Part IIA 2005

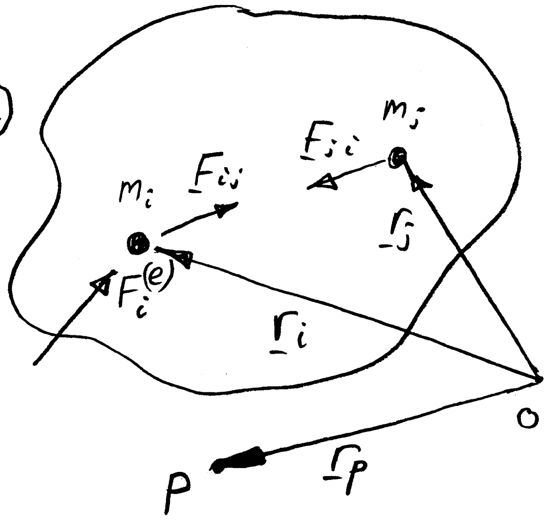
Answers

PART IIA 2005
3C5: Dynamics
Principal Assessor: Dr H E M Hunt

Datasheet: Dynamics & Vibration

1.(a) Newton's law for a particle : $\underline{F} = m\ddot{\underline{r}}$ ①

(i) A rigid body is a collection of particles m_i each obeying ①



$$\underline{F}_i^{(e)} + \sum_{j \neq i} \underline{F}_{ij} = m_i \ddot{\underline{r}}_i \quad ②$$

total external force acting on m_i

sum of all internal forces between m_i and all other particles

Sum ② over all particles i

$$\therefore \sum_i \underline{F}_i^{(e)} + \underbrace{\sum_i \sum_{j \neq i} \underline{F}_{ij}}_{} = \sum m_i \ddot{\underline{r}}_i \quad ③$$

This term is zero because internal forces cancel in pairs by Newton III

Define : $\underline{F}^{(e)} = \sum_i \underline{F}_i^{(e)}$ = total external force

and $M = \sum m_i$ = total mass

and $\underline{r}_G$ = position of centre of mass

such that $M\underline{r}_G = \sum_i m_i \underline{r}_i$

and $\underline{p} = \sum_i m_i \dot{\underline{r}}_i$ = total linear momentum ④

$= M\dot{\underline{r}}_G$

$\therefore \dot{\underline{p}} = \sum_i m_i \ddot{\underline{r}}_i$

So that (3) becomes $\underline{F}^{(e)} = \dot{\underline{p}}$

(ii) Also define $\underline{h}_P = \sum_i (\underline{r}_i - \underline{r}_P) \times (m_i \dot{\underline{r}}_i)$

= the total moment of momentum of all particles about an arbitrary point P

$$\therefore \dot{\underline{h}}_P = \sum_i (\dot{\underline{r}}_i - \dot{\underline{r}}_P) \times m_i \dot{\underline{r}}_i + \sum_i (\underline{r}_i - \underline{r}_P) \times (m_i \ddot{\underline{r}}_i)$$

and note that $\dot{\underline{r}}_i \times \dot{\underline{r}}_i = 0$, and also that $\dot{\underline{r}}_P$ can be taken out of the summation

$$\therefore \dot{\underline{h}}_P = -\dot{\underline{r}}_P \times \sum_i m_i \dot{\underline{r}}_i + \sum_i (\underline{r}_i - \underline{r}_P) \times (m_i \ddot{\underline{r}}_i)$$

$$\therefore \sum_i (\underline{r}_i - \underline{r}_P) \times (m_i \ddot{\underline{r}}_i) = \dot{\underline{h}}_P + \dot{\underline{r}}_P \times \underline{p} \qquad (5)$$

from (4)

Now take moments of (2) about P

$$\therefore (\underline{r}_i - \underline{r}_P) \times \underline{F}_i^{(e)} + \sum_{j \neq i} (\underline{r}_i - \underline{r}_P) \times \underline{F}_{ij} = (\underline{r}_i - \underline{r}_P) \times (m_i \ddot{\underline{r}}_i) \qquad (6)$$

and sum over all particles m_i noting that $\sum_i \sum_{j \neq i} (\underline{r}_i - \underline{r}_P) \times \underline{F}_{ij} = 0$ because the moments about P of all internal forces $\underline{F}_{ij}$ cancel in pairs

