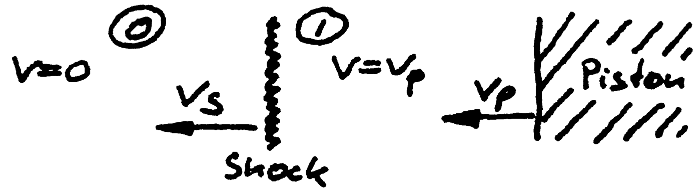
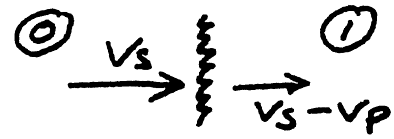
Module 3A3

Q1

a) In stationary frame of reference:



In shock frame of reference (steady):



Using conservation of mass (Area = const.):

$$\rho_0 V_S = \rho_1 (V_S - V_P)$$

$$\Rightarrow \frac{\rho_0}{\rho_1} = \frac{V_S - V_P}{V_S}$$

b) $a_0 = \sqrt{\gamma R T_0} = 340.5 \, \frac{m}{s}$

$\Rightarrow M_0 = \frac{V_S}{a_0} = 1.46$

use tables to get $\frac{\rho_1}{\rho_0}$ for shock Mach number of 1.46 and compare with $V_S / (V_S - V_P)$

$1.79 \equiv 1.79$ q.e.d

from tables: $p_1/p_0 = 2.32$ $(p_1 = 2.3 \cdot 10^5 \, Pa)$

$T_1/T_0 = 1.294$ $(T_1 = 372.7 \, K)$

pto.

c) Wave reflects as expansion to maintain $p_2 = p_1 = 1\,\text{bar}$

Across expansion : $\dfrac{T_1}{T_2} = \left(\dfrac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}}$ (isentropic)

$$\Rightarrow T_2 = 293\ \text{K}$$

$$a_2 = \sqrt{\gamma R T_2} = 343\ \tfrac{m}{s}$$

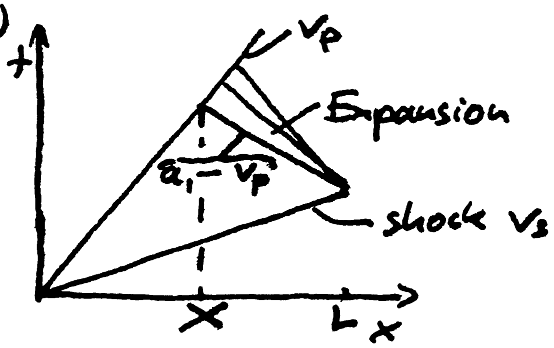
from b) $\quad a_1 = \sqrt{\gamma R T_1} = 387\ \tfrac{m}{s}$

Using Riemann invariant

$$v_2 + \frac{2a_2}{\gamma-1} = v_1 + \frac{2a_1}{\gamma-1} \qquad \text{with } v_1 = 220\ \tfrac{m}{s}$$

gives $\quad v_2 = 440\ \tfrac{m}{s}$

d)



Distance moved by piston $= v_p T$

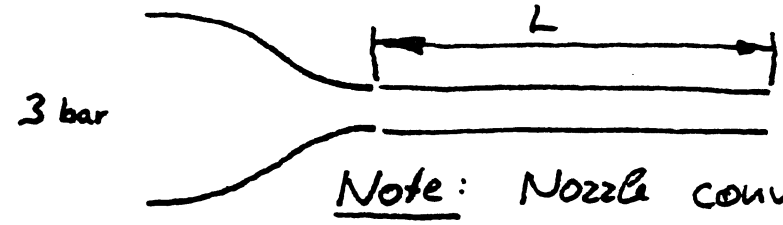
Time for shock to reach end : $\dfrac{L}{v_s}$

