

1 The 3C5 laboratory gyroscope is attached to a bicycle to keep it standing upright with no rider. Fig. 1 shows the gyro attached using its pivot at B. The gyro assembly is constrained to tilt only within the plane of the bicycle. This angle of tilt is denoted $\alpha$ as shown and the angle of tilt of the bicycle from the vertical is denoted $\phi$. Note that $\alpha$ can be written as $\pi/2 - \theta$, where $\theta$ is the conventional Euler angle for the gyroscope.

The bicycle has mass $m$ and the height of its centre of mass above the ground is $a$. The entire gyro assembly also has mass $m$ and its centre of mass is distance $a$ above B which itself is height $2a$ above the ground. The gyro rotor has polar moment of inertia $C$ and spins with 'fast' spin $\omega$. In a simple steady-state experiment, the handlebars are locked to prevent steering and the brakes prevent the wheels from rolling. The wheels never slip nor do they leave the ground. Assume throughout that angles $\phi$ and $\alpha$ remain small.

(a) For general angles $\alpha$ and $\phi$ determine the components of rotor angular velocity $\boldsymbol{\omega} = \omega_1\,\mathbf{i} + \omega_2\,\mathbf{j} + \omega_3\,\mathbf{k}$ in terms of the rates $\dot{\alpha}$ and $\dot{\phi}$. Also determine the components of the couple applied to the rotor. [40%]

(b) Show that steady-state small motion of the bicycle is described by the pair of equations $C\omega\dot{\alpha} - 4mga\phi = 0$ and $C\omega\dot{\phi} + mga\alpha = 0$. [35%]

(c) Hence find the frequency of oscillation of the bicycle and the ratio between peak gyro tilt $\alpha_{max}$ and peak bicycle tilt $\phi_{max}$. [25%]

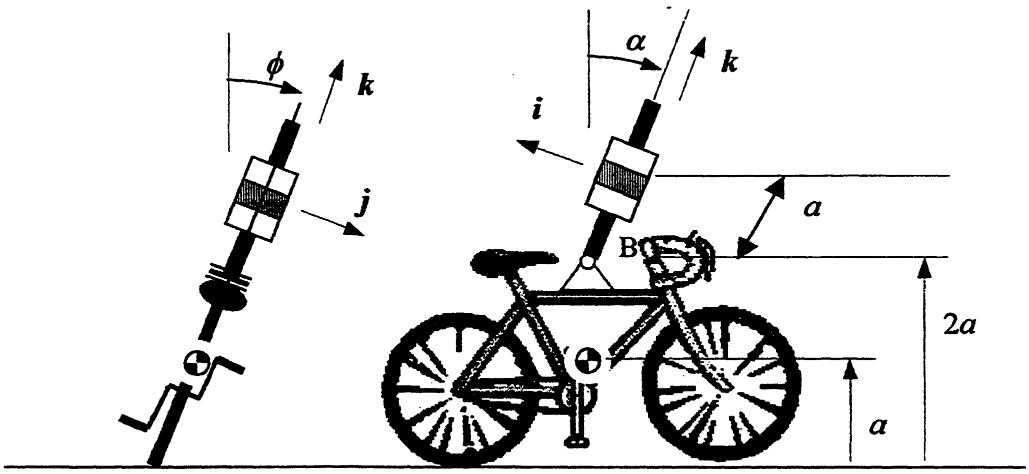


Fig. 1

2 A small thin coin of mass $m$ and radius $a$ is rolling without slip inside a conical vessel of cone angle $\theta$ as shown in Fig. 2. The centre of the coin moves at a steady speed $V$ on a horizontal circular path of radius $R>>a$ and the plane of the coin is tilted at an angle $\theta$ to the horizontal. An **i**, **j**, **k** reference frame moves with the coin.

(a) Sketch a free-body diagram of the coin showing all forces. [25%]

(b) Deduce the **i**, **j**, **k** components of the couple acting on the coin in steady-state. [25%]

(c) Use a no-slip condition to determine constraints on the angular velocities of the coin and the reference frame. [25%]

(d) What is $V$ for given $\theta$ and $R$? [25%]

