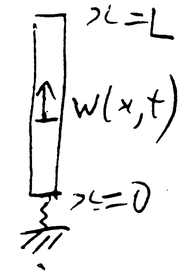


1. (a) Equations from data sheet:

$$EA \frac{\partial^2 w}{\partial x^2} = \rho A \frac{\partial^2 w}{\partial t^2}$$

i.e. $$\frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}, \quad c = \sqrt{\frac{E}{\rho}}$$

Boundary conditions:

At $x = L$, no vertical force $\therefore \frac{\partial w}{\partial x} = 0$

At $x = 0$, force balance with spring

$$\therefore \quad EA\frac{\partial w}{\partial x} = Kw$$

(b) For a mode, $w = u(x)e^{i\omega t}$

Substitute in governing equation:

$$u'' = -\frac{\omega^2}{c^2}u \qquad (\text{"SHM" equation})$$

$\therefore$ general solution $u = P\cos\frac{\omega x}{c} + Q\sin\frac{\omega x}{c}$

where P and Q are constants.

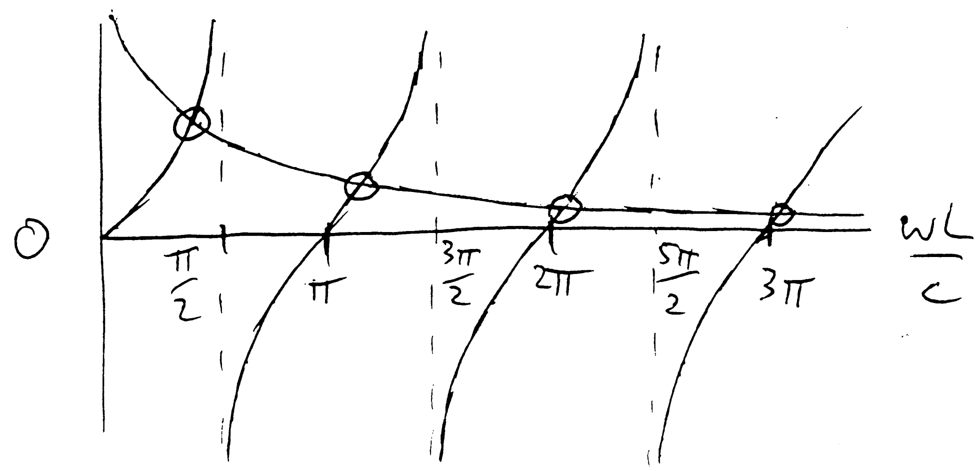
At $x = 0$: $EA\,Q\frac{\omega}{c} = KP$

At $x = L$: $-\frac{P\omega}{c}\sin\frac{\omega L}{c} + \frac{Q\omega}{c}\cos\frac{\omega L}{c} = 0$

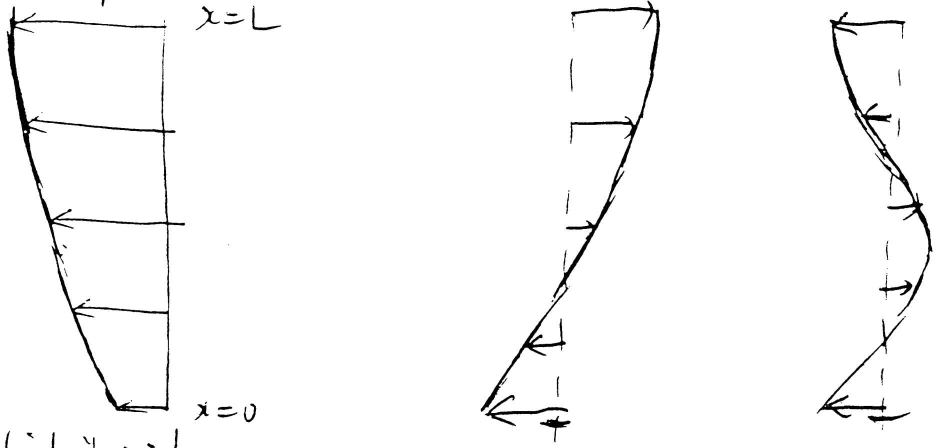
Eliminate Q: $\frac{P\omega}{c}\sin\frac{\omega L}{c} = \frac{\omega}{c}\left(\frac{KPc}{EA\omega}\right)\cos\frac{\omega L}{c}$

$$\therefore \quad \tan\frac{\omega L}{c} = \frac{Kc}{EA\omega} \quad \text{as given.}$$

(c) Plot $\tan\frac{\omega L}{c}$ and $\frac{Kc}{EA\omega}$ as functions of ω, then the natural frequencies are given by the intersections.



