3C7 Mechanics of solids

Datasheet: Elasticity & Plasticity

1. (a) $\sigma_{xx} = \sigma, \quad \sigma_{yy} = -\sigma, \quad \sigma_{zz} = 0$

$$\varepsilon_{xx} = \frac{\Delta a}{a}, \quad \varepsilon_{yy} = -\frac{\Delta a}{a}, \quad \varepsilon_{zz} = 0$$

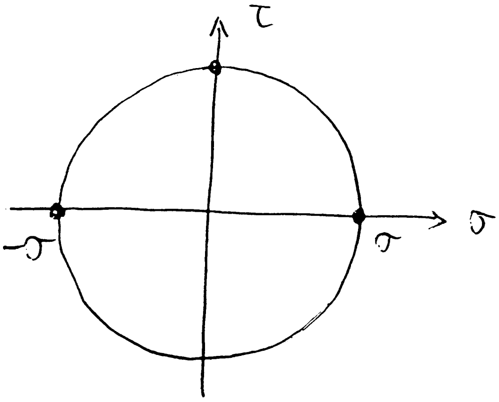
Assume plane stress, then

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu \frac{\sigma_{yy}}{E} = \frac{(1+\nu)\sigma}{E}$$

$$\Rightarrow \quad \frac{\Delta a}{a} = \frac{1+\nu}{E}\sigma$$

(20%)

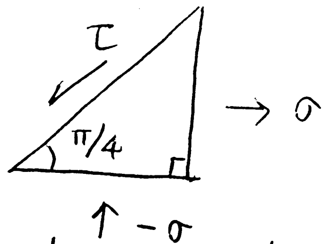
(b) Draw the Mohr's circle



The diagonal plane is $\frac{\pi}{4}$ angle from the x-axis. From the Mohr's circle, it then follows

$$\tau = \sigma$$

This can also be proved by considering equilibrium.



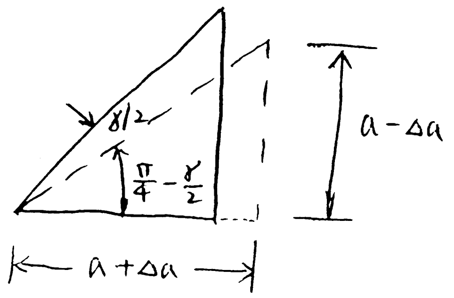
(15%)

(c) Engineering shear strain is defined as the change in angle between two lines that were originally perpendicular to each other. For the current problem, the shear strain along the diagonal plane is γ, and

(15%)

$$\tau = G\gamma$$

1. (d)



$$\tan\left(\frac{\pi}{4}-\frac{\gamma}{2}\right) = \frac{a-\Delta a}{a+\Delta a} = \frac{1-\frac{\Delta a}{a}}{1+\frac{\Delta a}{a}}$$

But

$$\tan\left(\frac{\pi}{4}-\frac{\gamma}{2}\right) = \frac{\sin\left(\frac{\pi}{4}-\frac{\gamma}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{\gamma}{2}\right)}$$

$$= \frac{\sin\frac{\pi}{4}\cos\frac{\gamma}{2} - \cos\frac{\pi}{4}\sin\frac{\gamma}{2}}{\cos\frac{\pi}{4}\cos\frac{\gamma}{2} + \sin\frac{\pi}{4}\sin\frac{\gamma}{2}}$$ (30%)

$$\left(\begin{array}{l} \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \\ \cos\frac{\gamma}{2} \approx 1,\ \sin\frac{\gamma}{2} \cong \frac{\gamma}{2} \end{array}\right)$$

$$= \frac{1-\frac{\gamma}{2}}{1+\frac{\gamma}{2}} = \frac{1-\frac{\Delta a}{a}}{1+\frac{\Delta a}{a}}$$

$$\Rightarrow \frac{\gamma}{2} = \frac{\Delta a}{a} \Rightarrow \boxed{\gamma = 2\frac{\Delta a}{a}}$$

(e)

$$G = \frac{\tau}{\gamma} = \frac{\sigma}{2\Delta a/a} = \frac{\cancel{\sigma}}{2\frac{(1+\nu)}{E}\cancel{\sigma}}$$

$$\Rightarrow \boxed{G = \frac{E}{2(1+\nu)}}$$ (10%)

(f) Bulk modulus

There are 2 independent material parameters.

