

ENGINEERING TRIPOS PART IIA

Friday 7 May 9.00 to 10.30

Module 3C6

VIBRATION

*Answer not more than **three** questions.**All questions carry the same number of marks.**The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.**Attachment:**Datasheet S32: 3C5 Dynamics and 3C6 Vibration (5 pages)*

<p>You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator</p>

(TURN OVER)

1 (a) A wheelset for a railway wagon consists of a solid circular shaft of length $2L$, with identical wheels of polar moment of inertia K , rigidly fixed to the two ends. The shaft has radius a and is made of material with shear modulus G and density ρ . The wheelset undergoes free torsional vibration of small amplitude. Write down the governing differential equation, and the boundary conditions which must be satisfied at the ends of the shaft. Assume that the shaft is aligned parallel to the x -axis, with its centre at $x = 0$. [15%]

(b) Show that the natural frequencies ω for free torsional vibration of the wheelset must satisfy the equation:

$$\tan \omega L \sqrt{\frac{\rho}{G}} = \begin{cases} \frac{\pi a^4 \sqrt{G\rho}}{2K\omega} & \text{(antisymmetric modes)} \\ -\frac{2K\omega}{\pi a^4 \sqrt{G\rho}} & \text{(symmetric modes)} \end{cases}$$

[30%]

(c) Sketch a graphical construction to solve this equation, and sketch the first three vibration modes. [30%]

(d) Describe what happens to the natural frequencies in the limiting cases:
(i) $K \rightarrow 0$ and (ii) $K \rightarrow \infty$.

Explain the physical reasons for the behaviour in both cases.

[25%]

2 (a) An Euler beam of length L and uniform rectangular cross-section undergoes bending vibration satisfying the differential equation

$$\rho \frac{\partial^2 w}{\partial t^2} + \frac{Eh^2}{12} \frac{\partial^4 w}{\partial x^4} = 0$$

where w is the lateral displacement at position x (measured from one end) and time t , E is the Young's modulus, ρ is the density, and h is the depth of the beam. Both ends of the beam are freely pinned to a rigid base. Write down the appropriate boundary conditions, and hence find the modes of vibration and corresponding natural frequencies of the pinned-pinned beam. [35%]

(b) Explain briefly how Rayleigh's principle can be used to estimate the shift in natural frequencies of a system which is slightly altered, assuming that the modes and natural frequencies of the original system are known. [15%]

(c) Apply this method to the bending vibration of a pinned-pinned Euler beam with rectangular cross-section, with a constant width but a slightly non-uniform depth $h(x) = h_0 + \Delta h(x)$ where h_0 is a constant and $|\Delta h| \ll h_0$. The potential energy V and kinetic energy T per unit width of beam are given by

$$V = \frac{E}{24} \int_0^L h^3 \left[\frac{\partial^2 w}{\partial x^2} \right]^2 dx, \quad T = \frac{\rho}{2} \int_0^L h \left[\frac{\partial w}{\partial t} \right]^2 dx.$$

Hence show that an approximate expression for the natural frequency ω_n of the non-uniform beam can be written in the form

$$\omega_n^2 \approx \frac{Eh_0^2}{12\rho} \left(\frac{n\pi}{L} \right)^4 \left[1 + \frac{4}{Lh_0} \int_0^L \Delta h(x) \sin^2 \left(\frac{n\pi x}{L} \right) dx \right]. \quad [35\%]$$

(d) What does this approximate theory predict for the natural frequencies of a beam with a linear taper $\Delta h = \epsilon x$? [15%]

(TURN OVER)

3 Three particles of mass m are attached to a tightly-stretched, light wire as shown in Figure 1. The tensile force P in the wire is large, so that it does not change appreciably for small lateral displacements of the particles. The length of each segment of the wire is L as shown.

- (a) Show that the potential energy of the system is given by:

$$V = \frac{P}{L} [y_1^2 + y_2^2 + y_3^2 - y_1 y_2 - y_2 y_3],$$

and write down the stiffness and mass matrices.

[25%]

- (b) Sketch the natural mode shapes. Hence, or otherwise, calculate the natural frequencies.

[40%]

- (c) If the middle mass is increased by 20%, use Rayleigh's quotient to estimate the percentage change in the lowest natural frequency. Explain how you could use Rayleigh's quotient to obtain the exact answer.

[35%]

(Cont.

