UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIA

Saturday 8 May 2004 9.00 to 10.30

Module 3D7

FINITE ELEMENT METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
Special datasheets (3 pages)

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 Fig. 1 shows a pin-jointed structure consisting of four bars meeting at joint B. The bars are linear elastic, with cross-sectional area A and Young's modulus E. The structure is initially unstressed. A force F, inclined at 45° to the horizontal, is applied at joint B as shown in the figure.

(a) Set up the stiffness matrix relating general X and Y force components at joint B with the corresponding displacement components. [40%]

(b) Find the displacement components of joint B due to the applied force F. [20%]

(c) Set out a general matrix procedure for recovering the extension and bar force in a bar of a pin-jointed structure, given the complete set of nodal displacement components. Hence derive an expression for the extension of bar III. [20%]

(d) Explain how the formulation in Part (a) could be modified to analyse the structure if, instead of applying a force at joint B, the load to be considered was an initial, uniform strain ε_o in bar I (due, for example, to thermal effects). [20%]

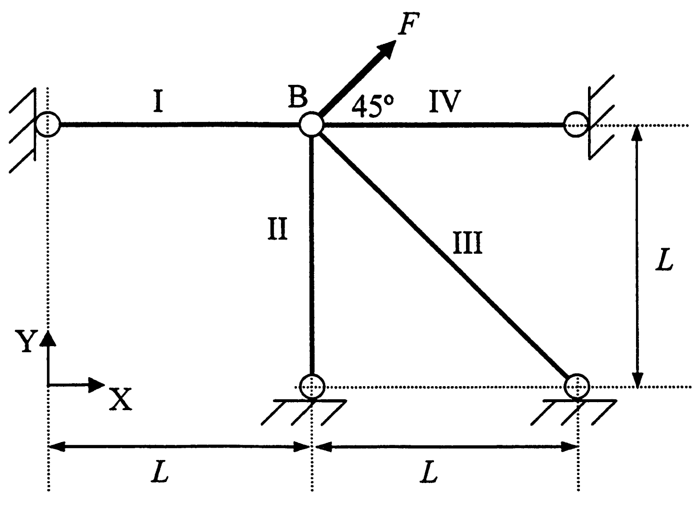


Fig. 1

2 A thin, linear-elastic plate is modelled by the 6-noded triangular plane-stress finite element shown in Fig. 2. The plate has Young's modulus E = 200 kN mm^{-2} and Poisson's ratio ν = 0.3

(a) Sketch the shape functions n_E and n_D and obtain analytical expressions for them in the x, y coordinate system. [20%]

(b) The nodal displacement components are as follows:

u_{Dx} = 3 x 10^{-4} mm u_{Dy} = 1 x 10^{-4} mm
u_{Ex} = -2 x 10^{-4} mm u_{Ey} = -2 x 10^{-4} mm
all other nodal displacement components are equal to zero.

Calculate the stress components σ_x, σ_y and τ_{xy} at a point P with coordinates (1, 0.5). [60%]

(c) Describe, qualitatively, how a *consistent* mass matrix could be obtained that could be used in a dynamic analysis. [20%]

