PART IIA 2005
3D7: Finite element methods
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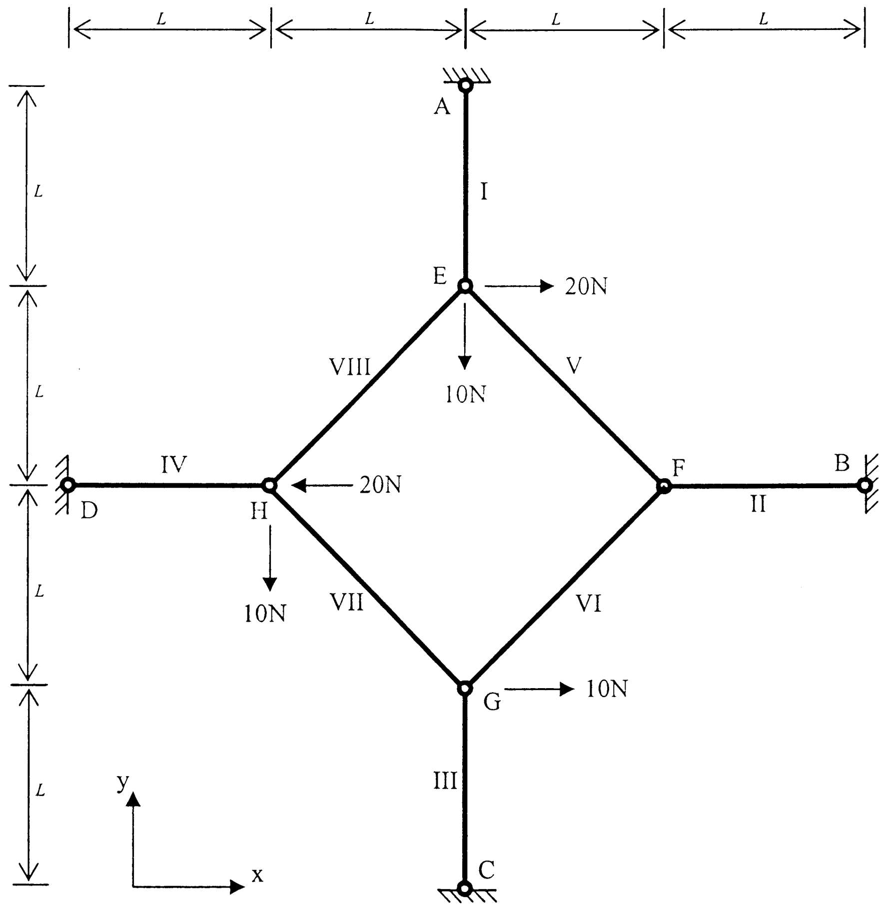
Datasheet: Finite Element Methods Formulae

2005 3D7

Q. 1.



(a) $\mathbf{Hr} = \mathbf{p}$ Find **H**. Joint equilibrium gives:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ -1 & 0 & 0 & 0 & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} t_I \\ t_{II} \\ t_{III} \\ t_{IV} \\ t_V \\ t_{VI} \\ t_{VII} \\ t_{VIII} \end{bmatrix} = \begin{bmatrix} p_{EX} \\ p_{EY} \\ p_{FX} \\ p_{FY} \\ p_{GX} \\ p_{GY} \\ p_{HX} \\ p_{HY} \end{bmatrix}$$

[6 marks]

(b) The external loads given in the Figure are:

$\mathbf{p} = \{20 \quad -10 \quad 0 \quad 0 \quad 10 \quad 0 \quad -20 \quad -10\}^T$ N

Given

$\mathbf{r}_0 = \{30 \quad 0 \quad 10 \quad 10 \quad 0 \quad 0 \quad 10\sqrt{2} \quad 20\sqrt{2}\}^T$ N

it is straightforward to show that $\mathbf{Hr}_0 = \mathbf{p}$

[2 marks]

(c) A suitable state of self stress is

$$\mathrm{S} = \{1 \quad 1 \quad 1 \quad 1 \quad 1/\sqrt{2} \quad 1/\sqrt{2} \quad 1/\sqrt{2} \quad 1/\sqrt{2}\}^T \text{ N}$$

Because **H** is square the existence of this single state of self-stress means that the structure contains a mechanism. In this case the mechanism is infinitesimal and involves rotation of the central square EFGH.

