PART IIA 2005
3D4: Structural analysis and stability
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ENGINEERING TRIPOS PART IIA
3D4
Crib

Datasheet: None

Q1.
a) Warping in beams loaded in torsion refers to axial movement out of the plane of the cross-section , such that "planes sections remain plane" is NOT true.

If warping is restrained, this will set up axial stresses in the section. If those stresses change there will be associated shear stresses, and these shear stresses will be associated with torques in some types of cross-section. These associated torques are called "restrained warping torsion". Restrained warping torsion also occurs when the warping varies along the length of the beam due to varying amounts of applied torque.

St Venant's torsion refers to the system of torsional shear stresses set up if restrained warping torsion is ignored (i.e. if it is assumed the section is completely free to warp such that no restrained warping torsion shear stresses are set up, or the section is one which naturally does not warp anyway).

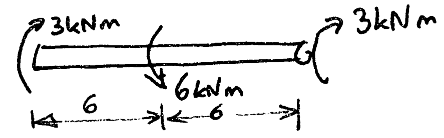
Restrained warping torsion adds torsional stiffness to a beam, hence when it is included in lateral-torsional buckling analysis it predicts <u>higher</u> buckling loads.

i) I beams warp, but warping is constant and unrestrained, thus RWT is not important.
ii) I beams warp, and warping is restrained at the ends, and thus RWT is important.
iii) Warping varies along the beam due to the variable torque, thus RWT is important.
iv) Circular sections do not warp, thus RWT is not important.
v) Circular sections do not warp, thus RWT is not important.
vi) Angle sections have shear centre at junction, thus RWT is not important. (Having only two legs it is generally easy to fit a plane through the section, thus they cannot easily warp.)
vii) As vi)
viii) Open boxes warp, but uniform here, so RWT is not important.
ix) Open boxes warp, and here it is restrained and so RWT is important.
x) In closed box sections, warping is negligible, so RWT is unimportant.
xi) As x)
xii) I beams warp and the warping varies in lateral torsional buckling, thus RWT is important.

b) $L = 12\,\text{m}$, $\Gamma = 2.6\times10^{11}\,\text{mm}^6$, $J = 3.6\times10^5\,\text{mm}^4$ $E = 210\,\text{GPa}$, $G = 81\,\text{GPa}$.

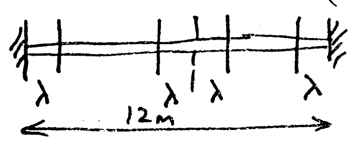
$\lambda = \sqrt{(210/81)(2.6\times10^{11}/3.6\times10^5)} = 1.37\,\text{m}$.

Ignoring warping restraint



$\phi' = \dfrac{3\times10^6}{81\times10^3 \cdot 3.6\times10^5} = 1.03\times10^{-4}$ rads/mm.

$\therefore$ Twist at centre $= (1.03\times10^{-4})(6000) = \underline{0.617 \text{ rads}}$.



Warping restraint at ends and also at centre by symmetry.
$\therefore$ Effective length $= 6\text{m} - 2\times1.37\text{m} = 3.26\text{m}$
$\therefore$ Twist $= (3.26/6)(0.617 \text{ rads}) = \underline{\underline{0.335 \text{ rads}}}$

In a) i-xii, some students confused warping with restrained warping torsion. In b) almost everyone had 6 kNm throughout the bar, rather than sending 3 kNm to either end and very few recognized that symmetry required 4λ to be subtracted from the actual length.

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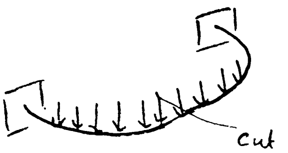
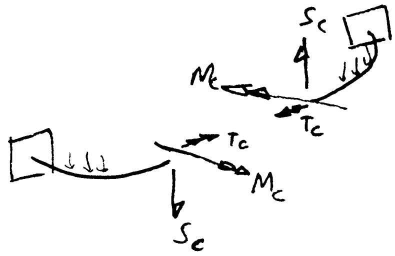
Q2

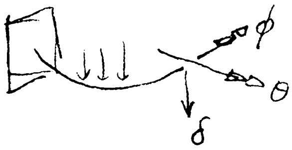
a)





By symmetry $\quad S_c = 0 \quad$ vertical shear.

$\qquad\qquad\quad T_c = 0 \quad$ (consider shear stresses flowing round opposing faces of cut)

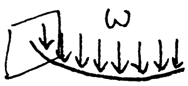


Also, $\theta = 0$ by symmetry.

b) One cut (eg at centre) makes it statically determinate, and at centre there is only one unknown stress resultant M_c.