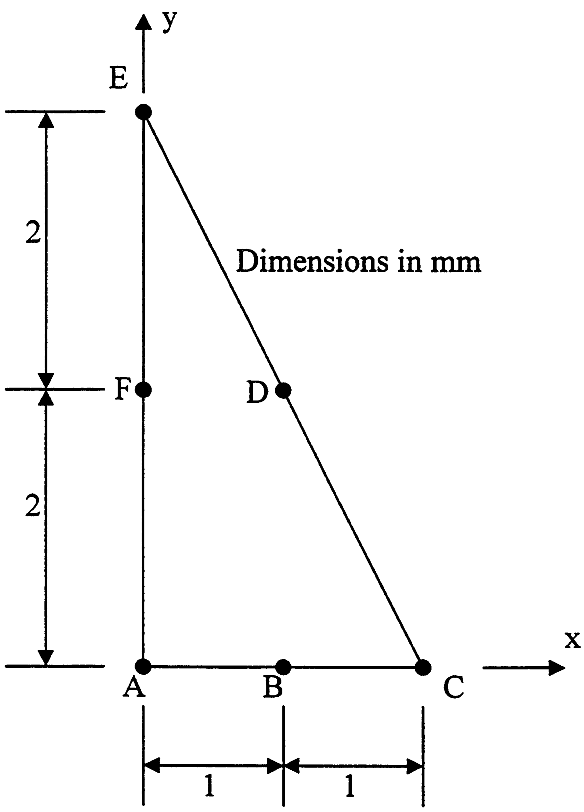


Fig. 2

3. Two pipes are connected end-to-end by flanged joints as shown in Fig. 3. A flange is welded to the end of each pipe section by two circumferential welds. For each joint, four bolts are installed into holes drilled in the flanges and they compress a gasket placed between the two pipe sections. The engineering interest is the temperature field inside the pipe-flange. For simplicity, the gasket is ignored and the attached sections are assumed to be continuous.

(a) The temperatures of the inner and outer surfaces of the pipe are measured to be 50 °C and 0 °C, respectively. The material of the pipe has thermal conductivity k_1, whereas that of the flange (including the weld and bolts) has k_2.

(i) Sketch the boundary of a finite element model of the pipe-flange section including the weld and bolts. Discuss any symmetry conditions adopted in your model. Define the boundary conditions of your model. [25%]

(ii) If $k_1 = k_2$, draw possible temperature contours inside the pipe-flange. There should be four contour lines that are defined by dividing 0 °C to 50 °C into five equal spacings. [10%]

(iii) If $k_1 < k_2$, draw possible temperature contours inside the pipe-flange. Use the same contour intervals defined in (ii). [10%]

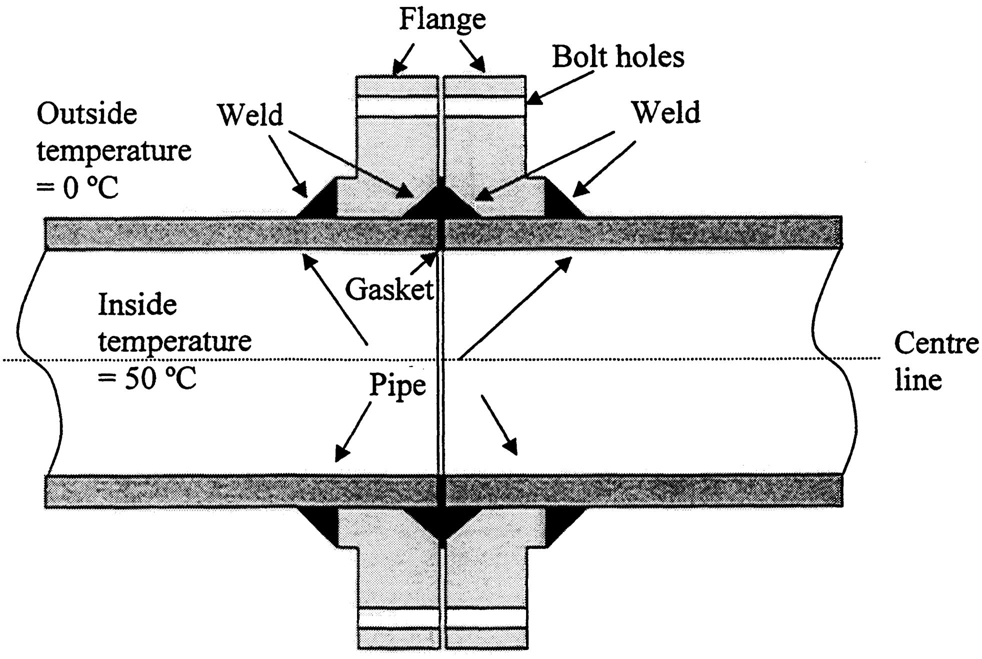
(iv) Instead of the known boundary temperatures, the temperatures of the fluid inside the pipe and the air outside the pipe are known. What are the changes in the boundary conditions and what other information is needed to proceed with the analysis? [10%]

(b) Once the temperature field inside the pipe is known, stress analysis can be performed.