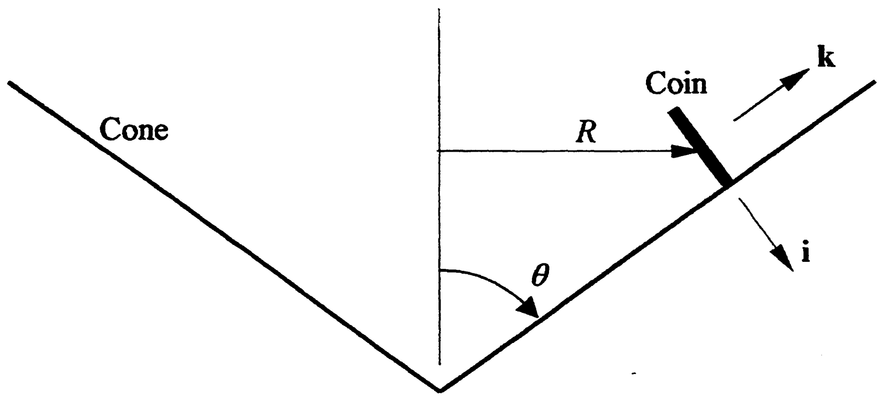


Fig. 2

(TURN OVER

3 A particle of mass $m$ is attached to one corner of a square flat plate of side $2a$ and mass $m$ . The total mass of the assembly is therefore $2m$, and the corner at which the mass is attached is labelled B.

(a) Find the principal moments of inertia at the centre of mass G of the assembly. [35%]

(b) At a certain instant the angular velocity vector makes an angle of 30° with the line GB and 60° with the normal to the plate. What is the angle between the moment-of-momentum vector and the plane of the plate? [30%]

(c) A shaft through G is used to generate the angular velocity described above. Bearings on the shaft are spaced distance $b$ apart. What is the magnitude of the force acting at one of these bearings due to dynamic imbalance? [35%]

4 A schematic of an orbiting "dumbbell" satellite is shown in Fig. 3. The satellite consists of two masses $m$ that are separated by a light rod of length $2a$. The position of the satellite is described by the three generalised coordinates $r$, $\theta$, and $\phi$ as shown in the Figure (the motion is planar). The potential energy of the satellite is given by

$$V = -GMm\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

where $G$ is the gravitational constant, $M$ is the mass of the earth, and $r_1$ and $r_2$ are the distances of the masses from the centre of the earth, A.

(a) Express the potential energy of the satellite in terms of the generalised coordinates $r$, $\theta$, and $\phi$. [15%]

(b) By using Lagrange's equation derive the three equations of motion of the satellite. [45%]

(c) For the case $a << r$ and steady orbital motion with $\dot{r} = \ddot{r} = 0$, show from your answer to part (b) that the orbital rotation rate is given by $\dot{\theta} = \sqrt{GM / r^3}$. [20%]

(d) For the case described in part (c), derive the equation of motion for small oscillations in $\phi$ around the mean position $\phi = 0$. Hence show that the natural frequency of these oscillations is given by $\omega_n = \sqrt{3}\dot{\theta}$. [20%]

Hint: you will need to show that, under the conditions listed in parts (c) and (d), $\partial V / \partial \phi \approx 6GMma^2\phi / r^3$.

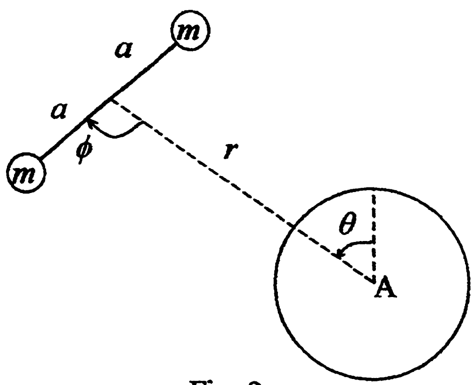


Fig. 3

5 A schematic of a pendulum vibration absorber is shown in Fig. 4. A pendulum consisting of a light rod of length $a$ and an end mass $M$ is attached to the edge of a spinning disk. The disk has polar moment of inertia $I$ and radius $R$, and rotates about the centre, A. The motion of the system is described by the two degrees of freedom $\theta$ and $\psi$ shown in the figure. A torque $Q$ is applied to the disk, and the effects of gravity can be ignored.

