UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIA

Thursday 6 May 2004 9.00 to 10.30

Module 3C7

MECHANICS OF SOLIDS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
Special datasheet(s) (2 pages).

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 A thin plate of height h and cross-sectional area A (viewed along the y axis) stands vertically on the ground. as shown in Fig. 1. There is frictionless contact between the bottom of the plate and the ground, and the plate is free to move horizontally. The plate is made of an elastic material with density ρ, Young's modulus E, Poisson's ratio ν and a coefficient of thermal expansion α. The plate is subjected to gravitational forces only, and hence is stress free at all surfaces other than $y = 0$.

(a) Assuming plane stress conditions, show that the stress field

$$\sigma_{xx} = 0, \ \sigma_{yy} = \rho g y + c, \ \sigma_{xy} = 0$$

satisfies the equilibrium equations, and hence determine the constant c from boundary conditions. [25%]

(b) Again assuming plane stress conditions, determine the horizontal displacement u and vertical displacement v of the plate as functions of x and y. [25%]

(c) Calculate the total strain energy stored in the plate. [25%]

(d) The temperature of the plate is now increased uniformly by ΔT. Assuming that the stress field is still given by that determined in (a), obtain the new displacement field of the plate. [25%]

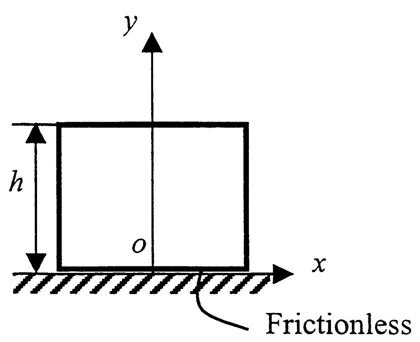

Fig. 1

2 A long cylindrical copper wire of radius b is heated by an electric current. The wire loses heat to the environment through convection at the wire surface. When the temperature in the wire reaches a steady state, the temperature of the wire T is only a function of the radial coordinate r :

$$T = (T_0 - T_\infty)(1 - r^2 / b^2) + T_\infty$$

where $T = T_0$ at the centre of the wire $(r = 0)$ and $T = T_\infty$ at the surface of the wire $(r = b)$. The temperature T_0 can be increased by increasing the electric current. For simplicity, assume $T_\infty = 0$. The Young's modulus E , Poisson's ratio ν , coefficient of thermal expansion α and yield stress in tension Y of copper can be taken as temperature independent.

(a) Assuming plane strain conditions $(\varepsilon_{zz} = 0)$ and neglecting any possible plasticity, determine the complete stress field in the copper wire. [25%]

(b) Assuming $\nu = 1/3$, sketch the stress components as functions of r/b. [25%]

(c) Calculate the radial displacement at the surface of the wire. [25%]

(d) Assuming that the material yields according to Tresca's criterion, determine the maximum T_0 that can be tolerated without causing yielding anywhere in the wire.

[25%]

3 A force P (per unit depth into the page) acts normal to the surface of an elastic half-space as shown in Fig. 2a. The stress field in the half-space can be obtained from an Airy stress function

$$\phi = Ar\theta \sin\theta$$

(a) Show that the stress components $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}$ derived from ϕ satisfy the boundary conditions and hence determine the constant A in terms of the applied load P. [10%]

(b) Find the Cartesian components $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ of the stress at an arbitary point (x, y). Hence, write an expression for the stress $\sigma_{yy}(x,a)$ on a plane at a depth a below the surface. [30%]

(c) Show that $\sigma_{rr} = -P/(\pi a)$ on a circle of radius a which is tangent to the point of load application. [20%]

(d) A thin disk of radius a is compressed by two diametrically opposite forces P as shown in Fig 2b. The remaining boundary of the disk is traction free. Determine the stress distribution $\sigma_{yy}(x,a)$ along the horizontal diameter AB. (Hint: superimpose a hydrostatic tensile field on the stress fields obtained in the previous parts of the question.) [40%]

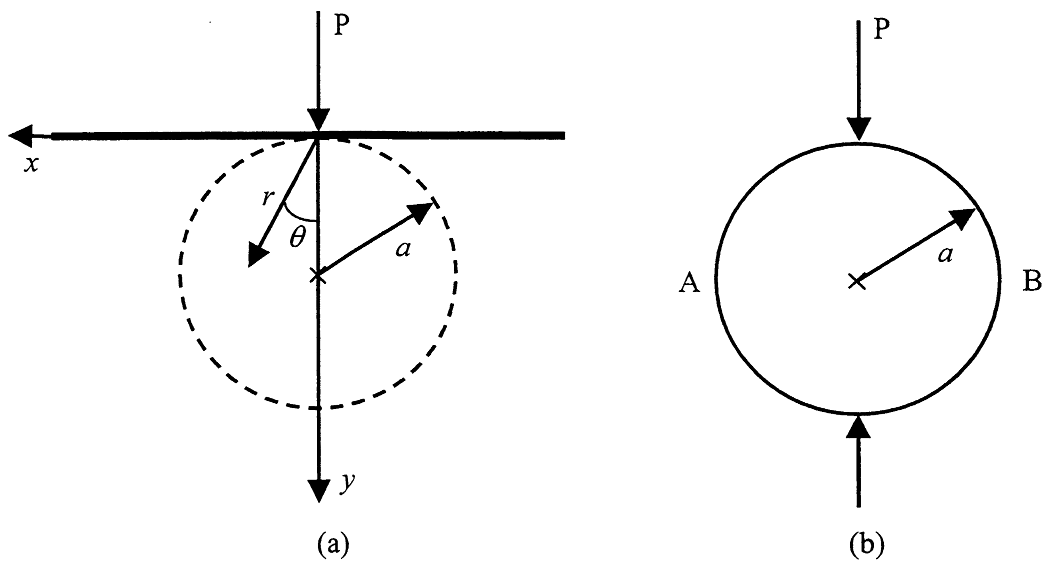

Fig. 2

4 (a) An elastic ideally-plastic material with yield strength σ_Y and Poisson's ratio $\nu = 1/2$ is loaded in plane strain (ie. $\varepsilon_{33} = 0$).