Also define $\underline{Q}_P^{(e)} = \sum_i (\underline{r}_i - \underline{r}_P) \times \underline{F}_i^{(e)}$ = the total moment of external forces about P

so that (6) becomes

$$\sum_i (\underline{r}_i - \underline{r}_P) \times \underline{F}_i^{(e)} + 0 = \sum_i (\underline{r}_i - \underline{r}_P) \times (m_i \ddot{\underline{r}}_i)$$

$$\therefore \quad \underline{Q}_P^{(e)} = \dot{\underline{h}}_P + \dot{\underline{r}}_P \times \underline{p} \qquad (7)$$

using (5)

b/ If P and G are coincident then

$\dot{\underline{r}}_P = \dot{\underline{r}}_G \quad \therefore \quad \dot{\underline{r}}_P \times \underline{p}$

$$= \dot{\underline{r}}_G \times (M \dot{\underline{r}}_G)$$

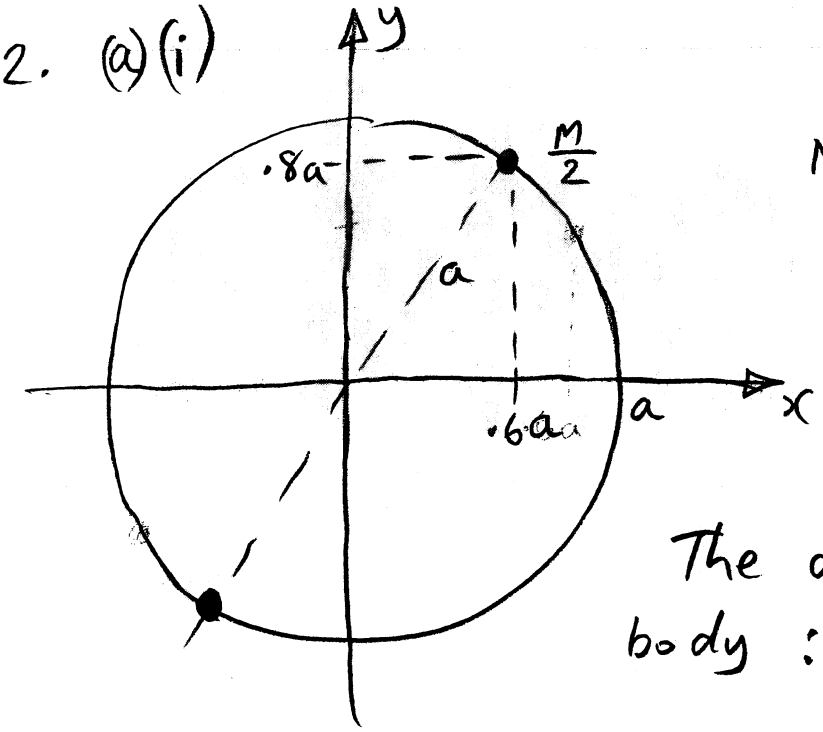
$$= 0$$

$$\therefore \quad \underline{Q}_P^{(e)} = \dot{\underline{h}}_P$$

This is a special result, easier to understand than (7)

This question was very well done by those that knew what to do. Careful attention to detail was required to get full marks and many got full marks. This question was a real time waster for those who didn't quite remember the right steps.

2. (a)(i)



Note : $0.6^2 + 0.8^2 = 1$

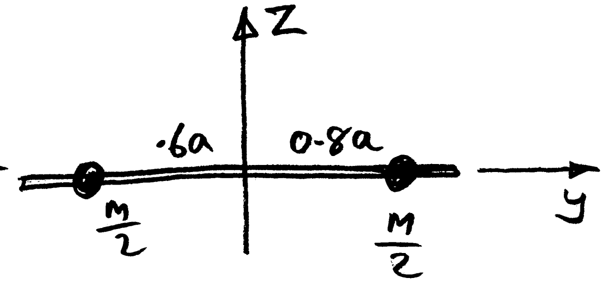
∴ particles are on the edge of the disc

The disc is an AAC body : $A = \frac{1}{4} m a^2$ (databook)