Use Virtual Work on one half of structure.

Consider: System ⓪ (with external forces applied)



System ① (unit moment applied at cut)

Then Real System = System ⓪ + M_c × System ①

Let Real Compatibility System be :- Real System
Virtual Equilibrium System be :- System ①.

and dot them together using Virtual Work.

REAL COMPAT = ⓪ + M_c × ①

EXTERNAL

$$"\sum M.\theta" + \ldots etc = "\int M.K\,ds" + "\int T.\phi'\,ds"$$

VIRTUAL EQUIL. = ①

INTERNAL.

LHS. The only external forces on the Virtual Equilibrium System ① is the unit end moment applied. The conjugate rotation there in the Real System, θ, is zero (from part (a)).

$\therefore$ LHS has $\quad 1.0 = $ RHS

$\qquad\qquad\qquad\quad 0 = $ RHS

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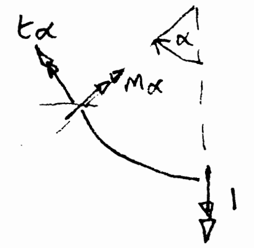
Q2 b) cont'd

RHS Internal moments and curvatures etc.

$$\text{LHS} \quad 0 = \int_0^{\pi/2} m(\alpha)\,.\,\kappa(\alpha)\,R\,d\alpha + \int_0^{\pi/2} t(\alpha)\,.\,\phi'(\alpha)\,R\,d\alpha$$

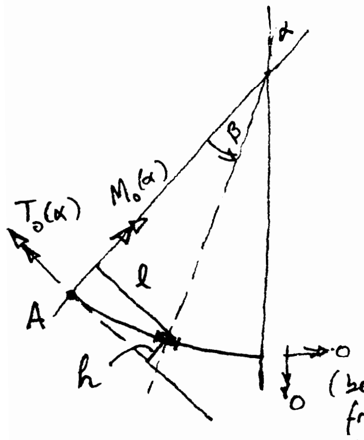
Internal bending and twisting moments $m(\alpha)$, $t(\alpha)$ in Virt. Eq. Sys 1



Resolve ↗ $\quad m(\alpha) = 1.\cos\alpha$

Resolve ↖ $\quad t(\alpha) = 1.\sin\alpha$

Need internal bending and twisting moments $M_0(\alpha)$, $T_0(\alpha)$ in ~~Sys~~ Sys 0.



(because free end in cut system) (also zero shear).

Resolve ↗ at A

$$M_0(\alpha) = -\int_0^{\alpha} w.R\,d\beta\,.\,l(\beta) \qquad l = R\sin\beta$$

$$= -wR^2\int_0^{\alpha}\sin\beta\,d\beta = -wR^2\Big[-\cos\beta\Big]_0^{\alpha}$$

$$\underline{\underline{M_0(\alpha) = -wR^2(1-\cos\alpha)}}$$

Resolve ↖ at A:

$$T_0(\alpha) = -\int_0^{\alpha} w.R\,d\beta\,.\,h(\beta) \qquad h = R - R\cos\beta$$

$$= -wR^2\int_0^{\alpha}(1-\cos\beta)\,d\beta = -wR^2\Big[\beta - \sin\beta\Big]_0^{\alpha}$$

$$\underline{\underline{T_0(\alpha) = -wR^2\big[\alpha - \sin\alpha\big]}}$$

Real curvatures and twists are thus

$$\kappa(\alpha) = \big[\underbrace{M_0(\alpha)}_{0} + M_c\,\underbrace{m(\alpha)}_{1}\big]/EI \qquad \text{i.e.} \quad = \kappa_0 + M_c\,\kappa_1$$

$$\phi'(\alpha) = \big[T_0(\alpha) + M_c\,t(\alpha)\big]/GJ \qquad \text{i.e.} \quad = \phi'_0 + M_c\,\phi'_1$$

V.W. at top of page says

$$0 = \int_0^{\pi/2} m(\alpha)\,.\,\frac{\big[M_0(\alpha) + M_c\,m(\alpha)\big]}{EI}\,R\,d\alpha + \int_0^{\pi/2} t(\alpha)\,.\,\frac{\big[T_0(\alpha) + M_c\,t(\alpha)\big]}{GJ}\,R\,d\alpha$$

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Q2(b) cont'd.