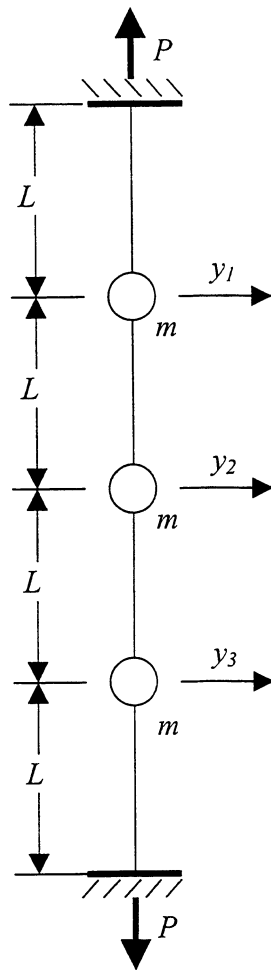


Fig. 1.

(TURN OVER

4 Figure 2 shows two rigid, uniform bars of length L and mass m that are hinged together at point B. The assembly is supported by three springs of stiffness k at points A, B, and C. It is constrained to move in the plane of the page. The small translation y_1 of point B and the rotations θ_2 and θ_3 are to be used as coordinates. Assume small angles.

(a) Show that the kinetic energy of the system is given by:

$$T = \frac{m}{2} \left[2\dot{y}_1^2 + \frac{L^2}{3}\dot{\theta}_2^2 + \frac{L^2}{3}\dot{\theta}_3^2 - L\dot{y}_1\dot{\theta}_2 + L\dot{y}_1\dot{\theta}_3 \right],$$

and the potential energy is given by:

$$V = \frac{k}{2} \left[3y_1^2 + L^2\theta_2^2 + L^2\theta_3^2 - 2Ly_1\theta_2 + 2Ly_1\theta_3 \right].$$

Hence write down the mass and stiffness matrices.

[30%]

(b) Calculate the natural frequencies of the system.

[20%]

(c) Show that one of the natural mode shapes has $y_1 = 0$. Sketch this mode shape. Explain why none of the other natural modes can satisfy this condition.

[20%]

(d) A transverse force $F\sin\omega t$ is applied to the pin joint at B. Sketch the variation of the log amplitude of the steady-state displacement at B as a function of the frequency of the input force. Briefly explain the main features.

[30%]

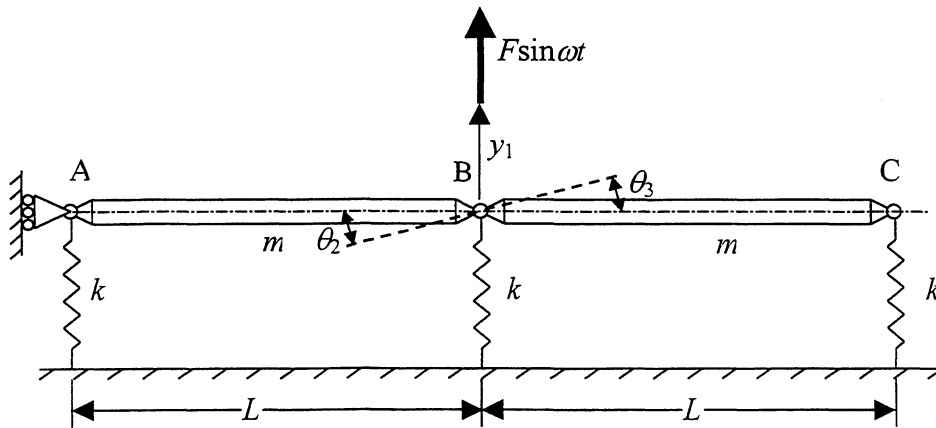


Fig. 2.

END OF PAPER

Part IIA Data sheet
Module 3C5 Dynamics
Module 3C6 Vibration

S32

Dynamics in three dimensions

Axes fixed in direction

- (a) Linear momentum for a general collection of particles m_i :

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}^{(e)}$$

where $\mathbf{p} = M \mathbf{v}_G$, M is the total mass, \mathbf{v}_G is the velocity of the centre of mass and $\mathbf{F}^{(e)}$ the total external force applied to the system.

- (b) Moment of momentum about a general point P

$$\begin{aligned} \mathbf{Q}^{(e)} &= (\mathbf{r}_G - \mathbf{r}_P) \times \dot{\mathbf{p}} + \dot{\mathbf{h}}_G \\ &= \dot{\mathbf{h}}_P + \dot{\mathbf{r}}_P \times \mathbf{p} \end{aligned}$$

where $\mathbf{Q}^{(e)}$ is the total moment of external forces about P. Here, \mathbf{h}_P and \mathbf{h}_G are the moments of momentum about P and G respectively, so that for example

$$\begin{aligned} \mathbf{h}_P &= \sum_i (\mathbf{r}_i - \mathbf{r}_P) \times m_i \dot{\mathbf{r}}_i \\ &= \mathbf{h}_G + (\mathbf{r}_G - \mathbf{r}_P) \times \mathbf{p} \end{aligned}$$

where the summation is over all the mass particles making up the system.