$C = \frac{1}{2} m a^2$

since $A + A = C$ by perpendicular axes theorem

<u>View along x axis</u>



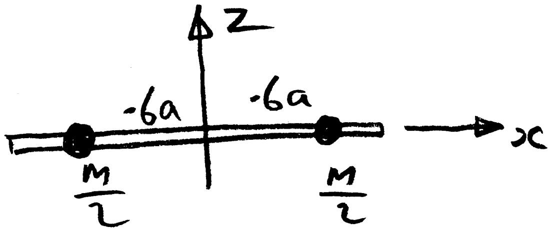
$I_{xx} = \frac{1}{4} m a^2 + 2 \cdot \frac{M}{2} \cdot (0.8a)^2$

$= (0.25 + 0.64) m a^2$

$= 0.89\, m a^2$

$I_{yz} = 0$ since all mass is along line $z = 0$

<u>View along y axis</u>



$I_{yy} = \frac{1}{4} m a^2 + 2 \cdot \frac{M}{2} \cdot (0.6a)^2$

$= (0.25 + 0.36) m a^2$

$= 0.61\, m a^2$

$I_{xz} = 0$

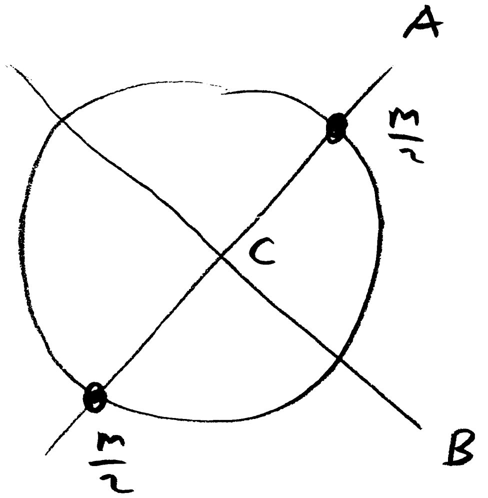
<u>View along z axis</u> (see diagram above)

$I_{zz} = \frac{1}{2} m a^2 + 2 \cdot \frac{M}{2} a^2 = (0.5 + 1) m a^2 = 1.5\, m a^2$

$I_{xy} = \frac{M}{2} \cdot (0.6a)(0.8a) + \frac{M}{2}(-0.6a)(-0.8a) = 0.48\, m a^2$

$$\therefore \quad I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} = ma^2 \begin{bmatrix} 0.89 & -0.48 & 0 \\ -0.48 & 0.61 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

(ii) Either use eigenvalue analysis to get principal inertias & axes or (and much more quickly) do it by inspection: The disc is AAC so the axis through the two particles is principal. Principal axis A B & C are as shown on the diagram with



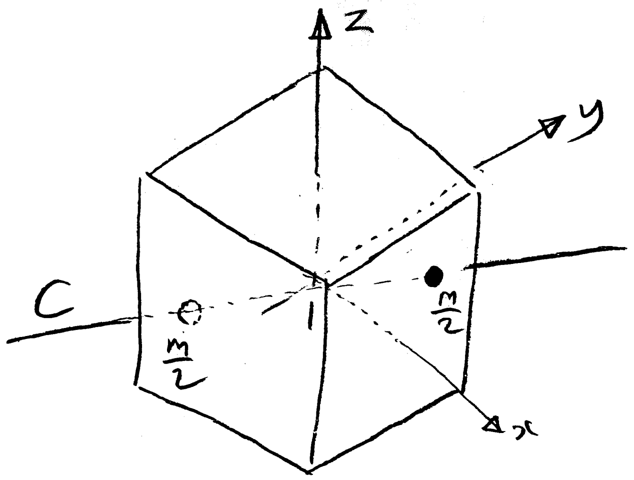
$$A = \frac{1}{4} ma^2$$

$$B = \frac{1}{4} ma^2 + ma^2 = \frac{5}{4} ma^2$$

$$C = \frac{1}{2} ma^2 + ma^2 = \frac{3}{2} ma^2$$

All but one candidate did this question. Many chose to do this the long way (eigenvalues) which is fine but it wastes time, and it's easy to make mistakes. Some (few) used "inspection" to see that they'd made mistakes & were rewarded with extra marks