Mode shapes:



Modified "rigid body mode" No nodes

One node

Two nodes

(d) If $K \to 0$, intersections move to $\frac{\omega L}{c} = 0, \pi, 2\pi \cdots$

These correspond to a rigid displacement at $\omega = 0$ then a series of free-free modes

If $K \to \infty$, intersections move to $\frac{\omega L}{c} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \cdots$

These are frequencies of a fixed-free column. But for a given K, however large, at very high frequencies the intersections always approach the values from the $K \to 0$ case, so high modes always look like free-free modes whatever the spring strength.

2\. (a) Natural frequencies ω_n depend only on L, EI, m.
So try $\omega_n \propto L^\alpha (EI)^\beta m^\gamma$
Now compare dimensions.

$$[T^{-1}] = [L]^\alpha [M L^{-1} T^{-2} . L^4]^\beta \left[\frac{M}{L}\right]^\gamma$$

so:
$$L : \quad 0 = \alpha + 3\beta - \gamma$$
$$M : \quad 0 = \beta + \gamma$$
$$T : \quad -1 = -2\beta$$

$\therefore \beta = 1/2, \; \gamma = -1/2, \; \alpha = -3/2 - 1/2 = -2$

$$\therefore \quad \omega_n = \frac{\lambda_n}{L^2}\left[\frac{EI}{m}\right]^{1/2}, \quad \lambda_n \text{ dimensionless}.$$

(b) Equation of motion (data sheet) is

$$EI \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} = 0 \qquad (1)$$

Boundary conditions: $x = 0$ (clamped end)

$$\begin{cases} w = 0 \\ \dfrac{\partial w}{\partial x} = 0 \end{cases}$$

$x = L$ (free end)

$$\begin{cases} \dfrac{\partial^2 w}{\partial x^2} = 0 \\ \dfrac{\partial^3 w}{\partial x^3} = 0 \end{cases}$$

For a mode, let $w = u(x) e^{i\omega t}$.
Then (1) $\rightarrow$ $EI u'''' = m\omega^2 u$
General solution is

$$u(x) = K_1 \cos kx + K_2 \sin kx + K_3 \cosh kx + K_4 \sinh kx$$

where $k^4 = \dfrac{m \omega^2}{EI}$

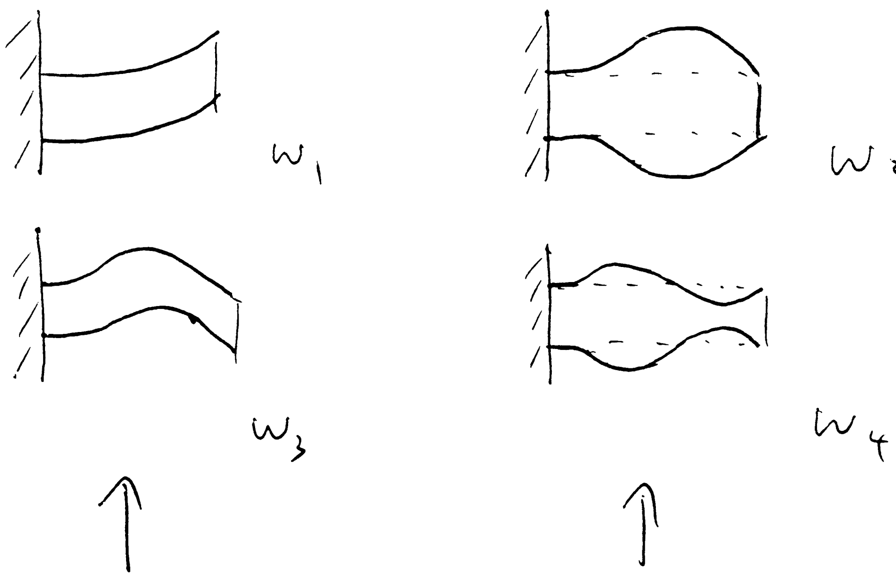
Now write down the four boundary conditions. This gives 4 simultaneous equations for K_1, K_2, K_3, K_4, all with zero on the RHS. So the determinant of the coefficients must be zero, and this is the condition which determines the natural frequencies.