Speed of returning wave : $a_1 - v_1$
$= 167\ \tfrac{m}{s}$

Equal time : $\quad \dfrac{X}{V_p} = \dfrac{L}{V_s} + \dfrac{L - X}{a_1 - V_p}$

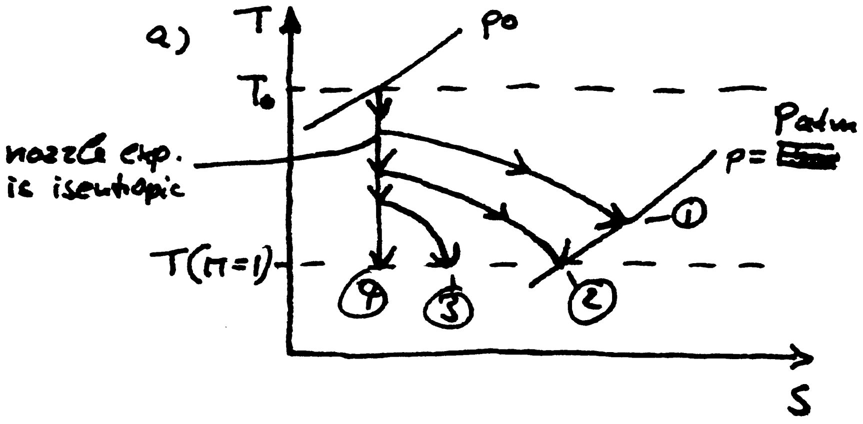
$$\Rightarrow X = 0.76\ L$$

Q2



Note: Nozzle converging, flow subsonic

a)



With decreasing friction:

1. $M<1$ at exit
2. $M=1$ and $p = p_{atm}$ at exit
3. $M=1$ at exit, underexpanded
4. No friction in pipe
   $\Rightarrow$ constant condition in pipe
   $\Rightarrow$ all expansion in nozzle, up to $M=1$.
   Underexpanded on exit

b) Equivalent to case 2 above

$p_{exit} = p_{atm}$, $M_{exit} = 1$

from table: $$\frac{\dot{m}\sqrt{c_p T_0}}{A\, p_{exit}} = 2.425$$

At inlet to pipe: $$\frac{\dot{m}\sqrt{c_p T_0}}{A\, p_0} = \frac{\dot{m}\sqrt{c_p T_0}}{A\, p_{exit}} \cdot \frac{p_{exit}}{p_0}$$

$$= 2.425 \cdot \frac{1}{2.5} = 0.97$$

hence from tables: $\underline{M_{inlet} = 0.57}$

from tables: $$\frac{4 C_f L_{max}}{D} = 0.9904$$

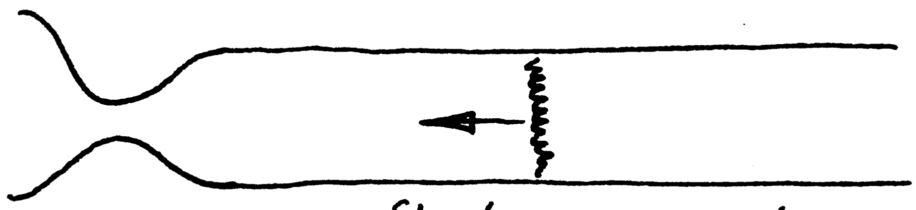
here $L \hat{=} L_{max}$, hence $\underline{C_f = 0.0025}$

Q2 continued

c) Now, nozzle is con-diverging, hence supersonic flow is possible (and shocks)

If there is no shock (but exit is choked), then $L \hat{=} L_{max}$, C_f as in b) but using $M=2$