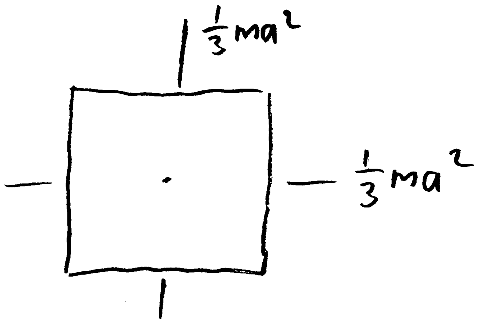
2(b) Having practised the "inspection" method in (a) it's easy now to do it here. A cube is an AAA body so the axis through the particles is principal, call it C



Any axis $\perp$ C is principal because the resulting body is "AAC"

For the cube, $I = \frac{2}{3} m a^2$

This is found by viewing along z (say) and from the data book for a square plate $I_{xx} = I_{yy} = \frac{1}{3} m a^2$ and by perpendicular axes theorem $I_{zz} = \frac{2}{3} m a^2$



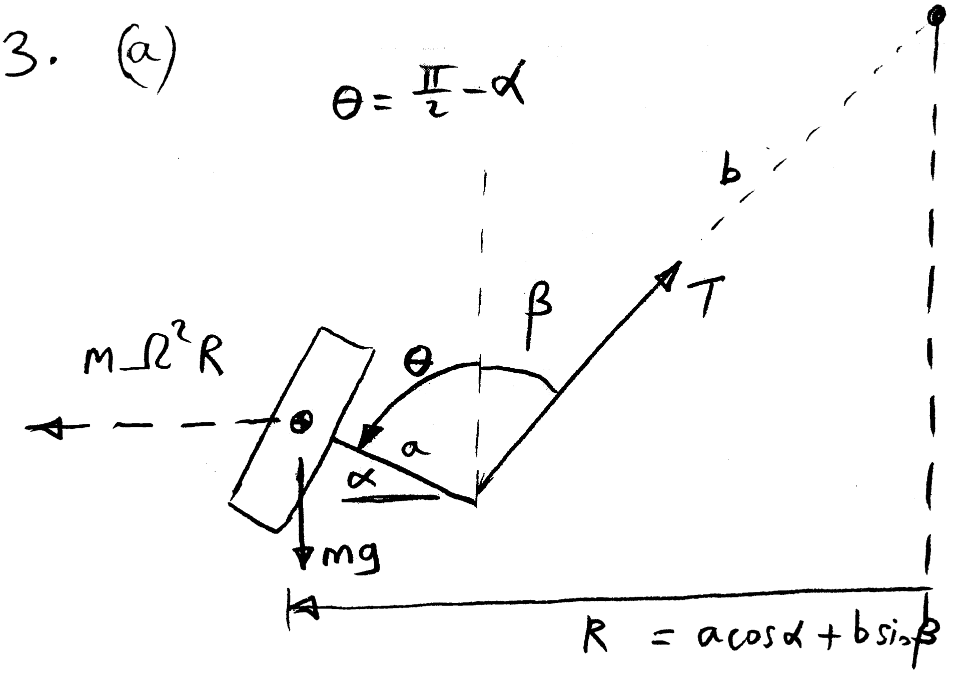
So $\underline{\underline{C = \frac{2}{3} m a^2}}$

$$A = \frac{2}{3} m a^2 + 2 \cdot \frac{m}{2} \left(a^2 + \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2 \right)$$

$$= \left(\frac{2}{3} + \frac{3}{2} \right) m a^2 = \underline{\underline{\frac{13}{6} m a^2}}$$

Those who used eigenvalue analysis mostly got the right answers – much to their credit under exam conditions. But they wasted a <u>lot</u> of time which showed in other questions. Moral: think of a cube as a sphere.