$$\therefore \quad 0 = \frac{I_1}{EI} + M_c \frac{I_3}{EI} + \frac{I_2}{GJ} + M_c \frac{I_4}{GJ}$$

$$\text{so} \quad M_c = -\frac{\left(I_1/EI + I_2/GJ \right)}{I_3/EI + I_4/GJ}$$

$$\text{with} \quad I_1 = \int M_0 m = -wR^2 \int_0^{\pi/2} (1-\cos\alpha)\cos\alpha \, d\alpha$$

$$I_2 = \int T_0 t = -wR^2 \int_0^{\pi/2} (\alpha - \sin\alpha)\sin\alpha \, d\alpha$$

$$I_3 = \int m^2 = \int_0^{\pi/2} \cos^2\alpha \, d\alpha = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$$

$$I_4 = \int t^2 = \int_0^{\pi/2} \sin^2\alpha \, d\alpha = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$$

No need for double angle formulae $\cos^2 + \sin^2 = 1$ so half each over a quarter cycle $\rightarrow \frac{1}{2} \times \frac{\pi}{2}$ each.

$$I_1 = -wR^2 \left[\int_0^{\pi/2} \cos\alpha \, d\alpha - \int_0^{\pi/2} \cos^2\alpha \, d\alpha \right] = -wR^2 \left[[\sin\alpha]_0^{\pi/2} - \frac{\pi}{4} \right] = -wR^2 \left[1 - \frac{\pi}{4} \right]$$

$$I_2 = -wR^2 \left[\int_0^{\pi/2} \alpha \sin\alpha \, d\alpha - \int_0^{\pi/2} \sin^2\alpha \, d\alpha \right] = -wR^2 \left[I_5 - \frac{\pi}{4} \right]$$

$$I_5 = \int_0^{\pi/2} \alpha \sin\alpha \, d\alpha = \left[-\alpha\cos\alpha \right]_0^{\pi/2} - \int_0^{\pi/2} -\cos\alpha \, . d\alpha = 0 + \left[\sin\alpha \right]_0^{\pi/2} = 1$$

$$\therefore \quad I_2 = -wR^2 \left[1 - \pi/4 \right]$$

$$\therefore \quad M_c = \frac{wR^2 \left[1 - \frac{\pi}{4} \right] \left(+\frac{1}{EI} + \frac{1}{GJ} \right)}{\frac{\pi}{4} \left(\frac{1}{EI} + \frac{1}{GJ} \right)}$$

$$= wR^2 \left[\frac{4-\pi}{\pi} \right] \left\{ \frac{GJ + EI}{GJ + EI} \right\} \qquad = wR^2(4/\pi - 1)$$

2(c) Apply a unit point load at free end of half the structure and use this as Virtual Equilibrium System (2).

= System (2)

then Virtual Work, with Virt. Eq. Sys (2) dotted with Real Compat gives

$$1 . \delta_{Real} = \int m_2 \kappa_{real} \, ds + \int t_2 \phi'_{Real} \, ds$$

$$\text{with} \quad \kappa_{real} = \kappa_{0} + M_c \kappa_{1} , \quad \text{and} \quad \phi'_{Real} = \phi'_{0} + M_c \phi'_{1}$$

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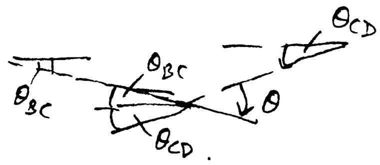
Q3 a) Eigenvalues interpreted as
- elastic critical loads in classical description
- stiffnesses in principal directions of quadratic form of local total potential energy function about equilibrium - in non-classical description.

b) $$\underbrace{\Pi(u_1, u_2)}_{\text{Total Potential Energy}} = \underbrace{\tfrac{1}{2} k u_1^2 + \tfrac{1}{2} G\theta^2}_{\text{Int. Strain Energy}} \underbrace{- P.y - \underline{Q}^T.\underline{u}}_{\text{External W.D.}}$$

i)
$$y = 3L - L(\cos\theta_{AB} + \cos\theta_{BC} + \cos\theta_{CD})$$
$$= 3L - L\left(1 - \frac{u_1^2}{2L^2}\ldots + 1 - \frac{(u_1-u_2)^2}{2L^2} + \ldots + 1 - \frac{u_2^2}{2L^2} + \ldots\right)$$
$$= \frac{1}{2L}\left(u_1^2 + (u_1 - u_2)^2 + u_2^2\right)$$
$$= \frac{1}{2L}\left(u_1^2 + u_1^2 - 2u_1u_2 + u_2^2 + u_2^2\right)$$
$$= \frac{1}{2L}\left(2u_1^2 - 2u_1u_2 + 2u_2^2\right) = \frac{1}{2L}\,\underline{u}^T\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\underline{u}$$