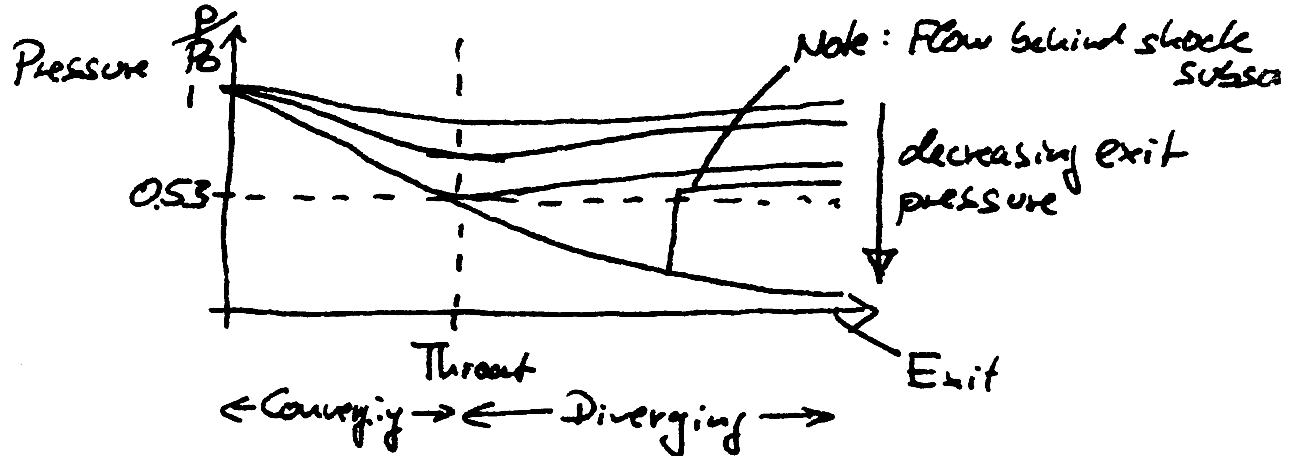
If shock is present then, with increasing C_f, the shock appears earlier.

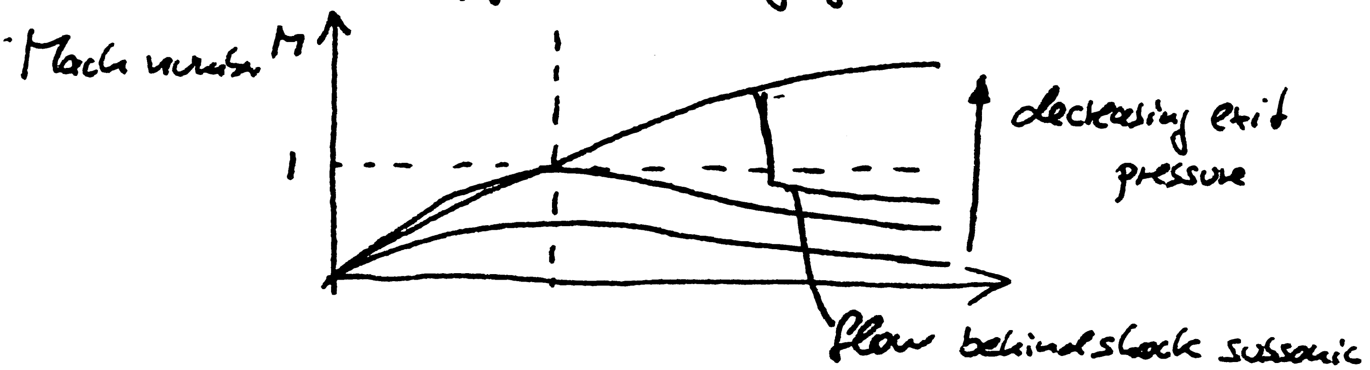


Shock moves upstream as C_f increases

Q3

a) Use standard figures and explanation as found in handout / textbooks :





b) At $h = 16000\,m$: $p = 0.1022 \cdot p_{atm}$ (databook)

$$p = 1.01 \cdot 10^4\ Pa$$

hence : $\frac{p}{p_0} = \left(\frac{1.01 \cdot 10^4}{2 \cdot 10^5}\right) = 0.0505$

from tables : $M_{exit} = 2.595$ (or use $M = 2.6$)

from tables : at throat : $\frac{\dot{m}\sqrt{c_p T_0}}{A^* p_0} = 1.281$

at exit : $\frac{\dot{m}\sqrt{c_p T_0}}{A_e p_0} = 0.444$

$$\Rightarrow \underline{\frac{A_e}{A^*} = 2.89}$$

p.t.o.

Q3 contd.