[4 marks]

(d) The structure is initially stress-free and therefore there is no lack-of-fit $\mathbf{e_0} = 0$.

To find the correct member forces **r** use the *Force Method* equations on the data sheet to apply compatibility.

$\mathbf{S^T F S}x = -\mathbf{S^T F r_0}$

F is a diagonal matrix $\mathbf{F} = \dfrac{L}{EA}\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & \sqrt{2} & & & \\ & & & & & \sqrt{2} & & \\ & & & & & & \sqrt{2} & \\ & & & & & & & \sqrt{2} \end{bmatrix}$

Hence $x = -\dfrac{\mathbf{S^T F r_0}}{\mathbf{S^T F S}} = -\dfrac{92.43}{6.828} = -13.54$

$\mathbf{r} = \mathbf{r_0} + \mathrm{S}x$

Therefore $\mathbf{r} = \{16.46 \quad -13.54 \quad -3.54 \quad -3.54 \quad -9.57 \quad -9.57 \quad 4.57 \quad 18.71\}^T$ N

[4 marks]

(e) To prevent any cable becoming slack under the prescribed loading it is necessary to add prestress, of a magnitude x' of the state of self-stress **S,** to **r** so that all **r** is greater than 50N i.e.

$$\begin{bmatrix} 16.46 \\ -13.54 \\ -3.54 \\ -3.54 \\ -9.57 \\ -9.57 \\ 4.57 \\ 18.71 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} x' = \begin{bmatrix} \\ \\ \\ \text{all} \\ > 50 \\ \\ \\ \\ \end{bmatrix}$$

Critical members are VI and VII, they require $x' = \dfrac{50+9.57}{1/\sqrt{2}} = 84.24$

(also check member II, it requires $x' = 50 + 13.54 = 63.54$).

We require $\mathbf{e}_0 = \{e_I \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^T$ so that $x' = 84.24$ in $\mathbf{S}^\mathbf{T}\mathbf{FS}x' = -\mathbf{S}^\mathbf{T}\mathbf{e}_0$

Therefore

$$6.828\frac{L}{EA} \text{ x } 84.24 = -e_I$$

$$e_I = -575\frac{L}{EA} \quad \text{(a shortening)}$$

[4 marks]

Q. 2. (a) $\mathbf{K}_i' = \int_\Omega \mathbf{B}_i^{\mathbf{T}}\mathbf{D}\mathbf{B}_i|\mathbf{J}|d\xi d\eta$

$\mathbf{K}_i'$ is the stiffness matrix of element i based on the local (element) node numbering

Ω is the area of the parent element

$\mathbf{B}_i$ is the strain shape function matrix:

$$\mathbf{B}_i = \begin{bmatrix} \frac{\partial n_A}{\partial X} & 0 & \frac{\partial n_B}{\partial X} & 0 & \frac{\partial n_C}{\partial X} & \\ 0 & \frac{\partial n_A}{\partial Y} & 0 & \frac{\partial n_B}{\partial Y} & 0 & etc....... \\ \frac{\partial n_A}{\partial Y} & \frac{\partial n_A}{\partial X} & \frac{\partial n_B}{\partial Y} & \frac{\partial n_B}{\partial X} & \frac{\partial n_C}{\partial Y} & \end{bmatrix}$$

D is the *material stiffness matrix* i.e. (for plane stress):

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

J is the Jacobian matrix of coordinate transformation from parent to iso-parametric element. It relates the derivatives w.r.t. ξ, η of any shape function n_i to its derivatives w.r.t. X, Y. $|\mathbf{J}|$ is a scale factor that multiplies $d\xi d\eta$ to produce the physical area increment $dXdY$.