3. (a)



(i) resolving vertically:

$$T\cos\beta = mg \qquad (1)$$

$$\therefore T = \frac{mg}{\cos\beta}$$

(ii) resolving horizontally: $T\sin\beta = m\Omega^2 R$

$$= m\Omega^2(a\cos\alpha + b\sin\beta) \qquad (2)$$

$$\frac{(2)}{(1)} \therefore \tan\beta = \frac{\Omega^2(a\cos\alpha + b\sin\beta)}{g} \qquad (3)$$

and for small α and β

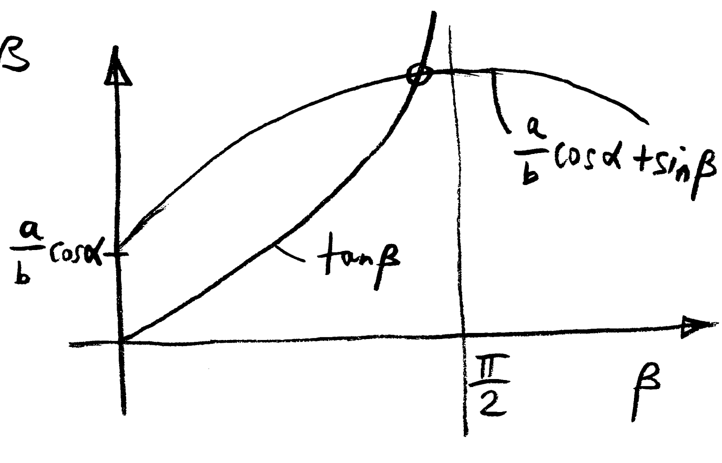
$$\beta \simeq \frac{\Omega^2}{g}(a + b\beta)$$

$$\therefore \beta(g - b\Omega^2) \approx a\Omega^2$$

$$\therefore \beta \approx \frac{a\Omega^2}{g - b\Omega^2} \qquad (4)$$

(b) putting $g = b\Omega^2$ into (4) gives $\beta = \infty$ which is stupid. Clearly β will be large but the small-β assumption no longer holds. From (3) we can see graphically that β will be near $90°$

$$\tan\beta = \frac{a}{b}\cos\alpha + \sin\beta$$



So the gyro pendulum will describe a large radius circle in steady state. [This is what Laithwaite did, but he released his gyro away from the steady state and the transient motion is, at first sight, curious]

(c) Gyro equation ② is the only useful one

$$Q_2 = A\dot{\omega}_2 + (A\Omega_3 - C\omega_3)\,\omega_1$$

with $\dot{\omega}_2 = 0$ (steady state)

& $A\Omega_3 \ll C\omega_3$ (fast spin $\omega_3 = \omega$)

& $\omega_1 = -\Omega \sin\theta = -\Omega\cos\alpha$

$$\therefore\ Q_2 = C\omega\Omega\cos\alpha = \tfrac{1}{2} m a^2 \omega\Omega\cos\alpha$$

and $Q_2 = T\cos\beta\, a\cos\alpha + T\sin\beta\, a\sin\alpha$

$$= mg\, a\cos\alpha + m\Omega^2 R\, a\sin\alpha$$

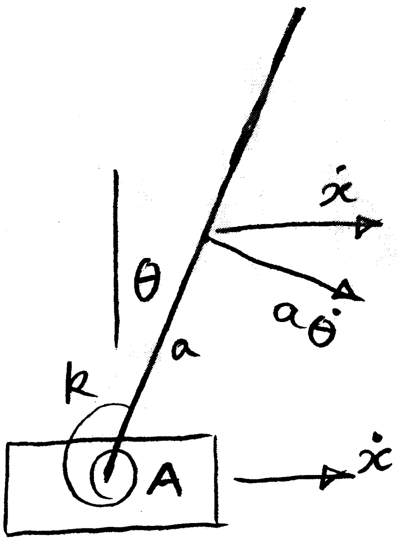
$$\therefore\ \tfrac{1}{2} a\omega\Omega = g + \Omega^2 R\tan\alpha$$

and for fast spin $\omega \gg \Omega$ we ignore Ω^2 in comparison to $\omega\Omega$

$$\therefore\ \Omega \simeq \frac{2g}{a\omega}$$

Not many attempted this question but those that did found it easy. Gyroscopes scare people off!