Thrust: use tables $\frac{F}{\dot{m}\sqrt{c_p T_0}} = 1.185$

$$\text{Thrust} = F - p_{atm} \cdot A_e$$

$$F = \frac{F}{\dot{m}\sqrt{c_p T_0}} \cdot \frac{\dot{m}\sqrt{c_p T_0}}{A^* p_0} \cdot A^* \cdot p_0 = \underline{23.9\,N}$$

$$\text{Thrust} = F - p_{atm} A_e = \underline{\underline{21.6\,N}}$$

c) $p_{exit} = p_{atm}$ $$\frac{\dot{m}\sqrt{c_p T_0}}{A_e \cdot p_e} = \underbrace{\frac{\dot{m}\sqrt{c_p T_0}}{A^* p_0}}_{1.281} \cdot \underbrace{\frac{A^*}{A_e}}_{\frac{1}{2.89}} \cdot \underbrace{\frac{p_0}{p_e}}_{2} = 0.877$$

hence: $M_{exit} = 0.39$

from tables: $\frac{\dot{m}\sqrt{c_p T_0}}{A_e\, p_{0e}} = 0.789$

hence: $$\frac{p_{0E}}{p_0} = \frac{\left(\frac{\dot{m}\sqrt{c_p T_0}}{A^* p_0}\right)}{\left(\frac{\dot{m}\sqrt{c_p T_0}}{A_e\, p_{0E}}\right)} \cdot \frac{A^*}{A_e} = 0.5618$$

from shock tables: $M_{\text{upstream of shock}} = 2.35$

Thrust:
$$F = \frac{F}{\dot{m}\sqrt{c_p T_0}} \cdot \frac{\dot{m}\sqrt{c_p T_0}}{A\, p_{0E}} \cdot p_{0E} \cdot A_e = 27.8\,N$$

$$\text{Thrust} = F - p_{atm} \cdot A_e = \underline{\underline{4.88\,N}}$$

p.t.o.

Q3 cont.)

To check that there is indeed a shock in the nozzle, prove that shock free expansion with the given area ratio $\left(\frac{A_e}{A^*}\right)$ gives exit conditions where $p_{exit} < p_{atm}$ (also check purely subsonic flow is impossible).

From b) $M_{exit} = 2.595$, $\frac{p_{exit}}{p_0} = 0.05 \ll \frac{1}{2}$

Note: Nozzle thrust is quite small – this is only a small nozzle.

Q4) a) $C_p = \dfrac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2}$

$$= \left(\frac{p}{p_\infty} - 1\right) \cdot \frac{1}{\frac{1}{2}\frac{\rho_\infty}{p_\infty} V_\infty^2}$$

using: $\dfrac{p}{\rho} = RT$ gives

$$C_p = \left(\frac{p}{p_\infty} - 1\right) \frac{2RT_\infty}{V_\infty^2}$$

using: $a_\infty^2 = \gamma R T_\infty$ gives

$$\underline{C_p = \left(\frac{p}{p_\infty} - 1\right) \frac{2}{\gamma M_\infty^2}}$$

b) Critical pressure coeff. C_p^* occurs when locally $M=1$

hence: $$\frac{p}{p_\infty} = \underbrace{\frac{p}{p_0}}_{M=1} \cdot \underbrace{\frac{p_0}{p_\infty}}_{f(M_\infty)}$$

$$= 0.528 \left(1 + \frac{\gamma-1}{2} M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}$$

hence: $$\underline{C_p^* = 1.056 \frac{\left(1 + \frac{\gamma-1}{2} M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}}{\gamma M_\infty^2} - \frac{2}{\gamma M_\infty^2}}$$

c) $C_{p,\min,\text{incompressible}} = -1.0$

Use Prandtl - Glauert rule to predict $C_{p,\min}$ under compressible flow conditions

$$C_{p_{\min}} = -\frac{1.0}{\sqrt{1 - M_\infty^2}}$$

At critical conditions: $C_{p,\min} \equiv C_p^*$

Q4, cont.

Best to use iterative method, eg:

M_∞	0.5	0.6	0.7	0.61
$C_{p,min}$	-1.16	-1.24	-1.39	-1.26
C_p^*	-2.13	-1.29	-0.78	-1.23