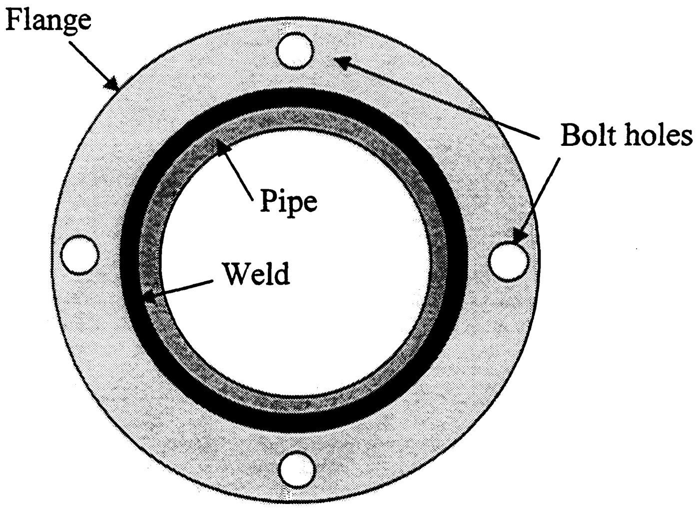
(i) Taking into account the bolt holes and the forces applied by the bolts, sketch the boundary of a finite element model of the pipe-flange section. Discuss any symmetry conditions adopted in your model. Define the boundary conditions of your model. [35%]

(ii) To save computational time, it is proposed to use reduced integration elements. What is the reduced integration element and what are its advantages and limitations? [10%]

(Cont.



(a) Cross section in longitudinal direction



(b) Cross section in diametrical direction

Fig. 3

4. In a two-dimensional finite element mesh the equivalent nodal loads at a node j are given by the following formulae:

$$p_{jX} = t \int_A p_X n_j \, dA$$

$$p_{jY} = t \int_A p_Y n_j \, dA$$

(a) Explain the meaning of the terms t, A, p_X, p_Y and n_j. [20%]

(b) A thin plate cantilever is modelled with a mesh of three-node constant strain triangles as shown in Fig. 4. The whole cantilever is subject to gravitational loading of 10 m s^{-2} in the direction shown. The cantilever plate has a thickness of 1 mm and a density of 2000 kg m^{-3}. Determine the equivalent nodal loads at node A. [40%]

(c) Hence determine the equivalent nodal loads at all the other nodes, B to J. [25%]

(d) Briefly comment on the suitability of this mesh for modelling this problem, and how it might be improved. [15%]