4. (a)



$$T = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} m (\dot{x} + a\dot{\theta}\cos\theta)^2 + \tfrac{1}{2} m (a\dot{\theta}\sin\theta)^2 + \tfrac{1}{2}\left(\tfrac{1}{3} m a^2\right)\dot{\theta}^2$$

(trolley) (rod)

Many got this wrong by not resolving $a\dot{\theta}$ into x & y directions. Many used parallel axis theorem to get I_A but the KE formula $\frac{1}{2} I_A \omega^2$ only works about A if A is <u>fixed</u>. It's <u>always</u> safest to use I_G. Some left out $\frac{1}{2} I \dot{\theta}^2$ altogether

$$\therefore T = m\dot{x}^2 + m a \dot{x}\dot{\theta}\cos\theta + \tfrac{2}{3} m a^2 \dot{\theta}^2$$

$$V = \tfrac{1}{2} k\theta^2 + mga(\cos\theta - 1)$$

zero PE at $\theta = 0$

some left this out

some said θ is small so ignore gravity

many had a minus sign here

Moral: take good care to get T & V right — the rest is easy!

(b) small θ $\quad \cos\theta \approx 1 - \frac{1}{2}\theta^2$

Keep only quadratic terms in T & V

$$T \simeq \frac{1}{2}\left(2M\dot{x}^2 + 2ma\dot{x}\dot{\theta} + \frac{4}{3}ma^2\dot{\theta}^2\right)$$

$$\therefore [M] = \begin{array}{cc} \begin{bmatrix} 2m & ma \\ ma & \frac{4}{3}ma^2 \end{bmatrix} & \begin{array}{c} x \\ \theta \end{array} \end{array}$$

$$V \simeq \frac{1}{2}\left(k\theta^2 - mga\,\theta^2\right)$$

$$\therefore [K] = \begin{array}{cc} \begin{bmatrix} 0 & 0 \\ 0 & k-mga \end{bmatrix} & \begin{array}{c} x \\ \theta \end{array} \end{array}$$

(c) Natural frequencies $\left|[K] - [M]\omega^2\right| = 0$

$$\begin{vmatrix} -2m\omega^2 & -ma\omega^2 \\ -ma\omega^2 & k-mga-\frac{4}{3}ma^2\omega^2 \end{vmatrix} = 0 \qquad (1)$$

$$\therefore 2m\omega^2\left(k-mga-\frac{4}{3}ma^2\omega^2\right) + m^2a^2\omega^4 = 0$$

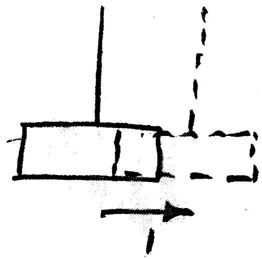
$$\therefore \frac{2(k-mga)}{ma^2} = \left(\frac{8}{3}-1\right)\omega^2 = \frac{5}{3}\omega^2$$

or $\quad \omega^2 = 0 \qquad$ (rigid body mode)

$$\therefore \omega^2 = \begin{cases} 0 \\ \frac{6}{5a^2}\left(\frac{k}{m} - ga\right) \end{cases}$$

Mode shapes : rigid body mode $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\omega = 0$



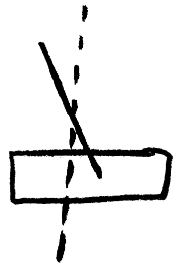
vibrating mode $\omega^2 = \frac{6}{5a^2}\left(\frac{k}{m} - ga\right)$

Use the first row of the eigenvalue equation ①

$$-2m\omega^2 x - ma\omega^2 \theta = 0$$

$$\therefore \quad \theta = -\frac{2x}{a}$$

$\begin{bmatrix} 1 \\ -\frac{2}{a} \end{bmatrix}$

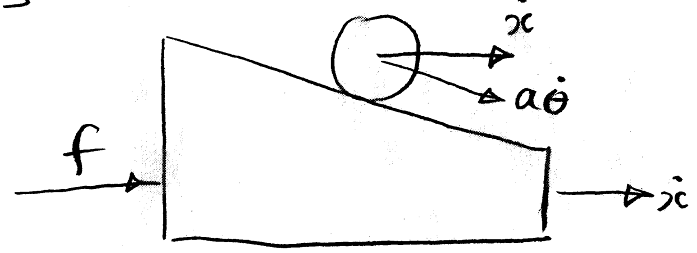


d/ stable solution exists provided that

$\omega^2 \geqslant 0 \quad \therefore \quad \underline{\underline{k \geqslant mga}}$