But ω only appears via k, and k only appears in the form kL. So the solutions take the form kL = constant, ie $\dfrac{m\omega_n^2}{EI} = \dfrac{\text{constant}}{L^4}$

$$\therefore \quad \omega_n = \frac{\text{constant}}{L^2}\sqrt{\frac{EI}{m}} \quad \text{as in (a).}$$

(c) Mode shapes:

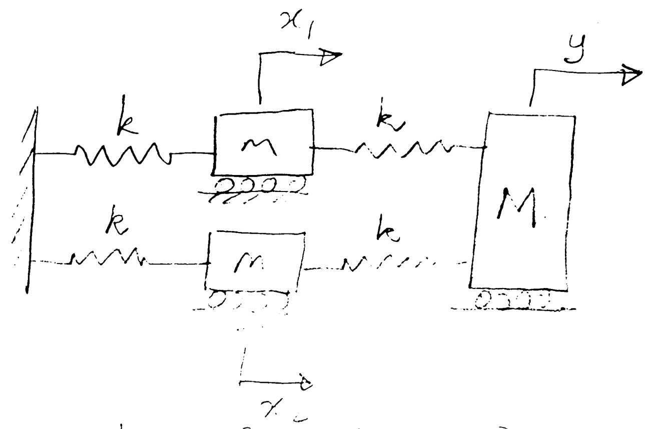
(d) System has a mirror symmetry, so all modes are either symmetric or antisymmetric.



Antisymmetric modes are exactly the same as in (c): link makes no difference

Symmetric modes are clamped–pinned modes of the beam, which have higher frequencies than clamped-free modes.

3.



$$\underset{\sim}{x} = \begin{Bmatrix} x_1 \\ x_2 \\ y \end{Bmatrix}$$

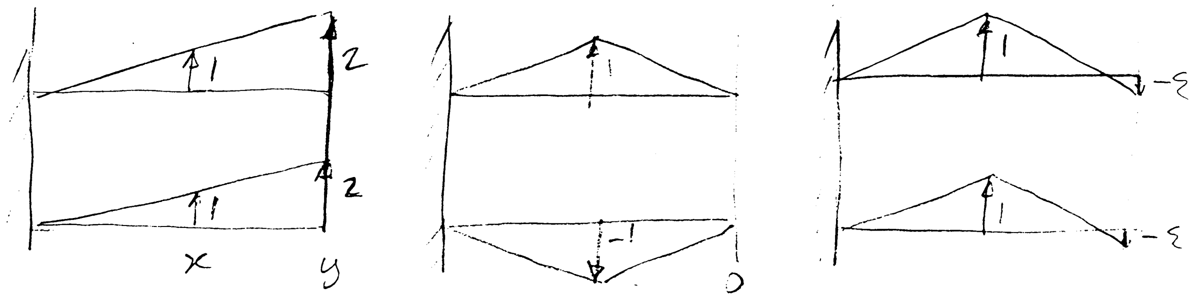
(a) $T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 + \frac{1}{2} M \dot{y}^2$

$$V = \tfrac{1}{2} k x_1^2 + \tfrac{1}{2} k x_2^2 + \tfrac{1}{2} k (y - x_1)^2 + \tfrac{1}{2} k (y - x_2)^2$$
$$= \tfrac{1}{2} k \left(2x_1^2 + 2x_2^2 + 2y^2 - 2x_1 y - 2x_2 y \right)$$

b) For $m/M \ll 1$, i.e. $m \ll M$:

- Modes are either symmetric or anti-symmetric
- Neglect m's for first mode $\Rightarrow$ $\boxed{M}$ $\Rightarrow \omega_1^2 = \sqrt{\frac{k}{M}}$
- Anti symm mode $\Rightarrow y = 0 \quad x_1 = -x_2$ $\quad \frac{k}{2} + \frac{k}{2}$ $\quad k \; m \; k$
- Symm mode $\Rightarrow x_1 = x_2 = 1, \; y = -\varepsilon$

So
$$\begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix} \qquad \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} \qquad \begin{Bmatrix} 1 \\ 1 \\ -\varepsilon \end{Bmatrix} \qquad (\varepsilon \ll 1)$$



$$\omega_1^2 \simeq \frac{k}{M} \qquad \omega_2^2 = \frac{2k}{m} \qquad \omega_3^2 \gtrsim \frac{2k}{m}$$

(c) Use Rayleigh to find modes 1 & 3 exactly.
Both of these modes can be represented by $(1, 1, \alpha)^T$

$$\omega^2 = \frac{V_{max}}{T^*} = \frac{\frac{1}{2} k \left(2 + 2 + 2\alpha^2 - 2\alpha - 2\alpha \right)}{\frac{1}{2} \left(m + m + M\alpha^2 \right)} = \frac{k \left(4 - 4\alpha + 2\alpha^2 \right)}{2m + M\alpha^2} \quad (1)$$

Find exact frequencies by minimising the Rayleigh quotient w.r.t α:

$$\frac{\partial \omega^2}{\partial \alpha} = \frac{(2m+M\alpha^2)(-4+4\alpha)k - k(4-4\alpha+2\alpha^2)\,2M\alpha}{(2m+M\alpha^2)^2}$$

$$\frac{d\omega^2}{d\alpha} = 0 \Rightarrow -2m + 2m\alpha - M\alpha^2 + \cancel{M\alpha^3} - 2M\alpha + 2M\alpha^2 - \cancel{M\alpha^3} = 0$$

$$\Rightarrow -2m + 2\alpha(m-M) + M\alpha^2 = 0$$

$$\Rightarrow \alpha = \frac{-2(m-M) \pm \sqrt{4(m-M)^2 + 8mM}}{2M}$$

$$\Rightarrow \alpha_{1,3} = 1 - \frac{m}{M} \pm \sqrt{\left(\frac{m}{M}\right)^2 + 1} \qquad \text{— (2)}$$

<u>Check</u>

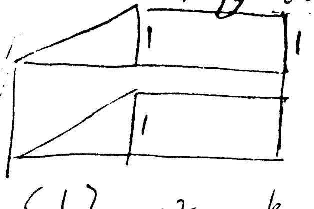
if $\frac{m}{M} \to 0$, $\alpha_{1,3} = 1 \pm 1 = 2, 0$ ✓

& (1) $\to$ $\omega^2 = \frac{k}{m}\frac{(4-4\alpha+2\alpha^2)}{(2+\frac{M}{m}\alpha^2)}$ $\Rightarrow$ $\alpha = 2 \to \omega_1^2 \approx k/M$ ✓

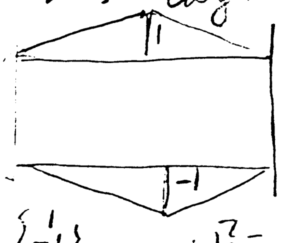
$\alpha = 0 \to \omega_3^2 = 2k/m$ ✓

For $m/M \gg 1$ ie $m \gg M$,

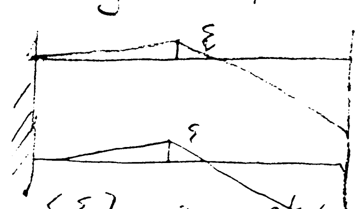
- asymmetric mode is unchanged
- lowest mode has no movement of y relative to $x_{1,2}$
- highest mode has large movement of y in antiphase with x