(10%)

2. (a) Lame solutions

$$\sigma_{rr} = A - \frac{B}{r^2}, \quad \sigma_{\theta\theta} = A + \frac{B}{r^2}$$

$$\varepsilon_{\theta\theta} = \frac{u}{r} = \frac{\sigma_{\theta\theta}}{E} - \frac{\nu\sigma_{rr}}{E}$$

(assume plane stress first)
change to plane strain later

$$= \frac{1}{E}\left\{ A + \frac{B}{r^2} - \nu A + \frac{\nu B}{r^2} \right\}$$

$$\Rightarrow \quad u = \frac{r}{E}\left\{ A(1-\nu) + \frac{B}{r^2}(1+\nu) \right\}$$

For plane strain, replace E by $\frac{E}{1-\nu^2}$ and ν by $\frac{\nu}{1-\nu}$ to get

$$u = \frac{r(1-\nu^2)}{E}\left\{ A\left(1-\frac{\nu}{1-\nu}\right) + \frac{B}{r^2}\left(1+\frac{\nu}{1-\nu}\right) \right\}$$

$$= \frac{r(1+\nu)}{E}\left\{ A(1-2\nu) + \frac{B}{r^2} \right\}$$

(40%)

(b) (i) At $r = a$, $\sigma_{rr} = -p_o = A - \frac{B}{a^2} \Rightarrow A = \frac{B}{a^2} - p_o$

At $r = b$

$$\sigma_{rr} = -p_e = \frac{E}{b}(u)_{r=b}$$

$$= \frac{\cancel{E}}{\cancel{b}} \cdot \frac{\cancel{b}(1+\nu)}{\cancel{E}}\left\{ A(1-2\nu) + \frac{B}{b^2} \right\}$$

$$= A(1+\nu)(1-2\nu) + \frac{B}{b^2}(1+\nu)$$

$$\Rightarrow -\frac{p_e}{1+\nu} = \left(\frac{B}{a^2} - p_o\right)(1-2\nu) + \frac{B}{b^2}$$

Rearrange gives

$$B = \frac{p_o(1-2\nu) - p_e/(1+\nu)}{\frac{1-2\nu}{a^2} + \frac{1}{b^2}}$$

$$A = \frac{p_o\frac{a^2}{b^2} - \frac{p_e}{1+\nu}}{(1-2\nu) + \frac{a^2}{b^2}}$$

2. (b) continued

Finally,

$$p_e = -(\sigma_{rr})_{r=b} = \frac{B}{b^2} - A$$

$$= \frac{p_o(1-2\nu) - \frac{p_e}{1+\nu}}{1+(1-2\nu)\frac{b^2}{a^2}} - \frac{p_o\frac{a^2}{b^2} - \frac{p_e}{1+\nu}}{(1-2\nu)+\frac{a^2}{b^2}}$$

$$= \frac{\frac{p_e}{1+\nu}\left(\frac{b^2}{a^2}-1\right) - 2\nu p_o}{1+(1-2\nu)\frac{b^2}{a^2}}$$

$$\Rightarrow \quad 2\nu p_o = p_e \left\{ \frac{\frac{b^2}{a^2}-1}{1+\nu} - 1 - (1-2\nu)\frac{b^2}{a^2} \right\}$$

$$\Rightarrow \quad p_e = \frac{2\nu p_o(1+\nu)}{\frac{b^2}{a^2}\left[1+(1-2\nu)(1+\nu)\right]+\nu}$$

(40%)

(ii) The change in the thickness is

$$\Delta = (u)_{r=b} - (u)_{r=a}$$

$$= \frac{b}{E}\left[A(1-\nu) + \frac{B}{b^2}(1+\nu)\right] - \frac{a}{E}\left[A(1-\nu) + \frac{B}{a^2}(1+\nu)\right]$$

$$= \frac{A(1-\nu)}{E}(b-a) + \frac{B(1+\nu)}{E}\left(\frac{1}{b} - \frac{1}{a}\right)$$

Where A and B have been found earlier. To be precise, the thickness change should be $|\Delta|$.