$\Big\}\ M_{\infty,crit} \approx \underline{\underline{0.605}}$

Assumptions made in Prandtl - Glauert rule:

inviscid

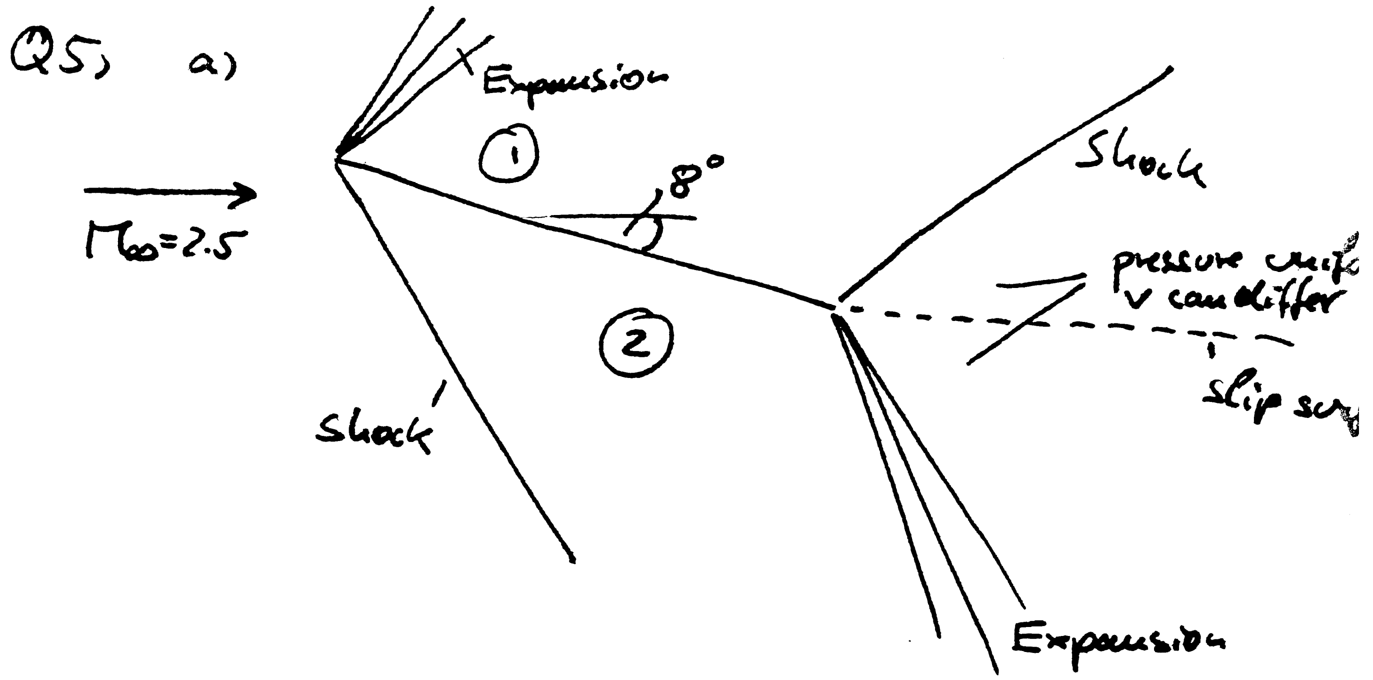
thin aerofoil

small α

$M < 1$ (or $M_\infty < 1$)

The last assumption is doubtful so $M_{\infty,crit}$ is only a (reasonable) estimate.

Q5) a)



b) In region 1, use Prandtl - Meyer angles

$M = 2.5 \quad \nu_0 = 39.12^\circ \implies \nu_1 = 47.12^\circ$

$\implies M_1 = 2.87 \qquad \frac{p}{p_0} = 0.0331$

p_0 is unchanged

In region 2, use oblique shock tables:

$M = 2.5, \ \theta = 8^\circ \implies M_2 = 2.17$

$\frac{p_2}{p_\infty} = 1.657$

Freestream conditions: $p_\infty = 100 \cdot 10^3 \text{ Pa}$

use tables $\implies p_0 = 1.709 \cdot 10^6 \text{ Pa}$

$\implies p_1 = 56.6 \cdot 10^3 \text{ Pa}$

$p_2 = 165.7 \cdot 10^3 \text{ Pa}$

Force on plate: $F = (p_2 - p_1) \cdot c \cdot w$

per unit width: $\frac{F}{w} = 218 \text{ kN}$

Q5 cont)