[6 marks]

(b) Coordinates of point G' in the parent element: ($\xi = 1/\sqrt{3}$, $\eta = -1/\sqrt{3}$)

Shape functions at G':

$n_q = (1-1/\sqrt{3})(1+1/\sqrt{3})/4 = 1/6 = 0.16667$

$n_r = (1+1/\sqrt{3})(1+1/\sqrt{3})/4 = 1/3 + 1/2\sqrt{3} = 0.62201$

$n_s = (1+1/\sqrt{3})(1-1/\sqrt{3})/4 = 1/6 = 0.16667$

$n_t = (1-1/\sqrt{3})(1-1/\sqrt{3})/4 = 1/3 - 1/2\sqrt{3} = 0.04466$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} n_q & 0 & n_r & 0 & n_s & 0 & n_t & 0 \\ 0 & n_q & 0 & n_r & 0 & n_s & 0 & n_t \end{bmatrix} \begin{bmatrix} X_Q \\ Y_Q \\ . \\ . \\ X_T \\ Y_T \end{bmatrix}$$

$$= \begin{bmatrix} 0.16667 & 0 & 0.62201 & 0 & 0.16667 & 0 & 0.04466 & 0 \\ 0 & 0.16667 & 0 & 0.62201 & 0 & 0.16667 & 0 & 0.04466 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 2 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1.57736 \\ 0.58933 \end{bmatrix} = \begin{bmatrix} 1 + 1/\sqrt{3} \\ 7/6 - 1/\sqrt{3} \end{bmatrix}$$

[7 marks]

(c) Elements of the Jacobian matrix **J** at G

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial X}{\partial \xi} & \dfrac{\partial Y}{\partial \xi} \\ \dfrac{\partial X}{\partial \eta} & \dfrac{\partial Y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial n_q}{\partial \xi} & \dfrac{\partial n_r}{\partial \xi} & \dfrac{\partial n_s}{\partial \xi} & \dfrac{\partial n_t}{\partial \xi} \\ \dfrac{\partial n_q}{\partial \eta} & \dfrac{\partial n_r}{\partial \eta} & \dfrac{\partial n_s}{\partial \eta} & \dfrac{\partial n_t}{\partial \eta} \end{bmatrix} \begin{bmatrix} X_Q & Y_Q \\ X_R & Y_R \\ X_S & Y_S \\ X_T & Y_T \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} \eta - 1 & 1 - \eta & 1 + \eta & -1 - \eta \\ \xi - 1 & -\xi - 1 & 1 + \xi & 1 - \xi \end{bmatrix} \begin{bmatrix} X_Q & Y_Q \\ X_R & Y_R \\ X_S & Y_S \\ X_T & Y_T \end{bmatrix} \qquad = \frac{1}{4} \begin{bmatrix} \eta - 1 & 1 - \eta & 1 + \eta & -1 - \eta \\ \xi - 1 & -\xi - 1 & 1 + \xi & 1 - \xi \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 2 & 3 \\ 0 & 2 \end{bmatrix}$$

$$\frac{1}{4} \begin{bmatrix} 4 & 1 + \eta \\ 0 & 5 + \xi \end{bmatrix} = \begin{bmatrix} 1 & 0.1056 \\ 0 & 1.394 \end{bmatrix}$$

[7 marks]

Q. 3. (a) Following from the requirement that any rigid body translation should produce no strains in the element $\sum_i n_i = 1$ **at any point within the element**

Displacement field in an element: $\mathbf{u} = \sum \mathbf{u}_i n_i$

Rigid body displacement **a**: $\mathbf{a} = \sum \mathbf{a}_i n_i$ where $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{a}_n$