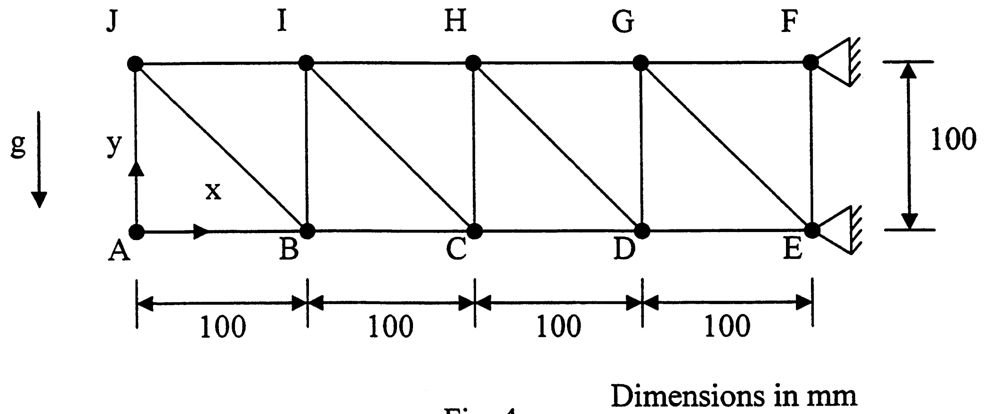


Fig. 4

END OF PAPER

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Part IIA: Module 3D7 2003-4

Finite Element Methods

Formulae

Force Method

Stress resultants: solve $\mathbf{Hr} = \mathbf{p}$ and find $\mathbf{r} = \mathbf{r}_0 + \mathbf{Sx}$;
then, solve $\mathbf{S}^T\mathbf{FSx} = -\mathbf{S}^T(\mathbf{Fr}_0 + \mathbf{e}_0)$ for $\mathbf{x}$.

Displacements: solve $\mathbf{H}^T\mathbf{d} = \mathbf{e}$, where $\mathbf{e} = \mathbf{Fr} + \mathbf{e}_0$.

Displacement Method

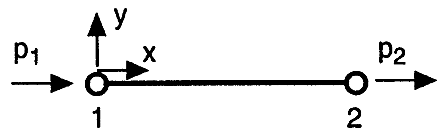
Displacements: solve $\mathbf{Kd} = \mathbf{p}$.

Stress resultants: for element i, solve $\mathbf{F}_i\mathbf{r}_i = \mathbf{e}_i$, where $\mathbf{e}_i = (\mathbf{H}_i')^T\mathbf{d}_i'$.

SP

February 2004

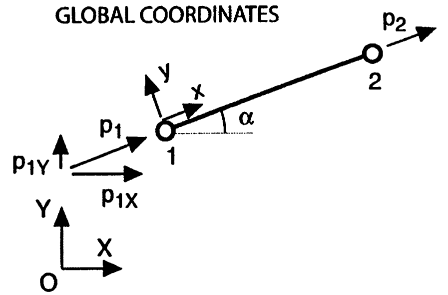
PIN-JOINTED BAR in LOCAL COORDINATES



Static variables	Kinematic variables	Equilibrium	Elasticity	Stiffness $\mathbf{K}_i = \mathbf{H}_i\mathbf{F}_i^{-1}\mathbf{H}_i^T$
$\mathbf{r}_i = [t]$ t = axial force $\mathbf{p}_i = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$	$\mathbf{e}_i = [e]$ e = extension $\mathbf{d}_i = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$	$\mathbf{H}_i = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\mathbf{F}_i = [a]$	$\mathbf{K}_i = \begin{bmatrix} 1/a & -1/a \\ -1/a & 1/a \end{bmatrix}$
Equilibrium Compatibility Constitutive Stiffness	$\mathbf{H}_i\mathbf{r}_i = \mathbf{p}_i$ $\mathbf{H}_i^T\mathbf{d}_i = \mathbf{e}_i$ $\mathbf{F}_i\mathbf{r}_i + \mathbf{e}_{i0} = \mathbf{e}_i$ $\mathbf{K}_i\mathbf{d}_i = \mathbf{p}_i$			