Assumptions made: inviscid flow
no reflected waves reach aerofoil (correct)

$$Lift = F \cdot \cos\alpha = 216\ kN$$
$$Drag = F \cdot \sin\alpha = 30\ kN$$

c) $$C_L = \frac{L}{\frac{1}{2}\rho_\infty U_\infty^2 \cdot C}$$

From tables: $\frac{\frac{1}{2}\rho V^2}{p_0} = 0.256$

$$C_L = 0.247$$

and similarly

$$C_D = \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 C} = 0.034$$

d) At low speed flow we might expect:

inviscid: $C_D = 0$

$C_L = 2\pi\alpha = 0.877$

$\Rightarrow$ significantly more lift at low speed

viscous: $C_D \neq 0$ but probably smaller than the wave drag

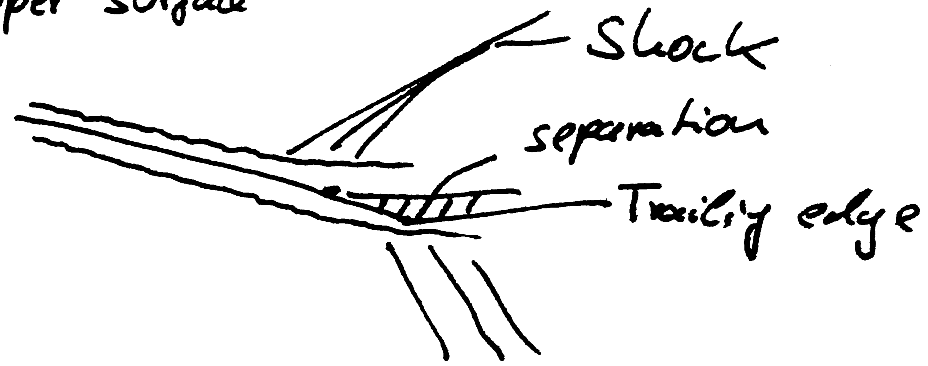
$C_L < 2\pi\alpha$ due to leading edge separation

e) There is no region of adverse pressure gradient on the aerofoil but the influence of the shock

Q5 cont

e) cont.

.. at the trailing edge may extend upstream to cause some TE separation on the upper surface



Note: This question did not explicitly specify γ, but since it considers an aerofoil it can be assumed that $\gamma = 1.4$.

Q6

a) A velocity potential can be used when the flow is irrotational.
The equation solved is the continuity equ.

b) For a 1D duct the continuity equ in differential form is:

$$\frac{d(\rho h V)}{dx} = 0$$

incompressible => ρ = const.

set $V = \frac{d\phi}{dx}$ gives $\frac{d}{dx}\left(h \frac{d\phi}{dx}\right) = 0$

$$\Rightarrow \underline{\frac{d^2\phi}{dx^2} = -\frac{1}{h}\frac{dh}{dx}\frac{d\phi}{dx}}$$

c) Central difference approximation:

$$\frac{\phi_{n+1} + \phi_{n-1} - 2\phi_n}{\Delta x^2} = -\frac{1}{h}\frac{dh}{dx}\,\frac{\phi_{n+1} - \phi_{n-1}}{2\Delta x}$$

$$\Rightarrow 2\phi_n = \phi_{n+1}\left(1 + \frac{\Delta x}{2h}\frac{dh}{dx}\right) + \phi_{n-1}\left(1 - \frac{\Delta x}{2h}\frac{dh}{dx}\right)$$

$$\beta = \frac{\Delta x}{2h}\frac{dh}{dx}$$

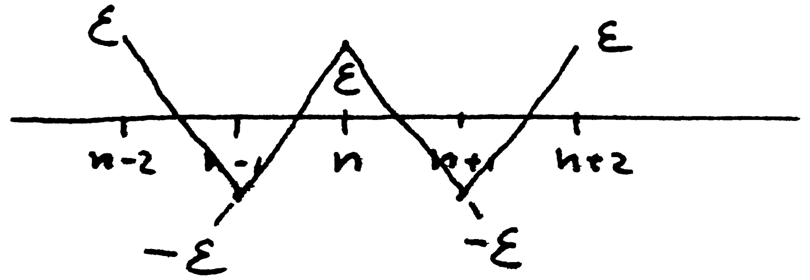
$$\Rightarrow \phi_n = \frac{1}{2}\left(\phi_{n+1}(1+\beta) + \phi_{n-1}(1-\beta)\right)$$