$\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ $\omega_1^2 \simeq k/m$

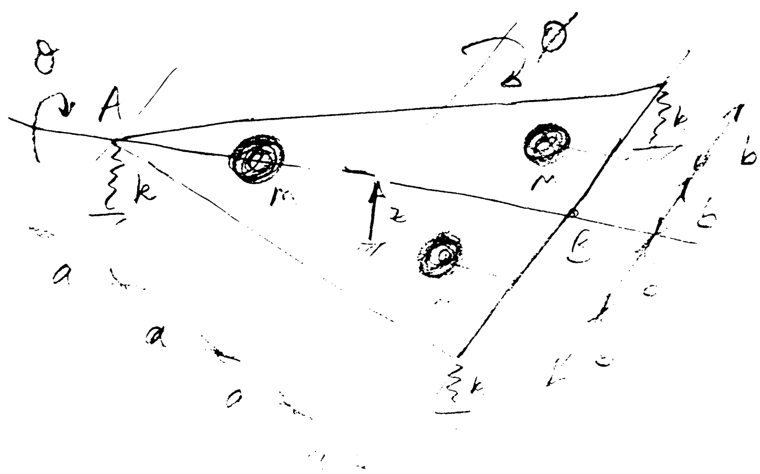


$\begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}$ $\omega_2^2 = 2k/m$



$\begin{Bmatrix} \varepsilon \\ \varepsilon \\ -1 \end{Bmatrix}$ $\omega_3^2 \simeq 2k/M$ Very high c/f part (i)

4.



(a) KE: $T = \frac{1}{2} m \dot{z}^2 + \ldots + \frac{1}{2} m [\dot{z} - \ldots]^2 + \ldots [\dot{z} - \ldots]^2$

$$= \frac{1}{2} m \left[3\dot{z}^2 + 2b^2\dot{\theta}^2 + 3a^2\dot{\phi}^2 - 2a\dot{z}\dot{\phi} \right]$$

$$\therefore [M] = m \begin{array}{c} \\ \left[\begin{matrix} 3 & 0 & -a \\ 0 & 2b^2 & 0 \\ -a & 0 & 3a^2 \end{matrix} \right] \end{array} \begin{matrix} z \\ \theta \\ \phi \end{matrix}$$

PE: $V = \frac{1}{2} k \left[z + 2a\phi \right]^2 + \frac{1}{2} k \left\{ \left[z - 2a\phi + 2b\theta \right]^2 + \left[z - 2a\phi - 2b\theta \right]^2 \right\}$

$$= \frac{1}{2} k \left[3z^2 + 12a^2\phi^2 + 8b^2\theta^2 - 4az\phi \right]$$

$$\therefore [K] = k \left[\begin{matrix} 3 & 0 & -2a \\ 0 & 8b^2 & 0 \\ -2a & 0 & 12a^2 \end{matrix} \right] \begin{matrix} z \\ \theta \\ \phi \end{matrix}$$

(b) The second row is an equation in θ only – ie θ is decoupled from z & ϕ.

$$\Rightarrow \quad 2b^2 m\ddot{\theta} + 8b^2 k\theta = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{8k}{2m}} = 2\sqrt{k/m}$$

This corresponds to roll rotation about the long'l axis.
For the other two modes, find ω^2 from $|[K] - \omega^2 [M]| = 0$

$$\Rightarrow \left| \begin{matrix} 3k - 3m\omega^2 & -2ak + am\omega^2 \\ -2ak + am\omega^2 & 12a^2k - 3a^2m\omega^2 \end{matrix} \right| = 0$$

4 Cont $\Rightarrow (3k-3m\omega^2)(12a^2k-3a^2m\omega^2)-(-2ak+am\omega^2)^2=0$

$$\Rightarrow \quad 8a^2m^2\omega^4 - 41a^2km\omega^2 + 32a^2k^2 = 0$$

$\div k^2 \quad \Rightarrow \quad 8\lambda^2 - 41\lambda + 32 = 0$ where $\lambda = \omega^2 m/k$

$$\lambda = \frac{41 \pm \sqrt{657}}{16} = 0.96,\ 4.16$$

So $\omega = 0.98\sqrt{k/m},\ 2.04\sqrt{k/m}$ //

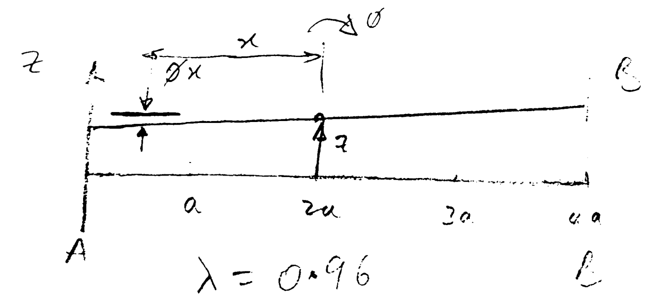
Mode shapes : $[K]\underset{\sim}{\phi} = \omega^2[M]\underset{\sim}{\phi}$