- (c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\mathbf{h}_P = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) dm = \mathbf{I} \boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\mathbf{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \quad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$\begin{aligned} \text{and} \quad A &= \int (y^2 + z^2) dm & B &= \int (z^2 + x^2) dm & C &= \int (x^2 + y^2) dm \\ D &= \int yz dm & E &= \int zx dm & F &= \int xy dm \end{aligned}$$

where all integrals are taken over the volume of the body.

Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the “rotating frame” form, so that for example

$$\dot{\mathbf{p}} + \boldsymbol{\Omega} \times \mathbf{p} = \mathbf{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector \mathbf{r} is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A \dot{\omega}_1 - (B - C) \omega_2 \omega_3 = Q_1$$

$$B \dot{\omega}_2 - (C - A) \omega_3 \omega_1 = Q_2$$

$$C \dot{\omega}_3 - (A - B) \omega_1 \omega_2 = Q_3$$

where A , B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\omega = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $Q = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P .

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A \dot{\Omega}_1 - (A \Omega_3 - C \omega_3) \Omega_2 = Q_1$$

$$A \dot{\Omega}_2 + (A \Omega_3 - C \omega_3) \Omega_1 = Q_2$$

$$C \dot{\omega}_3 = Q_3$$

where A , A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\omega = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $Q = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\Omega = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_i} \right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\frac{V}{T} = \frac{\underline{q}^T K \underline{q}}{\underline{q}^T M \underline{q}}$ where \underline{q} is the vector of

generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p5.

If this quantity is evaluated with any vector \underline{q} , the result will be

- (1) \geq the smallest squared frequency;
- (2) \leq the largest squared frequency;
- (3) a good approximation to ω_k^2 if \underline{q} is an approximation to $\underline{u}^{(k)}$.

(Formally, $\frac{V}{T}$ is stationary near each mode.)

VIBRATION MODES AND RESPONSE

Discrete systems

1. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K \underline{u}^{(n)} = \omega_n^2 M \underline{u}^{(n)}$$

where the M (mass matrix) and K (stiffness matrix) are both symmetric and positive definite.

2. Kinetic energy

$$T = \frac{1}{2} \dot{\underline{u}}^t M \dot{\underline{u}}$$

3. Orthogonality and normalisation

$$\underline{u}^{(j)t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

4. General response

The general response of the system can be written as a sum of modal responses

$$\underline{q}(t) = \sum_n a_n(t) \underline{u}^{(n)}$$

where \underline{q} is the vector of generalised coordinates and a_n gives the “amount” of the n th mode.

5. Transfer function

For (generalised) force F at frequency ω , applied at point (or generalised coordinate) j , and response q measured at point (or generalised coordinate) k the transfer function is

$$H(j, k, \omega) = \frac{q}{F} = \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

Continuous systems

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p5) and boundary conditions, assuming harmonic time dependence.

$$T = \frac{1}{2} \int \dot{u}^2 dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

$$\int u_j(x) u_k(x) dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

The general response of the system can be written as a sum of modal responses

$$w(x, t) = \sum_n a_n(t) u_n(x)$$

where $w(x, t)$ is the displacement and a_n gives the “amount” of the n th mode.

For force F at frequency ω applied at point x , and response w measured at point y , the transfer function is

$$H(x, y, \omega) = \frac{w}{F} = \sum_n \frac{u_n(x) u_n(y)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j, k, \omega) = \frac{q}{F} \approx \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

6. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

7. Impulse response

For a unit impulse applied at $t = 0$ at point (or generalised coordinate) j , the response at point (or generalised coordinate) k is

$$g(j, k, t) = \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t$$

(with no damping), or

$$g(j, k, t) \approx \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t e^{-\omega_n \zeta_n t}$$

(with small damping).