Using a relaxation factor R:

$$\phi_n^{new} = (1-R)\,\phi_n^{OLD} + R \cdot \frac{1}{2}\left[(1+\beta)\,\phi_{n+1}^{OLD} + (1-\beta)\,\phi_{n-1}^{OLD}\right]$$

Q6 cont.

d) Consider saw-tooth perturbation:



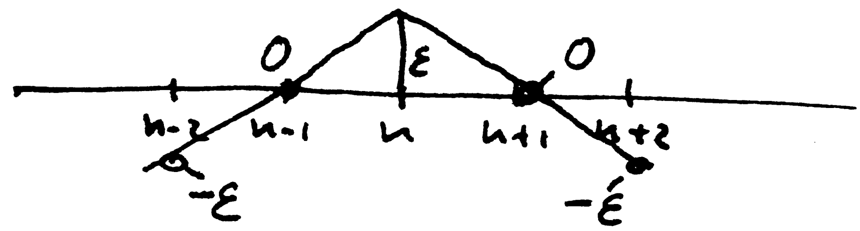
hence: $\phi_n^{NEW} = (1-R)\varepsilon + \frac{R}{2}\left[(1+\beta)(-\varepsilon)+(1-\beta)(-\varepsilon)\right]$

$$= (1-2R)\varepsilon$$

For stability: $|\phi_n^{NEW}| < \varepsilon$

$\Rightarrow R < 1$ (independent of β!)

e) Consider perturbation at twice the wave length:



hence $\phi_n^{NEW} = (1-R)\varepsilon$

but also $\phi_{n+1}^{NEW} = R\beta \cdot \varepsilon$

and $\phi_{n-1}^{NEW} = -R\beta\varepsilon$

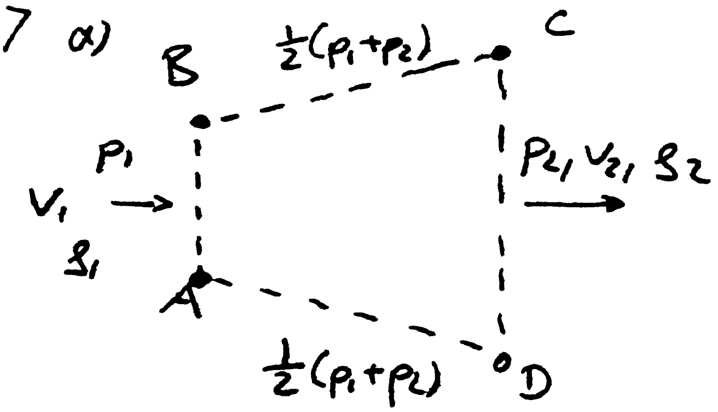
Max difference between two points: $(1-R+R\beta)\varepsilon$

for stability: $|1-R+R\beta| < 2$

$\Rightarrow R < \frac{3}{1-\beta}$

This is more unstable if $\beta < -2$

Q7 a)



Control volume equs: A: Area of cell ABCD

$$A \cdot \frac{\partial \rho}{\partial t} = \rho_1 V_1 h_1 - \rho_2 V_2 h_2 \qquad \text{mass flux}$$

$$A \frac{\partial(\rho V)}{\partial t} = p_1 h_1 - p_2 h_2 + \frac{p_1+p_2}{2}(h_2-h_1) + \rho_1 V_1^2 h_1 - \rho_2 V_2^2 h_2$$

mom. flux

$$A \frac{\partial \rho E}{\partial t} = \rho_1 V_1 h_1 h_{01} - \rho_2 V_2 h_2 h_{02} \qquad \text{energy flux}$$

An explanation for the time-marching method can be found in the notes.

b) By adding the mass and momentum flux equations one obtains:

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x} - \rho v \frac{\partial v}{\partial x}$$

using $dp = \pm \rho a \, dv$ gives:

$$\frac{\partial v}{\partial t} = \pm a \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial x} = -(v \mp a) \frac{\partial v}{\partial x}$$

set $c = -(v \mp a)$ and use finite differences:

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} = c \, \frac{v_{i+1}^n - v_{i-1}^n}{2\Delta x}$$

to give: $$v_i^{n+1} = \frac{c \, \Delta t}{2\Delta x}(v_{i+1}^n - v_{i-1}^n) + v_i^n$$