$\Rightarrow (3k-3\omega^2m)z + (-2ak+a\omega^2m)\phi = 0$

i.e. $(3-3\lambda)z + (-2a+\lambda a)\phi = 0$

$$\Rightarrow \frac{z}{a\phi} = \frac{\lambda-2}{3-3\lambda} = -8.7,\ -0.23$$

So modes are



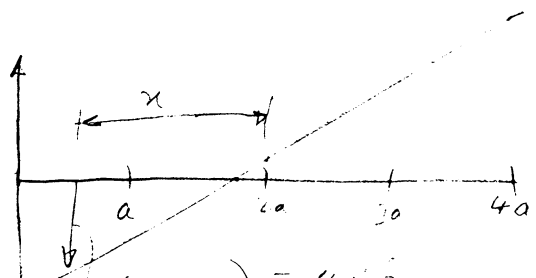
$\lambda = 0.96$

"bounce"

Node at $z - \phi x = 0$

$\Rightarrow x = z/\phi = -8.7a$

$\omega = 0.98\sqrt{k/m}$



$\lambda = 4.16$

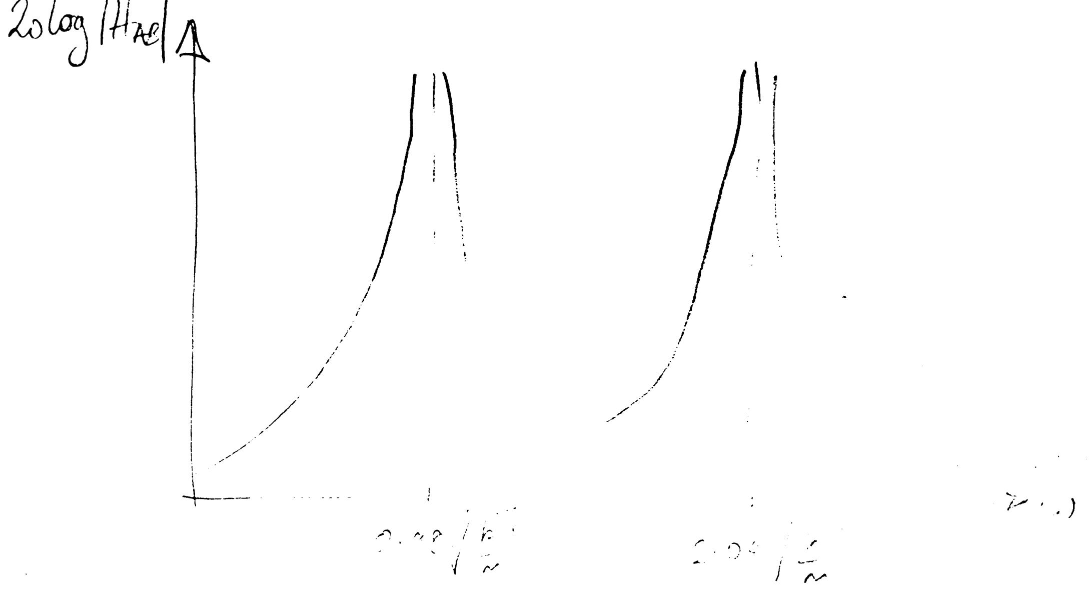
"pitch"

Node at $z - a\phi x = 0$

$\Rightarrow x = \frac{z}{\phi} = -0.23a$

$\omega = 2.04\sqrt{k/m}$

(c) θ mode doesn't show on transfer function for H_{AB} since both points lie on node line. $\Rightarrow$ only 2 resonances. Product of mode shapes is +ve for lower mode and −ve for higher mode. Since sign reverses, there is no anti-resonance between the two resonances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