UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIA

Wednesday 28 April 2004 9 to 10.30

Module 3F1

SIGNALS AND SYSTEMS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 A discrete-time system, with z-plane transfer function $G(z) = z^{-10}$, is to be controlled by an 'integral action' controller with transfer function $k\,C(z)$, as shown in Fig. 1, where $C(z) = \dfrac{1}{1 - z^{-1}}$ and k is a constant.

(a) Sketch the complete Nyquist diagram for $C(z)\,G(z)$. [40%]

(b) Deduce the range of values of k for which the feedback system is stable. [20%]

(c) Find the phase margin of the system if $k = 0.1$. [20%]

(d) Suppose that an extra delay of m sample periods is introduced into the loop and that $k = 0.1$. What is the largest value of m for which the loop remains stable? [20%]

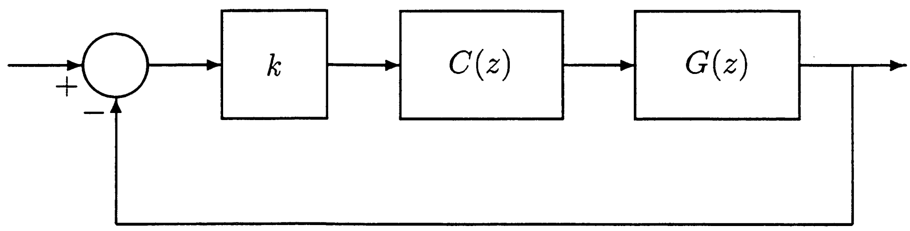

Fig. 1

2 (a) Describe the *forward difference* approximation, the *backward difference* approximation and the *Tustin transformation*, and explain their use for obtaining discrete-time filters from continuous-time ones. [25%]

Show by sketches how the left-half of the s-plane is mapped to the z-plane in each case, and comment on the stability properties of the resulting discrete-time filters. [25%]

(b) If random variable Y is related to random variable X by $Y = g(X)$, where g is a monotonically increasing function, show that the probability density functions (pdfs) of Y and X are related by:

$$f_Y(y) = \frac{f_X(x)}{g'(x)} \quad \text{where } g'(x) = \frac{d}{dx}\, g(x)$$ [20%]

Determine the function $g(X)$ which will convert a random variable X, with uniform pdf from -1 to 1, into the variable Y with a Rayleigh pdf, given by:

$$f_Y(y) = \begin{cases} y\, e^{-y^2/2} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$ [30%]

3 (a) The *characteristic function* of a random variable X is defined using the expectation operator as $\Phi_X(u) = E[e^{juX}]$. Show how the characteristic function relates to $f_X(x)$, the probability density function (pdf) of X, via the Fourier transform. [25%]

(b) Derive an expression for $E[X^n]$, the n^{th}-order moment of X, in terms of the n^{th} derivative of the characteristic function. [25%]

(c) The pdf of X is triangular and is given by:

$$f_X(x) = \begin{cases} a\left(1 - \frac{|x|}{b}\right) & \text{if } -b \leq x \leq b \\ 0 & \text{if } |x| > b \end{cases}$$

Determine (using data books) the characteristic function of X and express it as a power series in u. [25%]

(d) Hence obtain the moments of X from zeroth to fourth order and express them as functions of just the pdf width parameter b. Calculate the kurtosis of X and briefly explain the significance of this dimensionless parameter. [25%]

4 (a) Explain briefly what is meant by a *memoryless source* of data symbols. [10%]

(b) A memoryless source generates a stream of binary symbols (*zero* and *one*) with equal probability. These symbols are transmitted over an electrical wire using a simple transmitter which outputs a 0 volt signal to represent *zero* and outputs 1 volt to represent *one*. Due to electrical noise on the line, the probability density function for receiving a voltage V when a *one* is sent, $P(V|\,one)$, and the probability density function for receiving a voltage V when a *zero* is sent, $P(V|\,zero)$, are given by the following triangular distributions:

$$P(V|\,one) = \begin{cases} 0 & \text{if } V < 0 \\ 2V & \text{if } 0 \leq V \leq 1 \\ 0 & \text{if } V > 1 \end{cases} \qquad P(V|\,zero) = \begin{cases} 0 & \text{if } V < 0 \\ 2 - 2V & \text{if } 0 \leq V \leq 1 \\ 0 & \text{if } V > 1 \end{cases}$$

A simple decoder operates by testing to see if the received voltage is greater or less than 0.5 and assigns a decoded symbol according to:

$$\text{Output symbol} = \begin{cases} A & \text{if } V < 0.5 \\ B & \text{if } V \geq 0.5 \end{cases}$$

Calculate the mutual information between transmitted and received symbols. [30%]

(c) A soft decoder is now used. This decoder splits the voltage range up into four bins and generates an output symbol according to:

$$\text{Output symbol} = \begin{cases} A & \text{if } V < 0.25 \\ B & \text{if } 0.25 \leq V < 0.5 \\ C & \text{if } 0.5 \leq V < 0.75 \\ D & \text{if } V \geq 0.75 \end{cases}$$

Calculate the mutual information between transmitted and received symbols for this soft decoder and comment on the difference between this value and the answer to part (b). [35%]

(d) If the voltage range were split into a very large number of bins, what would be the maximum theoretical mutual information between input and output symbols? [25%]

$$\left[\text{Note:} \qquad \int x \ln(x)\, \mathrm{d}x = \frac{x^2 \ln(x)}{2} - \frac{x^2}{4} + C\right]$$

END OF PAPER