Discretisation exact at nodes if $\mathbf{a} = \mathbf{a}\sum n_i$ and therefore $1 = \sum n_i$

[4 marks]

(b) Start by finding shape functions for a linear displacement field for the 3-node element ACD:

$n_A = 1$ @ (-1,1) and zero along the line $X = Y$ Hence $n_A = \dfrac{Y - X}{2}$

$n_C = 1$ @ (1,1) and zero along the line $X = -Y$ Hence $n_C = \dfrac{X + Y}{2}$

$n_D = 1$ @ (0,0) and zero along the line $Y = 1$ Hence $n_D = 1 - Y$

Now find a quadratic shape function for B:

$n_B = 1$ @ (0,1) and is zero along both the lines $X = Y$ and $X = -Y$

Therefore $n_B = (Y - X)(Y + X) = Y^2 - X^2$

Now modify linear shape functions for A, C and D by adding a component of n_B:

$n_A = \dfrac{Y - X}{2} + c_A n_B$ $\quad n_A(0,1) = \dfrac{1}{2} + c_A = 0$

$n_C = \dfrac{X + Y}{2} + c_C n_B$ $\quad n_C(0,1) = \dfrac{1}{2} + c_C = 0$

$n_D = 1 - Y + c_D n_B$ $\quad n_D(0,1) = 0 + c_D = 0$

Therefore:

$$n_A = \frac{Y - X}{2} - \frac{Y^2 - X^2}{2} \qquad n_B = Y^2 - X^2 \qquad n_C = \frac{X + Y}{2} - \frac{Y^2 - X^2}{2} \qquad n_D = 1 - Y$$

These shape functions are linear along AD and CD and quadratic along ABC as required.

Also check : $\sum_i n_i = n_A + n_B + n_C + n_D = 1$

[8 marks]

(c) Nodal displacement components are:

$$\mathbf{d} = \begin{bmatrix} d_{AX} & d_{AY} & d_{BX} & d_{BY} & d_{CX} & d_{CY} & d_{DX} & d_{DY} \end{bmatrix}^T$$
$$= \frac{1}{10}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 \end{bmatrix}^T$$

Displacement field:

$$u = n_A d_{AX} + n_B d_{BX} + n_C d_{CX} + n_D d_{DX}$$
$$v = n_A d_{AY} + n_B d_{BY} + n_C d_{CY} + n_D d_{DY}$$

$$u = \frac{1}{10}\left\{\frac{X+Y}{2} - \frac{Y^2 - X^2}{2} + 2(1-Y)\right\}$$

$$v = \frac{1}{10}\left\{\frac{X+Y}{2} - \frac{Y^2 - X^2}{2}\right\}$$

Find the state of strain:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{XX} \\ \varepsilon_{YY} \\ \gamma_{XY} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial X} \\ \dfrac{\partial v}{\partial Y} \\ \dfrac{\partial u}{\partial Y} + \dfrac{\partial v}{\partial X} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} \dfrac{1}{2} + X \\ \dfrac{1}{2} - Y \\ X - Y - 1 \end{bmatrix}$$

Find the strain at point P (0,1/2):

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{XX} \\ \varepsilon_{YY} \\ \gamma_{XY} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1/2 \\ 0 \\ -3/2 \end{bmatrix}$$

[8 marks]

Q. 4. (a) Applying a weight function v to the governing equation and integrate between x= a and x=b.

$$v\left[\frac{d}{dx}(Ak\frac{dT}{dx})+Q\right]=0$$

$$\int_b^a v\left[\frac{d}{dx}(Ak\frac{dT}{dx})+Q\right]dx=0$$

Integrate by parts to derive the weak form.

$$\left[vAk\frac{dT}{dx}\right]_b^a-\int_b^a\frac{dv}{dx}Ak\frac{dT}{dx}dx+\int_b^a vQdx=0$$