(i) Write the von Mises yield criterion in terms of the stress components $\sigma_{11}, \sigma_{22}, \sigma_{12}$. [20%]

(ii) The material is subjected to a stress state $\sigma_{11} = S,\ \sigma_{22} = \sigma_{12} = 0$. Find the maximum value of S/σ_Y , assuming the von Mises yield condition. [20%]

(iii) What is the maximum value of S/σ_Y , if the material yields according to the Tresca yield criterion? [10%]

(b) A new yield criterion assumes that an isotropic material yields when the magnitude of any principal deviatoric stress reaches a critical value.

(i) Write down expressions for this criterion in terms of the principal stresses $\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}, \sigma_{\mathrm{III}}$ and the uniaxial yield strength σ_Y . [30%]

(ii) Assuming plane stress $(\sigma_{\mathrm{III}} = 0)$, plot the von Mises and the new yield criteria in $\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}}$ space. [20%]

END OF PAPER

UNIVERSITY OF CAMBRIDGE

Paper G4: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{\mathrm{d}(r\sigma_{rr})}{\mathrm{d}r} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{\mathrm{d}(r^2\sigma_{rr})}{\mathrm{d}r}$
Lamé's equations (in elasticity)	$\sigma_{rr} = \mathrm{A} - \dfrac{\mathrm{B}}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\displaystyle\int_c^r rT\,\mathrm{d}r$	$\sigma_{rr} = \mathrm{A} - \dfrac{\mathrm{B}}{r^3}$
	$\sigma_{\theta\theta} = \mathrm{A} + \dfrac{\mathrm{B}}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\displaystyle\int_c^r rT\,\mathrm{d}r - E\alpha T$	$\sigma_{\theta\theta} = \mathrm{A} + \dfrac{\mathrm{B}}{2r^3}$

2. Plane stress and plane strain

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \dfrac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2 \varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial \gamma_{r\theta}}{\partial \theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial \varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial \varepsilon_{rr}}{\partial r} + \dfrac{\partial^2 \varepsilon_{rr}}{\partial \theta^2}$
or (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial \sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right\} \times \left\{\dfrac{\partial^2 \phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2 \phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2 \phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2 \phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2 \phi}{\partial x \partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial \phi}{\mathrm{d}\theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (=\tau_x) = \frac{dF}{dy}$, $\sigma_{zy}\ (=\tau_y) = -\frac{dF}{dx}$

Equilibrium: $T = 2\int_A F\,dA$

Governing equation for elastic torsion: $\nabla^2 F = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D]\, \underset{\sim}{\varepsilon}\, dV$, $W = \underset{\sim}{P}^T \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx}-\sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1\sigma_P^2 + I_2\sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}}\left\{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}}\left\{d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\right\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1-\sigma_2|$, $|\sigma_2-\sigma_3|$ or $|\sigma_3-\sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1-d\varepsilon_2}{\sigma_1-\sigma_2} = \frac{d\varepsilon_2-d\varepsilon_3}{\sigma_2-\sigma_3} = \frac{d\varepsilon_3-d\varepsilon_1}{\sigma_3-\sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}}$$

Answers to 3C7: Mechanics of Solids: 2004

1. (b) $u = \frac{\nu\rho g}{E}(h-y)x, \; v = \frac{\rho g}{E}(\frac{y^2}{2} - hy) + \frac{\nu\rho g x^2}{2E}$

(c) $U = \frac{A\rho^2 g^2 h^3}{6E}$

(d) $u = \frac{\nu\rho g}{E}(h-y)x + \alpha\Delta T x, \; v = \frac{\rho g}{E}(\frac{y^2}{2} - hy) + \frac{\nu\rho g x^2}{2E} + \alpha\Delta T y$

2 (a)

$$\sigma_{rr} = -\frac{E\alpha T_o}{4(1-\nu)}\left(1 - \frac{r^2}{b^2}\right),$$

$$\sigma_{\theta\theta} = -\frac{E\alpha T_o}{4(1-\nu)}\left(1 - \frac{3r^2}{b^2}\right)$$

$$\sigma_{zz} = -\frac{\nu E\alpha T_o}{2(1-\nu)}\left(1 - \frac{2r^2}{b^2}\right)$$

(b) $u = \frac{\alpha b T_o(1+\nu)}{2}$

(c) $T_o = \frac{2(1-\nu)Y}{E\alpha}$

3 (a) $A = -\frac{P}{\pi}$

(d) $\sigma_{yy}(x,a) = \frac{F}{\pi a}\left(1 - \frac{4}{\left[1 + x^2/a^2\right]^2}\right)$

4(a) (ii) $S = \frac{2}{\sqrt{3}}\sigma_Y$

(iii) $S = \sigma_Y$