$a = L/AE$, AE = axial stiffness

PIN-JOINTED BAR in GLOBAL COORDINATES



Static variables	Kinematic variables	Coordinate transformation	Equilibrium	Stiffness
$\mathbf{r}_i$ $\mathbf{p}'_i = \begin{bmatrix} p_{1X} \\ p_{1Y} \\ p_{2X} \\ p_{2Y} \end{bmatrix}$	$\mathbf{e}_i$ $\mathbf{d}'_i = \begin{bmatrix} d_{1X} \\ d_{1Y} \\ d_{2X} \\ d_{2Y} \end{bmatrix}$	$\mathbf{T}_i = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$ $\mathbf{R} = \begin{bmatrix} u \\ v \end{bmatrix}$	$\mathbf{H}'_i = \begin{bmatrix} -u \\ -v \\ u \\ v \end{bmatrix}$	$\mathbf{K}'_i = \frac{1}{a}\begin{bmatrix} u^2 & uv & -u^2 & -uv \\ & v^2 & -uv & -v^2 \\ & & u^2 & uv \\ \text{symm.} & & & v^2 \end{bmatrix}$
Equilibrium Compatibility Constitutive Stiffness Transformations	$\mathbf{H}'_i\mathbf{r}_i = \mathbf{p}'_i$ $\mathbf{H}_i'^T\mathbf{d}'_i = \mathbf{e}_i$ $\mathbf{F}_i\mathbf{r}_i + \mathbf{e}_{i0} = \mathbf{e}_i$ $\mathbf{K}'_i\mathbf{d}'_i = \mathbf{p}'_i$ $\mathbf{T}_i\mathbf{p}_i = \mathbf{p}'_i$ $\mathbf{T}_i\mathbf{d}_i = \mathbf{d}'_i$ $\mathbf{T}_i\mathbf{H}_i = \mathbf{H}'_i$ $\mathbf{T}_i\mathbf{K}_i\mathbf{T}_i^T = \mathbf{K}'_i$			

$a = L/AE$, AE = axial stiffness, $u = \cos\alpha$, $v = \sin\alpha$

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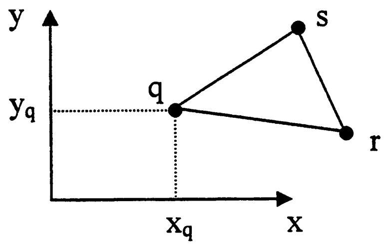
Basic relationships, for element j :

displacements	$\mathbf{u} = \mathbf{N}^j\,\mathbf{d}^j$
strains	$\boldsymbol{\varepsilon} = \mathbf{B}^j\,\mathbf{d}^j$
stresses	$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{D}\,\mathbf{B}^j\,\mathbf{d}^j$
stiffness matrix	$\mathbf{K}^j = \int (\mathbf{B}^j)^T\,\mathbf{D}\,\mathbf{B}^j\,dV$
stiffness equations	$\mathbf{K}^j\,\mathbf{d}^j = \mathbf{p}^j$

Material stiffness (for plane stress)

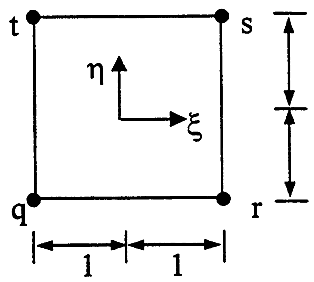
$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Shape functions of some simple plane stress elements

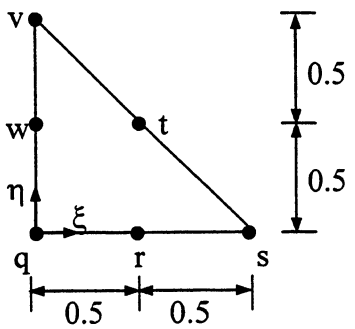


$$n_q = [(x_r y_s - x_s y_r) + (y_r - y_s)x + (x_s - x_r)y]/2A$$
$$n_r = [(x_s y_q - x_q y_s) + (y_s - y_q)x + (x_q - x_s)y]/2A$$
$$n_s = [(x_q y_r - x_r y_q) + (y_q - y_r)x + (x_r - x_q)y]/2A$$

A = area of triangle



$$n_q = (1-\xi)(1-\eta)/4$$
$$n_r = (1+\xi)(1-\eta)/4$$
$$n_s = (1+\xi)(1+\eta)/4$$
$$n_t = (1-\xi)(1+\eta)/4$$



$$n_q = (1-\xi-\eta)(1-2\xi-2\eta)$$
$$n_r = 4\xi\,(1-\xi-\eta)$$
$$n_s = \xi\,(2\xi-1)$$
$$n_t = 4\,\xi\,\eta$$
$$n_v = \eta\,(2\eta-1)$$
$$n_w = 4\eta\,(1-\xi-\eta)$$