8. Step response

For a unit step force applied at $t = 0$ at point (or generalised coordinate) j , the response at point (or generalised coordinate) k is

$$h(j, k, t) = \sum_n u_j^{(n)} u_k^{(n)} [1 - \cos \omega_n t]$$

(with no damping), or

$$h(j, k, t) \approx \sum_n u_j^{(n)} u_k^{(n)} [1 - \cos \omega_n t e^{-\omega_n \zeta_n t}]$$

(with small damping).

$$H(x, y, \omega) = \frac{w}{F} \approx \sum_n \frac{u_n(x) u_n(y)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For a system with low modal overlap, if the factor $u_n(x) u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a unit impulse applied at $t = 0$ at point x , the response at point y is

$$g(x, y, t) = \sum_n \frac{u_n(x) u_n(y)}{\omega_n} \sin \omega_n t$$

(with no damping), or

$$g(x, y, t) \approx \sum_n \frac{u_n(x) u_n(y)}{\omega_n} \sin \omega_n t e^{-\omega_n \zeta_n t}$$

(with small damping).

For a unit step force applied at $t = 0$ at point x , the response at point y is

$$h(x, y, t) = \sum_n u_n(x) u_n(y) [1 - \cos \omega_n t]$$

(with no damping), or

$$h(t) \approx \sum_n u_n(x) u_n(y) [1 - \cos \omega_n t e^{-\omega_n \zeta_n t}]$$

(with small damping).

Governing equations for continuous systems

Transverse vibration of a stretched string

Tension P , mass per unit length m , transverse displacement $w(x, t)$, applied lateral force $f(x, t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m \frac{\partial^2 w}{\partial t^2} - P \frac{\partial^2 w}{\partial x^2} = f(x, t)$	$V = \frac{1}{2} P \int \left(\frac{\partial w}{\partial x} \right)^2 dx$	$T = \frac{1}{2} m \int \left(\frac{\partial w}{\partial t} \right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G , density ρ , external radius a , internal radius b if shaft is hollow, angular displacement $\theta(x, t)$, applied torque $f(x, t)$ per unit length.

Polar moment of area is $J = (\pi / 2)(a^4 - b^4)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J \frac{\partial^2 \theta}{\partial t^2} - GJ \frac{\partial^2 \theta}{\partial x^2} = f(x, t)$	$V = \frac{1}{2} GJ \int \left(\frac{\partial \theta}{\partial x} \right)^2 dx$	$T = \frac{1}{2} \rho J \int \left(\frac{\partial \theta}{\partial t} \right)^2 dx$

Axial vibration of a rod or column

Young's modulus E , density ρ , cross-sectional area A , axial displacement $w(x, t)$, applied axial force $f(x, t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A \frac{\partial^2 w}{\partial t^2} - EA \frac{\partial^2 w}{\partial x^2} = f(x, t)$	$V = \frac{1}{2} EA \int \left(\frac{\partial w}{\partial x} \right)^2 dx$	$T = \frac{1}{2} \rho A \int \left(\frac{\partial w}{\partial t} \right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E , density ρ , cross-sectional area A , second moment of area of cross-section I , transverse displacement $w(x, t)$, applied transverse force $f(x, t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = f(x, t)$	$V = \frac{1}{2} EI \int \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx$	$T = \frac{1}{2} \rho A \int \left(\frac{\partial w}{\partial t} \right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

ENGINEERING TRIPOS PART IIB

Module 3C6 Examination, 2004

Answers

1. See crib

2. (a) $w = 0$ and $\frac{\partial^2 w}{\partial x^2} = 0$ at $x = 0$ and $x = L$;

$$\text{Modes: } u = \sin\left(\frac{n\pi x}{L}\right); \quad \omega_n = \left(\frac{n\pi}{L}\right)^4 \frac{Eh^2}{12\rho}$$

(b), (c) See crib

(d) With this approximation, a tapered beam has same natural frequencies as a uniform beam of thickness $h_0 + \epsilon L/2$.

3. (a) $[M] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \quad [K] = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \text{with } k = P/L$

(b) $\omega_1^2 = (2 - \sqrt{2})\frac{k}{m}; \quad \omega_2^2 = 2\frac{k}{m}; \quad \omega_3^2 = (2 + \sqrt{2})\frac{k}{m}$

(c) 4.7% decrease in lowest natural frequency

4. (a) $[M] = m \begin{bmatrix} 2 & -L/2 & L/2 \\ -L/2 & L^2/3 & 0 \\ L/2 & 0 & L^2/3 \end{bmatrix} \quad [K] = k \begin{bmatrix} 3 & -L & L \\ -L & L^2 & 0 \\ L & 0 & L^2 \end{bmatrix}$

(b) $\omega_1^2 = (3 - \sqrt{3})\frac{k}{m}; \quad \omega_2^2 = 3\frac{k}{m}; \quad \omega_3^2 = (3 + \sqrt{3})\frac{k}{m}$

(c), (d) See crib

(TURN OVER)