Q8.

a) Compressors increase p ⇒ adverse pressure gradients ⇒ danger of separation ⇒ need to limit work per stage

Turbines reduce p ⇒ favourable pressure gradients.

b) $p_{02}/p_{01} = 10 \qquad \frac{T_{02is}}{T_{01}} = 10^{\frac{\gamma-1}{\gamma}} = 1.9306$

$\Rightarrow \Delta T_{0is} = 0.9306\, T_{01}$

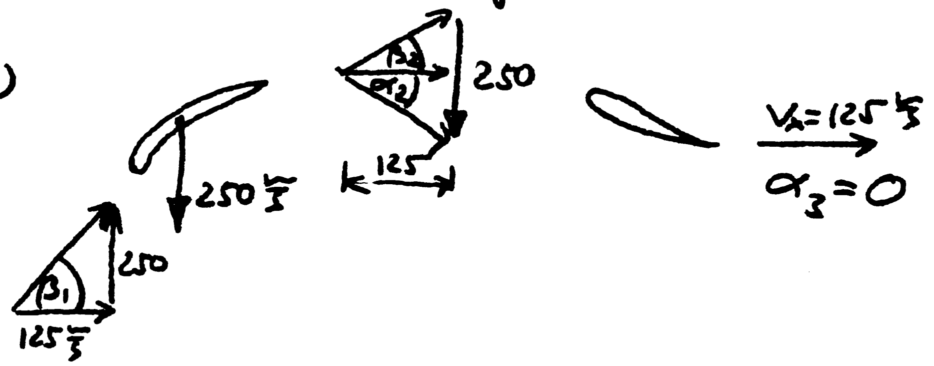
$\Delta T_{0actual} = \frac{1}{0.86} \cdot 0.9306 \cdot 300 = 324.64\,K$

$\Delta h_{0stage} = 0.35 \cdot 250^2$ (from stage loading coeff.)

$\Rightarrow \Delta T_{0stage} = \frac{\Delta h_{0stage}}{c_p} = 21.8\,K$

$\Rightarrow N_{stages} = \frac{\Delta T_0}{\Delta T_{0stage}} = 14.915 \approx 15$ Stages

c)



Euler's eqn.: $\Delta h_0 = U \Delta V_\theta$

$\Rightarrow \Delta V_\theta = 0.35 \cdot 250 = 87.5\,\frac{m}{s}$

$V_{\theta_2} = 87.5\,\frac{m}{s}$ since $V_{\theta_1} = 0$

$\Rightarrow \tan\beta_1 = \frac{250}{125} \quad \Rightarrow \beta_1 = 63.4^\circ$

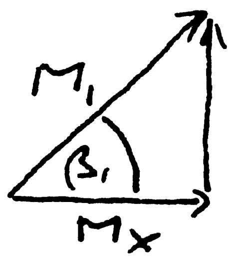
$\tan\beta_2 = \frac{87.5 - 250}{125} \quad \Rightarrow \beta_2 = 52.4^\circ$ (upwards)

$\tan\alpha_2 = \frac{87.5}{125} \quad \Rightarrow \alpha_2 = 35.0^\circ$

$\alpha_3 = 0^\circ$ (no swirl)

Q8 cont.)

d) Inflow triangle :



β_1 unchanged , $M_1 = 0.9$

$\Rightarrow M_x = 0.9 \cos\beta_1 = 0.4025$ (in stationary frame)

$\Rightarrow T_1/T_{01} = 0.9686$

$\Rightarrow T_1 = 290.58$

(Note : speed of sound varies with T !)

$\Rightarrow a_1 = \sqrt{\gamma R T_1} = 342 \frac{m}{s}$

$\Rightarrow$ in relative frame : $U = 0.9 \cdot 342 \frac{m}{s} \cdot \sin\beta_1$

$U = 275.3 \frac{m}{s}$

$\frac{\Delta h_0}{U^2}$ is unchanged because velocity triangles are similar

$\Rightarrow \Delta h_{0\,stage} = 0.35 \cdot (275.3 \frac{m}{s})^2$

$\Rightarrow \Delta T_{0\,stage} = 26.39 K$

as before : $N_{stages} = 12.3$

$\Rightarrow$ 13 stages needed