$$\theta = \theta_{BC} + \theta_{CD}$$



$$\simeq \frac{(u_2 - u_1)}{L} + \frac{u_2}{L} = \frac{(2u_2 - u_1)}{L}$$

$$\tfrac{1}{2}G\theta^2 = \tfrac{1}{2}G\left(\frac{2u_2 - u_1}{L}\right)^2 = \frac{1}{2}\frac{G}{L^2}\left(4u_2^2 - 4u_2u_1 + u_1^2\right)$$
$$= \frac{1}{2}\frac{G}{L^2}\,\underline{u}^T\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\underline{u}$$

$$\tfrac{1}{2}ku_1^2 = \frac{1}{2}\,k\,\underline{u}^T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\underline{u}$$

$$\therefore \quad \Pi = -\underline{Q}^T.\underline{u} + \tfrac{1}{2}\underline{u}^T\underline{\underline{K}}\,\underline{u} \quad \text{where}$$

$$K = \left[\begin{array}{c|c} k + \frac{G}{L^2} - \frac{2P}{L} & -\frac{2G}{L^2} + \frac{P}{L} \\ \hline -\frac{2G}{L^2} + \frac{P}{L} & \frac{4G}{L^2} - \frac{2P}{L} \end{array}\right]$$

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Q3 b) ii) $G = kL^2 \Rightarrow a = 2k - 2P/L \quad b = -2k + P/L, \quad c = 4k - 2P/L$

Let $P/L = p$. Buckling occurs when $\det K = 0 \rightarrow ac = b^2$

so, method 1.

$$(2k-2p)(4k-2p) = (-2k+p)^2$$
$$8k^2 - 8kp - \cancel{4kp} + 4p^2 = 4k^2 - \cancel{4kp} + p^2$$
$$4k^2 - 8kp + 3p^2 = 0$$
$$\therefore p = k\left[\frac{8 \pm \sqrt{64 - 4.4.3}}{6}\right] = k\left[\frac{8 \pm \sqrt{16}}{6}\right] = k\left[\frac{8 \pm 4}{6}\right]$$
$$= k\left(\frac{12}{6} \text{ or } \frac{4}{6}\right) = 2k \text{ or } \underline{\underline{\frac{2k}{3}}} \qquad \text{QED.}$$

or method 2. Insert $p = 2k/3$ into K

$$\Rightarrow a = 2k/3, \quad b = -4k/3, \quad c = 8k/3$$

and $\det K = \left(\frac{k}{3}\right)^2 \begin{vmatrix} 2 & -4 \\ -4 & 8 \end{vmatrix} = 0$ QED.

Eigenvector: $(K - \lambda I)u = 0$.

1st eqn gives $(a-\lambda)u_1 + bu_2 = 0$, but $\lambda = 0$

$$\therefore a u_1 + b u_2 = 0$$
$$\therefore 2u_1 - 4u_2 = 0 \qquad \therefore u_2 = u_1/2$$

So evec in direction $\begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$

b iii) Equilib: $0 = -\underline{Q} + \underline{\underline{K}}\,\underline{u} \rightarrow \underline{u} = \underline{K}^{-1}\underline{Q}$

When $P = \frac{kL}{4}$,

$$a = 2k - \frac{k}{2} = \frac{3k}{2} = \frac{6k}{4}$$
$$b = -2k + \frac{k}{4} = -\frac{7k}{4}$$
$$c = 4k - \frac{k}{2} = \frac{7k}{2} = \frac{14k}{4}$$

$K = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ $\quad K^{-1} = \frac{1}{(ac-b^2)}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix} = \frac{k}{4}\begin{bmatrix} 14 & 7 \\ 7 & 6 \end{bmatrix} \frac{1}{\frac{k^2}{4^2}(6.14 - 7^2)}$

$$= \frac{4}{k.35}\begin{bmatrix} 14 & 7 \\ 7 & 6 \end{bmatrix}$$

Apply to Q

$$\underline{u} = \frac{4}{k.35}\begin{bmatrix} 14 & 7 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix}\frac{kL}{400}$$
$$= \frac{L}{3500}\begin{bmatrix} 35 \\ 15 \end{bmatrix} = \underline{\underline{\frac{L}{700}\begin{bmatrix} 7 \\ 3 \end{bmatrix}}}$$