(20%)

3

a) $$\phi = \frac{Cr^2}{\tan\alpha - \alpha}\left[\alpha - \theta + \frac{\sin 2\theta}{2} - \tan\alpha\cos^2\theta\right]$$

$$\frac{\partial\phi}{\partial r} = \frac{2Cr}{\tan\alpha - \alpha}\left[\alpha - \theta + \frac{\sin 2\theta}{2} - \tan\alpha\cos^2\theta\right]$$

$$\frac{\partial^2\phi}{\partial r^2} = \frac{2C}{\tan\alpha - \alpha}\left[\alpha - \theta + \frac{\sin 2\theta}{2} - \tan\alpha\cos^2\theta\right]$$

$$\frac{\partial\phi}{\partial\theta} = \frac{Cr^2}{\tan\alpha - \alpha}\left[-1 + \cos 2\theta + 2\tan\alpha\sin\theta\cos\theta\right]$$

$$\frac{\partial^2\phi}{\partial\theta^2} = \frac{Cr^2}{\tan\alpha - \alpha}\left[-2\sin 2\theta + 2\tan\alpha\left(\cos^2\theta - \sin^2\theta\right)\right]$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right] = \frac{C}{\tan\alpha - \alpha}\left[-1 + \cos 2\theta + 2\tan\alpha\sin\theta\cos\theta\right]$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$= \frac{C}{\tan\alpha - \alpha}\left[2\alpha - 2\theta - \sin 2\theta - 2\tan\alpha\sin^2\theta\right]$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = \frac{C}{\tan\alpha - \alpha}\left[2\alpha - 2\theta + \sin 2\theta - 2\tan\alpha\cos^2\theta\right] \qquad (30\%)$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right] = \frac{C}{\tan\alpha - \alpha}\left[1 - \cos 2\theta - 2\tan\alpha\sin\theta\cos\theta\right]$$

b) Along OB : $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$

Along OA $\sigma_{r\theta} = 0, \quad \sigma_{\theta\theta} = -p$.

$$\sigma_{\theta\theta}\Big|_{\theta=\alpha} = \frac{C}{\tan\alpha - \alpha}\left[2\alpha - 2\alpha - \sin 2\alpha - 2\sin\alpha\cos\alpha\right] = 0$$

$$\sigma_{r\theta}\Big|_{\theta=\alpha} = \frac{C}{\tan\alpha - \alpha}\left[1 - \cos 2\alpha - 2\sin^2\alpha\right] = 0$$

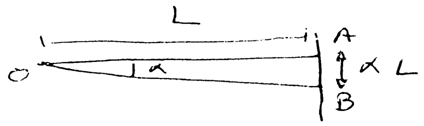
$\Rightarrow$ BC along OB satisfied.

$$\sigma_{\theta\theta}\Big|_{\theta=0} = \frac{C}{\tan\alpha - \alpha}\left[2\alpha - 2\tan\alpha\right] = -2C = -p$$

$$C = \frac{p}{2}$$

$$\sigma_{r\theta}\Big|_{\theta=0} = \frac{C}{\tan\alpha - \alpha}\left[1 - 1 - 0\right] = 0 \qquad (30\%)$$

(c)
(i)



consider cantilever of unit depth

$$M = pL\,\frac{L}{2} = \frac{pL^2}{2}$$

$$I \text{ at } AB = \frac{bd^3}{12} = \frac{1(\alpha L)^3}{12}$$

$$\sigma \text{ at } A = \frac{My}{I} = \frac{pL^2}{2}\,\frac{\alpha L}{2}\cdot\frac{12}{\alpha^3 L^3}$$

$$= \frac{3p}{\alpha^2} \qquad (20\%)$$

(ii)

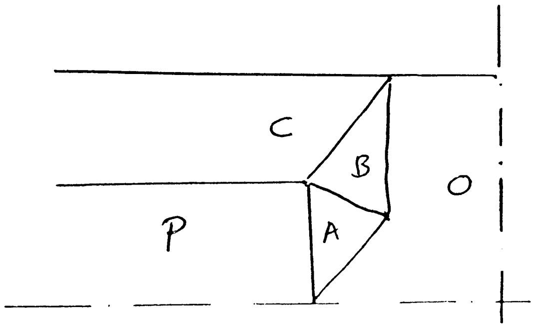
$$\sigma_{rr}\Big|_{\theta=0} = \frac{p}{2(\tan\alpha - \alpha)}\left[2\alpha\right]$$