(a) Derive an expression for the kinetic energy of the system. Do not assume that either $\theta$ or $\psi$ is a small angle. [20%]

(b) By using Lagrange's equation, derive the equations of motion of the system. [35%]

(c) Show that for $Q = 0$ the equation of motion for $\psi$ corresponds to conservation of angular momentum about the centre of the disk, A. [20%]

(d) If $Q$ is such that the disk is spun at a constant rate $\dot{\psi} = \Omega$, derive an expression for the natural frequency of the pendulum for small oscillations around the position $\theta = 0$. [25%]

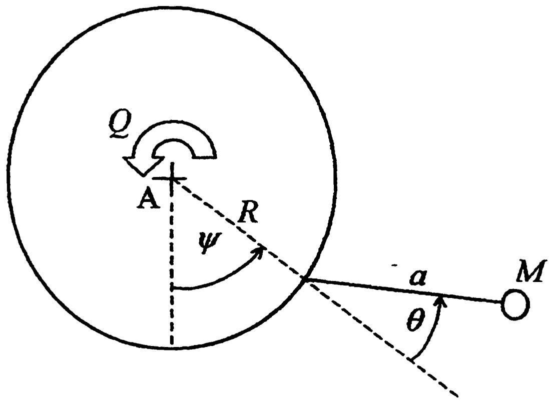


Fig. 4

**END OF PAPER**

**Part IIA Data sheet** **S32**

**Module 3C5 Dynamics**

**Module 3C6 Vibration**

## Dynamics in three dimensions

### Axes fixed in direction

(a) Linear momentum for a general collection of particles $m_i$:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\,\boldsymbol{v}_G$, $M$ is the total mass, $\boldsymbol{v}_G$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_G$$
$$= \dot{\boldsymbol{h}}_P + \dot{\boldsymbol{r}}_P \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here, $\boldsymbol{h}_P$ and $\boldsymbol{h}_G$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_P) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_G + (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})\, dm = \boldsymbol{I}\,\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\boldsymbol{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and

$$A = \int (y^2 + z^2)\,dm \qquad B = \int (z^2 + x^2)\,dm \qquad C = \int (x^2 + y^2)\,dm$$
$$D = \int yz\,dm \qquad E = \int zx\,dm \qquad F = \int xy\,dm$$

where all integrals are taken over the volume of the body.

### Axes rotating with angular velocity $\Omega$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

**Euler's dynamic equations** (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B-C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C-A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A-B)\,\omega_1\,\omega_2 = Q_3$$

where $A$, $B$ and $C$ are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\omega = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $Q = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A\,\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 = Q_1$$
$$A\,\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 = Q_2$$
$$C\,\dot{\omega}_3 = Q_3$$

where $A$, $A$ and $C$ are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\omega = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $Q = [Q_1, Q_2, Q_3]$ using axes such that $\omega_3$ and $Q_3$ are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\Omega = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

**Lagrange's equations**

For a holonomic system with generalised coordinates $q_i$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where $T$ is the total kinetic energy, $V$ is the total potential energy, and $Q_i$ are the non-conservative generalised forces.

**Rayleigh's principle for small vibrations**

The "Rayleigh quotient" for a discrete system is $\dfrac{V}{\tilde{T}} = \dfrac{\underline{q}^t K \underline{q}}{\underline{q}^t M \underline{q}}$ where $\underline{q}$ is the vector of generalised coordinates, $M$ is the mass matrix and $K$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p5.

If this quantity is evaluated with any vector $\underline{q}$, the result will be

(1) $\geq$ the smallest squared frequency;

(2) $\leq$ the largest squared frequency;

(3) a good approximation to $\omega_k{}^2$ if $\underline{q}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.)

# VIBRATION MODES AND RESPONSE

## Discrete systems

**1.** The natural frequencies $\omega_n$ and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n{}^2 M\underline{u}^{(n)}$$

where the $M$ (mass matrix) and $K$ (stiffness matrix) are both symmetric and positive definite.

**2. Kinetic energy**

$$T = \frac{1}{2}\underline{\dot{u}}^t M\underline{\dot{u}}$$

**3. Orthogonality and normalisation**

$$\underline{u}^{(j)^t} M\underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)^t} K\underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

**4. General response**

The general response of the system can be written as a sum of modal responses

$$\underline{q}(t) = \sum_n a_n(t)\,\underline{u}^{(n)}$$

where $\underline{q}$ is the vector of generalised coordinates and $a_n$ gives the "amount" of the $n$th mode.

**5. Transfer function**

For (generalised) force $F$ at frequency $\omega$, applied at point (or generalised coordinate) $j$, and response $q$ measured at point (or generalised coordinate) $k$ the transfer function is

$$H(j,k,\omega) = \frac{q}{F} = \sum_n \frac{u_j{}^{(n)} u_k{}^{(n)}}{\omega_n{}^2 - \omega^2}$$

(with no damping), or

## Continuous systems

The natural frequencies $\omega_n$ and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p5) and boundary conditions, assuming harmonic time dependence.