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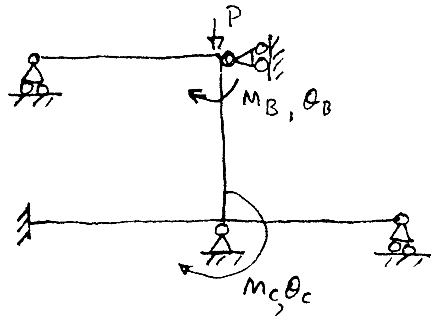
Q4. a) Shanley noticed that a column could bend <u>during</u> the application of an axial loading increment, such that is possible to develop some of the curvature without strain reversal. This leads to an infinite number of possible solutions which all satisfy equilibrium, compatibility and a nonlinear elasto-plastic material law.

b)



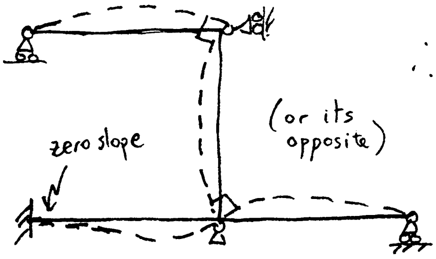
$$\begin{bmatrix} M_B \\ M_C \end{bmatrix} = \underline{\underline{K}} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}$$

By inspection

$$\underline{\underline{K}} = \underbrace{\begin{bmatrix} sk & +\ sck \\ sck & sk \end{bmatrix}}_{\text{from column}} + \underbrace{\begin{bmatrix} 3k & 0 \\ 0 & 4k+3k \end{bmatrix}}_{\text{from beams}} = \begin{bmatrix} k(s+3) & k(sc) \\ k(sc) & k(s+7) \end{bmatrix}$$

Instability when $\det K = 0 \Rightarrow \begin{vmatrix} s+3 & sc \\ sc & s+7 \end{vmatrix} = 0$

$=$
$$(s+3)(s+7) - s^2c^2 = 0$$
$$s^2 + 3s + 7s + 21 - s^2c^2 = 0$$



$$\therefore\ 0 = s^2(1-c^2) + 10s + 21$$

$$\therefore\ \underline{\underline{\beta = 10}}, \quad \underline{\underline{\gamma = 21}}$$

c) Perry-Robertson avoids the more complicated subtleties of inelastic behaviour by adopting first yield of the extreme fibre as the "failure" criterion.

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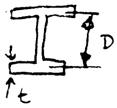
Q4 d) $$M_{cr} = \frac{\pi}{L}\sqrt{GJEI_y}\left(1 + \frac{\pi^2}{L^2}\frac{E\Gamma}{GJ}\right)^{1/2} \qquad \Gamma = \frac{I_y D^2}{4}$$

686 x 254 x 140 UB $\Rightarrow$

$$\begin{aligned}
D &= 683.5\text{ mm} &&= 0.6835\text{ m} \\
I_y &= 5183\text{ cm}^4 &&= 5183 \times 10^{-8}\text{ m}^4 \\
J &= 169\text{ cm}^4 &&= 169 \times 10^{-8}\text{ m}^4 \\
E &= 210 \times 10^9\text{ N/m}^2 \\
G &= 81 \times 10^9\text{ N/m}^2 \\
L &= 6\text{ m}
\end{aligned}$$

(or use "D" = D − t )

$$\begin{aligned}
\sqrt{GJEI_y} &= 10^{-8}\,10^{9}\sqrt{(81)(169)(210)(5183)} = \\
&= 10 \times 122 \times 10^3\text{ Nm}^2 \\
&= 1220\text{ kNm}^2
\end{aligned}$$

$$\Gamma = \frac{(5183 \times 10^{-8})(0.6835)^2}{4} = 605 \times 10^{-8}\text{ m}^6$$

$$\frac{\pi^2}{L^2}\frac{E\Gamma}{GJ} = \frac{\pi^2}{36}\left(\frac{210}{81}\right)\left(\frac{605\times10^{-8}}{169\times10^{-8}}\right) = 2.54$$

$$\left(1 + \frac{\pi^2 E\Gamma}{L^2 GJ}\right)^{1/2} = \sqrt{3.54} = 1.88$$

$$M_{cr} = \frac{\pi}{6}(1220)\,1.88 \quad \frac{\text{kNm}^2}{\text{m}} = 1203\text{ kNm}$$