$$\tan\alpha \sim \alpha + \frac{\alpha^3}{3}$$

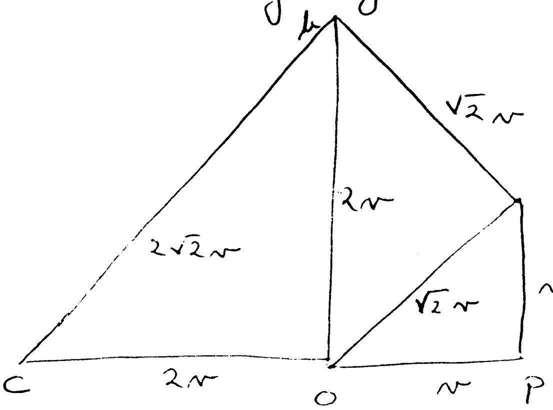
$$= \frac{p}{2\,\frac{\alpha^3}{3}}\;2\alpha = \frac{3p}{\alpha^2} \qquad (20\%)$$

4)

a)



Velocity diagram



Unit depth :

~~P~~ Input power $= 2 l v p$

power dissipated along OA : $(\sqrt{2}\, l\, k)\sqrt{2}\, v = 2\, k l v$

AB : $2 k l v$

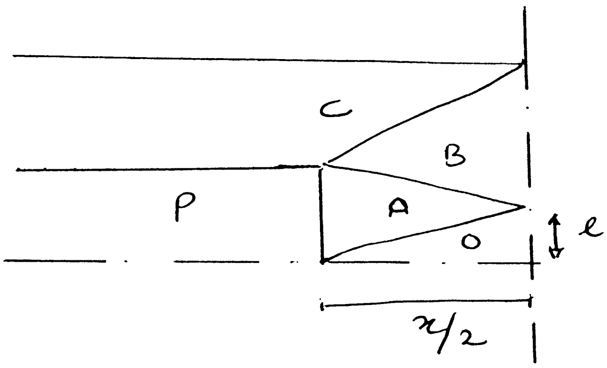
OB : $(2 l k)\, 2v = 4 k l v$

BC : $(\sqrt{2}\, k l)\, 2\sqrt{2}\, v = 4 k l v$

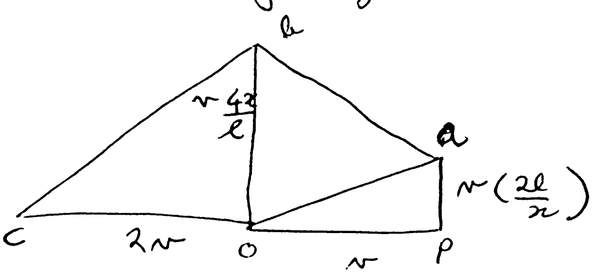
$\Rightarrow \quad 2 p l v = 12\, k l v$

$\underline{p = 6k}$ (40%)

b)



velocity diagram



I/P power $= 2 p l v$

power dissipated along OA : $k\sqrt{\ell^2+\frac{x^2}{4}}\ \cancel{2}v\sqrt{\left(\frac{2\ell}{x}\right)^2+1}$

$$= \frac{2vk}{x}\left(\ell^2+\left(\frac{x}{2}\right)^2\right)$$

AB : $\frac{2vk}{x}\left(\ell+\left(\frac{x}{2}\right)^2\right)$

BC : $k\left[\ell^2+\left(\frac{x}{2}\right)^2\right]^{\frac{1}{2}}\ 4vx\left[\ell^2+\left(\frac{x}{2}\right)^2\right]$

$$= \frac{4vk}{x}\left[\ell^2+\left(\frac{x}{2}\right)^2\right]$$

$$2p\ell v = \frac{8vk}{x}\left[\ell^2+\left(\frac{x}{2}\right)^2\right]$$

$$p = \frac{4k}{x\ell}\left[\ell^2+\frac{x^2}{4}\right]$$ (40%)

(c) Equate (a) & (b)

$$6k = \frac{4k}{x\ell}\left[\ell^2+\frac{x^2}{4}\right]$$

$$\frac{3x\ell}{2} = \ell^2+\frac{x^2}{4}$$

$$\frac{x}{\ell} = 3 \pm \sqrt{5}$$

$\frac{x}{2} > \ell$ for mech. (b) to be operative

$\Rightarrow \quad x = 5{\cdot}24\ell$ (20%)