Heat flux is defined as $q=-k(dT/dx)$

$$\int_b^a\frac{dv}{dx}Ak\frac{dT}{dx}dx=(vAq)_{x=b}-(vAq)_{x=a}+\int_b^a vQdx$$

$$\int_b^a\frac{dv}{dx}Ak\frac{dT}{dx}dx=(vA)_{x=b}q_b-(vA)_{x=a}q_a+\int_b^a vQdx$$

[6 marks]

(b) Substituting the shape functions given and noting that a is not a function of x,

$$\left(\int_b^a \mathbf{Bc}Ak\mathbf{B}dx\right)\mathbf{a}=(\mathbf{Nc}A)_{x=b}q_b-(\mathbf{Nc}A)_{x=a}q_a+\int_b^a\mathbf{Nc}Qdx$$

Using vector product manipulation, $\mathbf{Bc}=\mathbf{c}^T\mathbf{B}^T$, $\mathbf{Nc}=\mathbf{c}^T\mathbf{N}^T$

$$\left(\int_b^a \mathbf{c}^T\mathbf{B}^T Ak\mathbf{B}dx\right)\mathbf{a}=(\mathbf{c}^T\mathbf{N}^T A)_{x=b}q_b-(\mathbf{c}^T\mathbf{N}^T A)_{x=a}q_a+\int_b^a\mathbf{c}^T\mathbf{N}^T Qdx$$

Since $\mathbf{c}$ is not a function x,

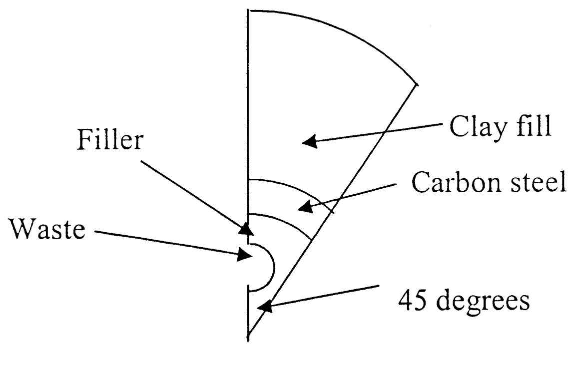
$$\mathbf{c}^T\left(\int_b^a \mathbf{B}^T Ak\mathbf{B}dx\right)\mathbf{a}=\mathbf{c}^T\left\{(\mathbf{N}^T A)_{x=b}q_b-(\mathbf{N}^T A)_{x=a}q_a+\int_b^a\mathbf{N}^T Qdx\right\}$$

Cancelling $\mathbf{c}^T$, the following finite element formulation can be obtained.

$$\left(\int_b^a \mathbf{B}^T Ak\mathbf{B}dx\right)\mathbf{a}=(\mathbf{N}^T A)_{x=b}q_b-(\mathbf{N}^T A)_{x=a}q_a+\int_b^a\mathbf{N}^T Qdx$$

[4 marks]

(c) Use 1/8 of the system considering the symmetry.



[4 marks]

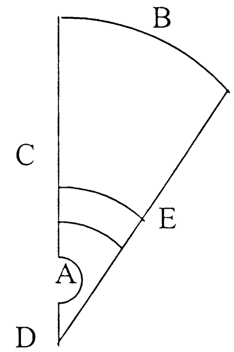
(d)
The heat flux is uniform at the clay-rock interface and is in the radial direction.

The conductivities of the materials (container, filler, carbon steel and clay fill) are isotropic and uniform.

No heat flux in the longitudinal direction (i.e. into the paper).

[3 marks]

(e)



A: Boundary between the waste and filler – Temperature Ts

B: Boundary between the rock and clay barrier, use conduction boundary, $q = h(T-T_0)$, where h is the convection coefficient and T_R is the temperature of the rock far away from the system.

C, D and E: Heat flux is zero.

[3 marks]