Some candidates used full Lagrange to find [K] & [M] matrices showing no knowledge of the $\frac{1}{2}\{q\}^T[k]\{q\}$ shortcut. Some didn't know how/when to use the small θ approximation. One wonders if they'd practised any of these types of questions at all! Well done by many, though.

5.



(a) Work done by f

$$\delta W = f\,\delta x$$

$$\therefore \quad Q_x = f$$

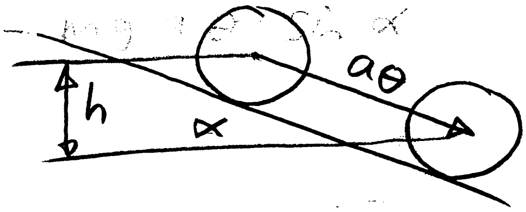
$$Q_\theta = 0$$

don't include work done by gravity because this is in the P.E.

many did, though

(b) $V = mgh$

$$\underline{\underline{V = -mga\theta\sin\alpha}}$$



$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m(\dot{x} + a\dot{\theta}\cos\alpha)^2 + \tfrac{1}{2}m(a\dot{\theta}\sin\alpha)^2 + \tfrac{1}{2}\left(\tfrac{1}{2}ma^2\right)\dot{\theta}^2$$

see notes for 4 (a)

$$\underline{\underline{T = m\dot{x}^2 + ma\dot{x}\dot{\theta}\cos\alpha + \tfrac{3}{4}ma^2\dot{\theta}^2}}$$

(c) $\dfrac{\partial T}{\partial \dot{x}} = 2m\dot{x} + ma\dot{\theta}\cos\alpha$

$\dfrac{\partial T}{\partial x} = 0 \qquad \dfrac{\partial V}{\partial x} = 0$

$$\therefore \quad \underline{\underline{\frac{d}{dt}\left(2m\dot{x} + ma\dot{\theta}\cos\alpha\right) = f}} \qquad (1)$$

$$\frac{\partial T}{\partial \dot\theta} = ma\dot{x}\cos\alpha + \frac{3}{2}ma^2\dot\theta$$

$$\frac{\partial T}{\partial \theta} = 0$$

note: some candidates with poor handwriting read $\cos\alpha$ as $\cos\theta$ and found this term was non-zero

$$\frac{\partial V}{\partial \theta} = -mga\sin\alpha$$

$$\therefore \quad \frac{d}{dt}\left(ma\dot{x}\cos\alpha + \frac{3}{2}ma^2\dot\theta\right) - mga\sin\alpha = 0$$

$$\therefore \quad \ddot{x}\cos\alpha + \frac{3}{2}a\ddot\theta = g\sin\alpha \qquad (2)$$

(d)(i) θ constant:

$$(2) \rightarrow \quad \ddot{x} = g\tan\alpha$$

then $(1) \rightarrow \quad f = 2m\ddot{x} = 2mg\tan\alpha$

Many simply said $f = 2m\ddot{x}$ which was not enough to earn marks

(ii) $f = 0$: $\quad (1) \rightarrow \quad \ddot\theta = -\dfrac{2\ddot{x}}{a\cos\alpha}$

$$(2) \rightarrow \quad \ddot{x}\cos\alpha + \frac{3}{2}\left(-\frac{2\ddot{x}}{\cos\alpha}\right) = g\sin\alpha$$

$$\therefore \quad \ddot{x}\left(\cos^2\alpha - 3\right) = g\sin\alpha\cos\alpha$$

$$\therefore \quad \ddot{x} = \frac{g\sin\alpha\cos\alpha}{\cos^2\alpha - 3}$$

which is to the left as expected.

HEMH May '05