**2. Kinetic energy**

$$T = \frac{1}{2}\int \dot{u}^2 dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

**3. Orthogonality and normalisation**

$$\int u_j(x)\,u_k(x)\,dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

**4. General response**

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_n a_n(t)\,u_n(x)$$

where $w(x,t)$ is the displacement and $a_n$ gives the "amount" of the $n$th mode.

**5. Transfer function**

For force $F$ at frequency $\omega$ applied at point $x$, and response $w$ measured at point $y$, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\,u_n(y)}{\omega_n{}^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega)=\frac{q}{F}\approx\sum_n \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2+2i\omega\omega_n\zeta_n-\omega^2}$$

(with small damping) where the damping factor $\zeta_n$ is as in the Mechanics Data Book for one-degree-of-freedom systems.

$$H(x,y,\omega)=\frac{w}{F}\approx\sum_n \frac{u_n(x)\,u_n(y)}{\omega_n^2+2i\omega\omega_n\zeta_n-\omega^2}$$

(with small damping) where the damping factor $\zeta_n$ is as in the Mechanics Data Book for one-degree-of-freedom systems.

## 6. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)}u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\,u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

## 7. Impulse response

For a unit impulse applied at $t=0$ at point (or generalised coordinate) $j$, the response at point (or generalised coordinate) $k$ is

$$g(j,k,t)=\sum_n \frac{u_j^{(n)}u_k^{(n)}}{\omega_n}\sin\omega_n t$$

(with no damping), or

$$g(j,k,t)\approx\sum_n \frac{u_j^{(n)}u_k^{(n)}}{\omega_n}\sin\omega_n t\; e^{-\omega_n\zeta_n t}$$

(with small damping).

For a unit impulse applied at $t=0$ at point $x$, the response at point $y$ is

$$g(x,y,t)=\sum_n \frac{u_n(x)\,u_n(y)}{\omega_n}\sin\omega_n t$$

(with no damping), or

$$g(x,y,t)\approx\sum_n \frac{u_n(x)\,u_n(y)}{\omega_n}\sin\omega_n t\; e^{-\omega_n\zeta_n t}$$

(with small damping).

## 8. Step response

For a unit step force applied at $t=0$ at point (or generalised coordinate) $j$, the response at point (or generalised coordinate) $k$ is

$$h(j,k,t)=\sum_n u_j^{(n)}u_k^{(n)}\left[1-\cos\omega_n t\right]$$

(with no damping), or

$$h(j,k,t)\approx\sum_n u_j^{(n)}u_k^{(n)}\left[1-\cos\omega_n t\; e^{-\omega_n\zeta_n t}\right]$$

(with small damping).

For a unit step force applied at $t=0$ at point $x$, the response at point $y$ is

$$h(x,y,t)=\sum_n u_n(x)\,u_n(y)\left[1-\cos\omega_n t\right]$$

(with no damping), or

$$h(t)\approx\sum_n u_n(x)\,u_n(y)\left[1-\cos\omega_n t\; e^{-\omega_n\zeta_n t}\right]$$

(with small damping).

## Governing equations for continuous systems

### Transverse vibration of a stretched string

Tension $P$, mass per unit length $m$, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

| Equation of motion | Potential energy | Kinetic energy |
|---|---|---|
| $m\dfrac{\partial^2 w}{\partial t^2} - P\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$ | $V = \dfrac{1}{2}P\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$ | $T = \dfrac{1}{2}m\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$ |

### Torsional vibration of a circular shaft

Shear modulus $G$, density $\rho$, external radius $a$, internal radius $b$ if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)\left(a^4 - b^4\right)$.

| Equation of motion | Potential energy | Kinetic energy |
|---|---|---|
| $\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = f(x,t)$ | $V = \dfrac{1}{2}GJ\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$ | $T = \dfrac{1}{2}\rho J\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$ |

### Axial vibration of a rod or column

Young's modulus $E$, density $\rho$, cross-sectional area $A$, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

| Equation of motion | Potential energy | Kinetic energy |
|---|---|---|
| $\rho A\dfrac{\partial^2 w}{\partial t^2} - EA\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$ | $V = \dfrac{1}{2}EA\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$ | $T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$ |

### Bending vibration of an Euler beam

Young's modulus $E$, density $\rho$, cross-sectional area $A$, second moment of area of cross-section $I$, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

| Equation of motion | Potential energy | Kinetic energy |
|---|---|---|
| $\rho A\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = f(x,t)$ | $V = \dfrac{1}{2}EI\int\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 dx$ | $T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$ |

Note that values of $I$ can be found in the Mechanics Data Book.